SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 4

TRISTRAM DE PIRO

Abstract.

Definition 0.1. *We call $(\overline{E}_0, \overline{B}_0)$, a solution to Maxwell's equation in vacuum, good, if $(\overline{E}+\overline{E}_0)\times\overline{B}_0 = 0$, for some fundamental solution $(\overline{E},\overline{0})$ corresponding to $\{\rho,\overline{J}\}$ satisfying the conditions from Lemma 4.1 in* [12]*, with $\{\rho,\overline{J}\}$ not vacuum and $\{\rho,\overline{J}\}\subset S(\mathcal{R}^3\times\mathcal{R}_{>0})$. We call $(\overline{E}_0, \overline{B}_0)$ static if $\frac{\partial\overline{E}_0}{\partial t}=\frac{\partial\overline{B}_0}{\partial t}=\overline{0}$.*

Definition 0.2. *We say that a field $\overline{C}(\overline{x},t)$ is simple if all the components c_i, $1\leq i\leq 3$ are continuously fourth differentiable in the coordinates (x_1,x_2,x_3) and continuously twice differentiable in the coordinate t, such that the partial derivatives all belong to $L^1(\mathcal{R}^3)$ for fixed $t\geq 0$, and, the L^1-norm of the partial derivatives is uniformly bounded for $0\leq t<1$. We also require that the components c_i are in $L^2(\mathcal{R}^3)$ and, for some $a>0$, $e^{a|\overline{x}|}c_i(\overline{x},t)\in L^2(\mathcal{R}^3)$.*

Definition 0.3. *We say that a real pair $(\overline{E},\overline{B})$, satisfying Maxwell's equations for some $\{\rho,\overline{J}\}$, satisfies the strong no radiation condition if;*

$$P(r,t)=\int_{S(\overline{0},r)}(\overline{E}_t\times\overline{B}_t)\cdot d\overline{S}=0$$

for all $r>0$ and $t\in\mathcal{R}$. We say that it satisfies the no radiation condition if;

$$lim_{r\rightarrow\infty}P(r,t)=0$$

for all $t\in\mathcal{R}$

Lemma 0.4. *For any $\{\rho,\overline{J}\}$ satisfying the conditions from Lemma 4.1 in* [12]*, if $(\overline{E},\overline{0})$ denotes a fundamental solution, then a solution $\{\overline{E}+\overline{E}_0,\overline{B}_0\}$, with $(\rho,\overline{J},\overline{E}+\overline{E}_0,\overline{B}_0)$ satisfying Maxwell's equations, satisfies the no radiating condition, if $\overline{E}$,$\overline{E}_0$ and $\overline{B}_0$ are simple and $\{(\overline{E}+\overline{E}_0)_0,\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_0,(\overline{B}_0)_0,\frac{\partial\overline{B}_0}{\partial t}|_0\}\subset S(\mathcal{R}^3)$, $(*)$. Moreover, we have* *the explicit representation;*

$(\overline{E}+\overline{E}_0)(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b}(\overline{k})e^{ikct}+\overline{d}(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$\overline{B}_0(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b}_1(\overline{l})e^{ilct}+\overline{d}_1(\overline{l})e^{-ilct})e^{i\overline{l}.\overline{x}}d\overline{l}$

where $\{\overline{b},\overline{d},\overline{b}_1,\overline{d}_1\}\subset S(\mathcal{R}^3)$.

Proof. By Lemma 4.1 in [12], and the argument in [6], we have that;

$\square^2\overline{E}=\overline{0}$, $\overline{B}=\overline{0}$

$\square^2\overline{E}_0=\overline{0}$, $\square^2\overline{B}_0=\overline{0}$ $(*)$

Then, as $\overline{B}=\overline{0}$;

$lim_{r\rightarrow\infty}P(r)=lim_{r\rightarrow\infty}\int_{S(r)}((\overline{E}+\overline{E}_0)\times(\overline{B}+\overline{B}_0))d\overline{S}(r)$

$=lim_{r\rightarrow\infty}\int_{S(r)}(\overline{E}\times\overline{B})d\overline{S}(r)+lim_{r\rightarrow\infty}\int_{S(r)}((\overline{E}+\overline{E}_0)\times\overline{B}_0)d\overline{S}(r)$

$+lim_{r\rightarrow\infty}\int_{S(r)}(\overline{E}_0\times\overline{B})d\overline{S}(r)$

$=lim_{r\rightarrow\infty}\int_{S(r)}((\overline{E}+\overline{E}_0)\times\overline{B}_0)d\overline{S}(r)$

and, by $(*)$, we have that $\square^2(\overline{E}+\overline{E}_0)=\overline{0}$ as well, $(\dagger)$.

