EVOLUTE OF A QUADRATIC

TRISTRAM DE PIRO

For the quadratic $y - x^2 = 0$;

$\frac{dy}{dx} = 2x$

At a general point (a, a^2), we have that;

$m_{tangent} = 2a$

$m_{normal} = -\frac{1}{2a}$

The equation of the normal through (a, a^2) with gradient $-\frac{1}{2a}$ is given by;

$\frac{y-a^2}{x-a} = -\frac{1}{2a}$

Similarly, the equation of the normal through $(a+h, (a+h)^2)$ with gradient $-\frac{1}{2(a+h)}$ is given by;

$\frac{y-(a+h)^2}{x-(a+h)} = -\frac{1}{2(a+h)}$

These two lines intersect when;

$a^2 - \frac{(x-a)}{2a} = (a+h)^2 - \frac{x-(a+h)}{2(a+h)}$

and rearranging;

$\frac{x-(a+h)}{2(a+h)} - \frac{x-a}{2a} = (a+h)^2 - a^2 = 2ah + h^2$

and cross multiplying;

$2a(x-(a+h)) - 2(a+h)(x-a) = 2a(2(a+h))(2ah+h^2)$

Cancelling terms;

$-hx = 4a(a+h)(2ah+h^2)$

$x = -4a(a+h)(2a+h)$

Taking the limit as $h \rightarrow 0$;

$x = -8a^3$

$y = a^2 - \frac{(-8a^3-a)}{2a} = 5a^2 + \frac{1}{2}$

The centre of curvature of the point $(a, a^2$ is $(-8a^3, 5a^2 + \frac{1}{2})$

As a varies, we trace out the *evolute* of the quadratic $y = x^2$. It is a cubic with the equation;

$(y - \frac{1}{2})^3 = \frac{125x^2}{64}$

$y = \frac{1}{2} + \frac{5x^{\frac{2}{3}}}{4}$

Note that as a traces out the right hand side of the quadratic, a traces out the left hand side of the evolute curve and vice-versa.

We define the *involute* curve to be the locus of normals to the quadratic. The evolute is the *dual* curve to the involute.

Note on duality;

To any line $y = mx + c$, we can associate a point in the dual projective space $P^2(\mathcal{R})$, given by $[-m : 1 : -c] = [\frac{m}{c} : -\frac{1}{c} : 1]$ in projective coordinates. The dual curve is the curve defined by the tangents to the curve in $P^2(\mathcal{R})$. If we compute the dual of $y = nx^2$, we get $y = \frac{x^2}{4n}$. This follows as the equation of the tangent to the curve at (a, na^2) is given by;

$\frac{y-na^2}{x-a} = 2na$

$y = 2nax - 2na^2 + na^2 = 2nax - na^2$

which corresponds to the point $[-2na : 1 : na^2] = [-\frac{2}{a} : \frac{1}{na^2} : 1]$. It follows that the dual of the dual curve to $y = nx^2$ is;

$y = \frac{x^2}{4\frac{1}{4n}} = nx^2$

so the dual of the dual of $y = x^2$ is itself. This holds in general. Similarly, to a point $(a, b) \in \mathcal{R}^2$, we can associate a *line* in the dual space given by the locus of all the lines which pass through the point. We have the relation $R(p, l)$ if $p \in l$, and it is true that;

$R(p, l)$ iff $R(l^*, p^*)$

where $*$ is the dual. Given the involute is the locus of normals to the quadratic. If we fix a point p on the involute and draw a chord l' between p and p' on the involute, then $R(p, l')$ and $R(p', l')$, so that $R(l'^*, p^*)$ and $R(l'^*, p'^*)$, that is l'^* lies on the intersection of the normals corresponding to p and p' on the original quadratic. Taking the limit l'' of chords as $p' \to p$, the limit l''^* corresponds to the centre of curvature where the normal p^* intersects the quadratic. The limit of chords l'' is just the tangent to the involute, so the evolute is the dual curve to the involute, and conversely.

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