Assume that $\overline{E}$,$\overline{E}_0$ and $\overline{B}_0$ are simple, then, $\overline{E}+\overline{E}_0$ and $\overline{B}_0$ are simple, and we have that;

$\bigtriangledown^2(\overline{E}-\overline{E}_0)-\frac{1}{c^2}\frac{\partial^2(\overline{E}-\overline{E}_0)}{\partial t^2}=\overline{0}$

so that, applying the three dimensional Fourier transform $\mathcal{F}$ to the components, and using integration by parts, we have that;

$\mathcal{F}(\bigtriangledown^2(\overline{E}-\overline{E}_0))(\overline{k},t))-\frac{1}{c^2}\frac{\partial^2(\mathcal{F}(\overline{E}-\overline{E}_0))(\overline{k},t)}{\partial t^2}$

$=-k^2\mathcal{F}(\overline{E}-\overline{E}_0)(\overline{k},t)-\frac{1}{c^2}\frac{\partial^2(\mathcal{F}(\overline{E}-\overline{E}_0))(\overline{k},t)}{\partial t^2}$

$=-k^2\overline{a}(\overline{k},t)-\frac{1}{c^2}\frac{\partial^2\overline{a}(\overline{k},t)}{\partial t^2}$

$= \overline{0}$

where $k^2 = k_1^2 + k_2^2 + k_3^2$, $\overline{a} = \mathcal{F}(\overline{E} - \overline{E}_0)$. For fixed $\overline{k}$, we obtain the ordinary differential equation;

$\frac{d^2\overline{a}_{\overline{k}}}{dt^2} = -c^2k^2\overline{a}_{\overline{k}}$

so that;

$\overline{a}_{\overline{k}}(t) = \overline{C}_0(\overline{k})e^{ikct} + \overline{D}_0(\overline{k})e^{-ikct}$

with;

$\overline{a}_{\overline{k}}(0) = \overline{C}_0(\overline{k}) + \overline{D}_0(\overline{k})$

$\overline{a}'_{\overline{k}}(0) = ikc\overline{C}_0(\overline{k}) - ikc\overline{D}_0(\overline{k})$ (††)

and, solving the simultaneous equations (††), we obtain that;

$\overline{C}_0(\overline{k}) = \frac{1}{2}(\overline{a}_{\overline{k}}(0) + \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))$

$\overline{D}_0(\overline{k}) = \frac{1}{2}(\overline{a}_{\overline{k}}(0) - \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))$

and;

$\mathcal{F}(\overline{E} - \overline{E}_0)(\overline{k}, t) = \overline{a}(\overline{k}, t)$

$= \frac{1}{2}(\overline{a}_{\overline{k}}(0) + \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))e^{ikct} + \frac{1}{2}(\overline{a}_{\overline{k}}(0) - \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))e^{-ikct}$

$= \overline{b}(\overline{k})e^{ikct} + \overline{d}(\overline{k})e^{-ikct}$

where;

$\overline{b}(\overline{k}) = \frac{1}{2}(\mathcal{F}((\overline{E} + \overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)} + \frac{1}{ikc}\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$\overline{d}(\overline{k}) = \frac{1}{2}(\mathcal{F}((\overline{E} + \overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)} - \frac{1}{ikc}\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

Similarly;

$\mathcal{F}(\overline{B}_0)(\overline{l}, t) = \overline{a}_1(\overline{l}, t) = \overline{b}_1(\overline{l})e^{ilct} + \overline{d}_1(\overline{l})e^{-ilct}$

where;

$\bar{b}_1(\bar{l}) = \frac{1}{2}(\mathcal{F}((\overline{B}_0)|_{(\bar{x},0)})|_{(\bar{l},0)} + \frac{1}{ilc}\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\bar{x},0)})|_{(\bar{l},0)})$

$\bar{d}+1(\bar{l}) = \frac{1}{2}(\mathcal{F}((\overline{B}_0)|_{(\bar{x},0)})|_{(\bar{l},0)} - \frac{1}{ilc}\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\bar{x},0)})|_{(\bar{l},0)})$

and $l^2 = l_1^2+l_2^2+l_3^2$. Using the fact that $\{\bar{b}(\bar{k})e^{ikct}+\bar{d}(\bar{k})e^{-ikct}, \bar{b}_1(\bar{l})e^{ilct}+\bar{d}_1(\bar{l})e^{-ilct}\} \subset S(\mathcal{R}^3)$ for $t \in \mathcal{R}$, we can apply the inversion theorem, to obtain;

$(\overline{E}+\overline{E}_0)(\bar{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\bar{b}(\bar{k})e^{ikct}+\bar{d}(\bar{k})e^{-ikct})e^{i\bar{k}\centerdot\bar{x}}d\bar{k}$

$\overline{B}_0(\bar{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\bar{b}_1(\bar{l})e^{ilct}+\bar{d}_1(\bar{l})e^{-ilct})e^{i\bar{l}\centerdot\bar{x}}d\bar{l}$

As we noted above, $\{\bar{b}e^{ikct}+\bar{d}e^{-ikct}, \bar{b}_1e^{ilct}+\bar{d}_1e^{-ilct}\} \subset S(\mathcal{R}^3)$ for $t \in \mathcal{R}$, so that, by the fact that the Fourier transform preserves the Schwartz class, see [17], we must have that $\{(\overline{E}+\overline{E}_0)_t, (\overline{B}_0)_t\} \subset S(R^3)$ for $t \in \mathcal{R}$. Then, for $n \geq 3$ and the definition of the Schwartz class;

$|P(r,t)| = |\int_{S(r)}((\overline{E}+\overline{E}_0)_t \times (\overline{B}_0)_t)d\overline{S}|$

$\leq \int_{S(r)}|((\overline{E}+\overline{E}_0)_t \times (\overline{B}_0)_t)\centerdot\hat{\bar{n}}|dS(r)|$

$\leq \int_{S(r)}|(\overline{E}+\overline{E}_0)_t||(\overline{B}_0)_t|dS(r)$

$\leq 4\pi r^2\frac{C_{1,n,t}}{r^n}\frac{D_{1,n,t}}{r^n}$

$= \frac{4\pi C_{1,n,t}D_{1,n,t}}{r^{2n-2}}$

so clearly;

$lim_{r\rightarrow\infty}P(r,t) = 0$

□

Definition 0.5. *Fix a real propagation vector* $\bar{k}_0 \neq \bar{0}$ *and a real vector* $\bar{d}_0$ *with* $\bar{k}_0 \centerdot \bar{d}_0 = 0$*. Let;*

$\overline{E}_0(\bar{x},t) = \bar{d}_0e^{-ik_0ct}e^{i\bar{k}_0\centerdot\bar{x}}$

$\overline{B}_0(\bar{x},t) = \bar{d}_1e^{-ik_0ct}e^{i\bar{k}_0\centerdot\bar{x}}$

where $\overline{d}_1 = \frac{1}{k_0 c}(\overline{k}_0 \times \overline{d}_0)$. *Then, see* [6], *the pair* $(\overline{E}_0, \overline{B}_0)$ *solves Maxwell's equation in vacuum, and so does* $(Re(\overline{E}_0), Re(\overline{B}_0))$. *We call a real pair* $(\overline{E}_1, \overline{B}_1)$ *a monochromatic solution if it of the form* $(Re(\overline{E}_0), Re(\overline{B}_0))$ *as above, or* $(\overline{E}_1, \overline{B}_1)$ *are constants.*

Lemma 0.6. *For a monochromatic solution to Maxwell's equation in vacuum, we have that* $P(r,t) = O(r)$. *In particularly, the pair* $(\overline{E}_1, \overline{B}_1)$ *doesn't satisfy the no radiation condition unless* $\overline{E}_1$ *and* $\overline{B}_1$ *are constants. Any constant real solution* $(\overline{E}_1, \overline{B}_1)$ *satisfies the strong no radiation and no radiation conditions.*

Proof. We have, for a monochromatic solution, with $\overline{k}_0 \neq 0$, that;

$$Re(\overline{E}_0)(\overline{x}, t) = \frac{\overline{d}_0}{2}(e^{-ik_0 ct} e^{i\overline{k}_0 \centerdot \overline{x}} + e^{ik_0 ct} e^{-i\overline{k}_0 \centerdot \overline{x}})$$

$$Re(\overline{B}_0)(\overline{x}, t) = \frac{\overline{d}_1}{2}(e^{-ik_0 ct} e^{i\overline{k}_0 \centerdot \overline{x}} + e^{ik_0 ct} e^{-i\overline{k}_0 \centerdot \overline{x}})$$

so that;

$$Re(\overline{E}_0) \times Re(\overline{B}_0) = \frac{(\overline{d}_0 \times \overline{d}_0')}{4}(e^{-2ik_0 ct} e^{2i\overline{k}_0 \centerdot \overline{x}} + e^{2ik_0 ct} e^{-2i\overline{k}_0 \centerdot \overline{x}} + 2)$$

By the divergence theorem, using [3] and [8], we have that;

$$P(r,t) = \int_{S(\overline{0},r)} (Re(\overline{E}_0) \times Re(\overline{B}_0)) d\overline{S}(r)$$

$$= \int_{B(\overline{0},r)} \bigtriangledown \centerdot (\frac{(\overline{d}_0 \times \overline{d}_1)}{4}(e^{-2ik_0 ct} e^{2i\overline{k}_0 \centerdot \overline{x}} + e^{2ik_0 ct} e^{-2i\overline{k}_0 \centerdot \overline{x}} + 2)) dB(r)$$

$$= \int_{B(\overline{0},r)} \frac{(\overline{d}_0 \times \overline{d}_1)}{4} \centerdot 2i\overline{k}_0 (e^{-2ik_0 ct} e^{2i\overline{k}_0 \centerdot \overline{x}} - e^{2ik_0 ct} e^{-2i\overline{k}_0 \centerdot \overline{x}}) dB(r)$$

$$= \frac{(\overline{d}_0 \times \overline{d}_1)}{4} \centerdot 2i\overline{k}_0 (e^{-2ik_0 ct} - e^{2ik_0 ct})(\frac{2\pi r}{|2\overline{k}_0|})^{\frac{3}{2}} J_{\frac{3}{2}}(r|2\overline{k}_0|)$$

$$= \frac{(\overline{d}_0 \times \overline{d}_1)}{2} \centerdot i\overline{k}_0 (e^{-2ik_0 ct} - e^{2ik_0 ct})(\frac{\pi r}{|\overline{k}_0|})^{\frac{3}{2}} J_{\frac{3}{2}}(2r|\overline{k}_0|)$$

$$= \frac{(\overline{d}_0 \times \overline{d}_1)}{2} \centerdot i\overline{k}_0 (e^{-2ik_0 ct} - e^{2ik_0 ct})(\frac{\pi r}{|\overline{k}_0|})^{\frac{3}{2}} (\frac{1}{\pi r|\overline{k}_0|})^{\frac{1}{2}} (P_1(\frac{1}{2r|\overline{k}_0|}) sin(2r|\overline{k}_0|)$$

$$-Q_0(\frac{1}{2r|\overline{k}_0|}) cos(2r|\overline{k}_0|))$$

$$= \frac{(\overline{d}_0 \times \overline{d}_1)}{2} \centerdot i\overline{k}_0 (e^{-2ik_0 ct} - e^{2ik_0 ct})(\frac{\pi r}{|\overline{k}_0|})^{\frac{3}{2}} (\frac{1}{\pi r|\overline{k}_0|})^{\frac{1}{2}} ((\frac{P_{1,1}}{2r|\overline{k}_0|}) sin(2r|\overline{k}_0|)$$

$$-Q_{0,0} cos(2r|\overline{k}_0|))$$

$= \frac{(\overline{d}_0\times\overline{d}_1)}{2} \centerdot i\overline{k}_0(e^{-2ik_0ct}-e^{2ik_0ct})(\frac{\pi}{|\overline{k}_0|})^{\frac{3}{2}}(\frac{1}{\pi|\overline{k}_0|})^{\frac{1}{2}}((\frac{P_{1,1}}{2|\overline{k}_0|})sin(2r|\overline{k}_0|)$

$-Q_{0,0}rcos(2r|\overline{k}_0|))$

Clearly, $P(r,t)=O(r)$ unless $\overline{d}_0\times\overline{d}_1\centerdot\overline{k}_0=0$, in which case $\overline{d}_0=\overline{0}$, which gives a constant solution. The last claim is clear by the divergence theorem and the fact that $\bigtriangledown\centerdot(\overline{E}_1\times\overline{B}_1)=0$. $\square$

Lemma 0.7. *For any $\{\rho,\overline{J}\}$ satisfying the conditions from Lemma 4.1 in* [12]*, if $(\overline{E},\overline{0})$ denotes a fundamental solution, then a solution $\{\overline{E}+\overline{E}_0,\overline{B}_0\}$, with $(\rho,\overline{J},\overline{E}+\overline{E}_0,\overline{B}_0)$ satisfying Maxwell's equations such that $\{\overline{E},\overline{E}_0,\overline{B}_0\}$ are simple and $\{(\overline{E}+\overline{E}_0)_0,\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_0,(\overline{B}_0)_0,\frac{\partial\overline{B}_0}{\partial t}|_0\}\subset S(\mathcal{R}^3)$, satisfies the strong no-radiation condition, using the integral representation in Lemma 0.4, when;*

$\overline{b}(\overline{k},t)\times\overline{b}_1(\overline{l},t)=\overline{b}(\overline{k},t)\times\overline{d}_1(\overline{l},t)$

$=\overline{d}(\overline{k},t)\times\overline{b}_1(\overline{l},t)$

$=\overline{d}(\overline{k},t)\times\overline{d}_1(\overline{l},t)$

$=\overline{0}$, $(\dagger)$

or when $\overline{B}_0$ is parallel to $\overline{E}+\overline{E}_0$, in the sense that $\overline{B}_0=\lambda(\overline{E}+\overline{E}_0)$. In either of these cases, the no radiation condition holds as well.

If $\{\overline{E},\overline{E}_0,\overline{B}_0\}$ are simple and the components of $\{\overline{E}_0,\overline{B}_0\}$ are non oscillatory, then $\{\overline{E}+\overline{E}_0,\overline{B}_0\}$ satisfies the no-radiation condition.

Proof. Using the result of Lemma 0.4, we can use the integral representations of $\overline{E}+\overline{E}_0$ and $\overline{B}_0$ to compute;

$((\overline{E}+\overline{E}_0)\times\overline{B}_0)(\overline{x},t)$

$=\frac{1}{(2\pi)^3}\int_{\mathcal{R}^6}(\overline{b}(\overline{k})\times\overline{b}_1(\overline{l}))e^{i(\overline{k}+\overline{l})\centerdot\overline{x}}e^{i(k+l)ct}d\overline{k}d\overline{l}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^6}(\overline{b}(\overline{k})\times\overline{d}_1(\overline{l}))e^{i(\overline{k}+\overline{l})\centerdot\overline{x}}e^{i(k-l)ct}d\overline{k}d\overline{l}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^6}(\overline{d}(\overline{k})\times\overline{b}_1(\overline{l}))e^{i(\overline{k}+\overline{l})\centerdot\overline{x}}e^{i(l-k)ct}d\overline{k}d\overline{l}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^6}(\overline{d}(\overline{k})\times\overline{d}_1(\overline{l}))e^{i(\overline{k}+\overline{l})\centerdot\overline{x}}e^{-i(k+l)ct}d\overline{k}d\overline{l}$, $(\dagger\dagger)$

Clearly, if $(\dagger)$ is satisfied, then we obtain that $(\overline{E}+\overline{E}_0)\times\overline{B}_0=\overline{0}$, so that $\bigtriangledown\centerdot((\overline{E}+\overline{E}_0)\times\overline{B}_0)=0$, and using the divergence theorem, $P(r,t)=0$ for all $r>0$ and $t\in\mathcal{R}$, and $lim_{r\rightarrow\infty}P(r,t)=0$, for all $t\in\mathcal{R}$, so that the strong no radiation and no radiation conditions hold. Similarly, if $\overline{B}_0$ is parallel to $\overline{E}+\overline{E}_0$, then $(\overline{E}+\overline{E}_0)\times\overline{B}_0=\overline{0}$, so that $((\overline{E}+\overline{E}_0),\overline{B}_0)$ satisfies the strong no radiation and the no radiation conditions again.

If $\{\overline{E},\overline{E}_0,\overline{B}_0\}$ are simple, then, we have that;

$\mathcal{F}((\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2})^2(\overline{E}+\overline{E}_0))(\overline{k},t)=(k_1^2+k_2^2+k_3^2)^2\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)$

so that, for $|\overline{k}|\geq 1$, $1\leq i\leq 3$;

$|\mathcal{F}(\overline{E}+\overline{E}_0)_i(\overline{k},t)|\leq\frac{1}{|\overline{k}|^4}\int_{\mathcal{R}^3}|(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2})(\overline{E}+\overline{E}_0)_i|d\overline{x}$

$\leq\frac{C_{i,t}}{|\overline{k}|^4}$

and, similarly, for $|\overline{k}|>1$, $1\leq i\leq 3$;

$|\mathcal{F}(\overline{B}_0)_i(\overline{k},t)|\leq\frac{D_{i,t}}{|\overline{k}|^4}$

where $\{C_{i,t},D_{i,t}\}\subset\mathcal{R}_{\geq 0}$

Similarly;

$|\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)|$

$\leq\sum_{i=1}^3|\mathcal{F}(\overline{E}+\overline{E}_0)_i(\overline{k},t)|$

$\leq\frac{C_t}{|\overline{k}|^4}$

where $C_t=\sum_{i=1}^3 C_{i,t}$

and $|\mathcal{F}(\overline{B}_0)(\overline{k},t)|$

$\leq\frac{D_t}{|\overline{k}|^4}$ $(\sharp)$

We have that $\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)$ and $\mathcal{F}(\overline{B}_0)(\overline{k},t)$ are bounded on $B(\overline{0},1)$, as, for $|\overline{k}|\leq 1$;

$|\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)| = |\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{E}+\overline{E}_0)(\overline{x},t)e^{-i\overline{k}.\overline{x}}d\overline{x}|$

$\leq \frac{1}{2\pi}^{\frac{3}{2}}\sum_{i=1}^{3}\int_{\mathcal{R}^3}|(\overline{E}+\overline{E}_0)_i(\overline{x},t)|d\overline{x}$

$\leq \frac{1}{2\pi}^{\frac{3}{2}}\sum_{i=1}^{3}D_{i,t}$

where $D_{i,t}\in\mathcal{R}_{>0}$.

It follows that, using polar coordinates, with $k_1 = Rsin(\theta)cos(\phi)$, $k_2 = Rsin(\theta)sin(\phi)$, $k_3 = Rcos(\theta)$, and using $(\sharp)$
;

$|\int_{\mathcal{R}^3}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}d\overline{k}|$

$= |\int_{B(\overline{0},1)}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}d\overline{k}+\int_{\mathcal{R}^3\setminus B(\overline{0},1)}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}d\overline{k}|$

$\leq C_{i,t,1}+|\int_{R>1}\int_0^{\pi}\int_{-\pi}^{\pi}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}(R,\theta,\phi)R^2sin(\theta)dRd\theta d\phi$

$\leq C_{i,t,1}+\int_{R>1}\int_0^{\pi}\int_{-\pi}^{\pi}R^2\frac{C_{i,t}}{R^4}dR$

$\leq C_{i,t,1}+2\pi^2C_{i,t}\int_1^{\infty}\frac{1}{R^2}dR$

$= C_{i,t,1}+2\pi^2C_{i,t}$

so that, for $1\leq i\leq 3$, $\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}\in L^1(\mathcal{R}^3)$, and, similarly, $\mathcal{F}(\overline{B}_0)_{i,t}\in L^1(\mathcal{R}^3)$. A similar argument shows that for $1\leq i\leq 3$, $\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t})_{i,t}\in L^1(\mathcal{R}^3)$, and $\mathcal{F}(\frac{\partial\overline{B}_0}{\partial t})_{i,t}\in L^1(\mathcal{R}^3)$. We have, using polar coordinates, that;

$|\int_{B(\overline{0},1)}\frac{1}{ikc}\mathcal{F}(\frac{\partial\overline{E}+\overline{E}_0)_i}{\partial t}|_{\overline{x},0})(\overline{k})d\overline{k}|$

$\leq\int_0^1\int_0^{\pi}\int_{-\pi}^{\pi}|\mathcal{F}(\frac{\partial\overline{E}+\overline{E}_0)_i}{\partial t}|_{\overline{x},0})(R,\theta,\phi)|\frac{1}{R}R^2dRd\theta\dot{p}hi$

$= \frac{2\pi^2}{2} = \pi^2$

so that the components, $\frac{1}{ikc}\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)_i}{\partial t})|_{\overline{x},0})(\overline{k})$ for $1\leq i\leq 3$, are integrable on $B(\overline{0},1)$. Similarly, for $1\leq i\leq 3$, the components

$\frac{1}{ilc}\mathcal{F}(\frac{\partial(\overline{B}_0)_i}{\partial t})|_{\overline{x},0})(\overline{l})$ are integrable on $B(\overline{0},1)$. Therefore, so are the components of $\{\overline{b},\overline{b}_1,\overline{d},\overline{d}_1\}$, (B).

Applying the result $(\sharp)$, we obtain that, for $|\overline{k}|>1$;

$$|\overline{b}(\overline{k})+\overline{d}(\overline{k})|\leq\frac{C_0}{|\overline{k}|^4}$$

$$|e^{ikct}\overline{b}(\overline{k})+e^{ikct}\overline{d}(\overline{k})|\leq\frac{C_0}{|\overline{k}|^4}$$

$$|e^{ikct}\overline{b}(\overline{k})+e^{-ikct}\overline{d}(\overline{k})|\leq\frac{C_t}{|\overline{k}|^4}$$

$$|(e^{ikct}-e^{-ikct})\overline{d}(\overline{k})|$$

$$=2|sin(kct)\overline{d}(\overline{k})|$$

$$\leq\frac{C_0+C_t}{|k|^4}$$

so that at time $t=\frac{\pi}{2kc}$, we have that;

$$|\overline{d}(\overline{k})|\leq\frac{C_0+C_{\frac{\pi}{2kc}}}{|k|^4}$$

$$\leq\frac{C_0+E}{|k|^4}$$

where $E\in\mathcal{R}_{>0}$ is the uniform bound for $t\in[0,1]$, and, similarly, for $|\overline{k}|>1$;

$$max(|\overline{b}|,|\overline{b}_1|,|\overline{d}|,|\overline{d}_1|)(\overline{k})\leq\frac{F}{|k|^4}\ (A)$$

for some $F\in\mathcal{R}_{>0}$. In particularly, combining (A), (B), we have that the components of $\{\overline{b},\overline{b}_1,\overline{d},\overline{d}_1\}$ belong to $L^1(\overline{R}^3)$ and we can apply the calculation in $(\dagger\dagger)$.

We consider one term, by the divergence theorem, and using [8], we have that;

$$\int_{S(\overline{0},r)}(\overline{b}(\overline{k})\times\overline{d}_1(\overline{l}))e^{i(\overline{k}+\overline{l})\cdot\overline{x}}e^{i(k-l)ct}d\overline{S}(r)$$

$$=\int_{B(\overline{0},r)}\bigtriangledown\cdot((\overline{b}(\overline{k})\times\overline{d}_1(\overline{l}))e^{i(\overline{k}+\overline{l})\cdot\overline{x}}e^{i(k-l)ct})dB(r)$$

$$=\int_{B(\overline{0},r)}((\overline{b}(\overline{k})\times\overline{d}_1(\overline{l}))\cdot i(\overline{k}+\overline{l}))e^{i(\overline{k}+\overline{l})\cdot\overline{x}}e^{i(k-l)ct}dB(r)$$

$$=((\overline{b}(\overline{k})\times\overline{d}_1(\overline{l}))\cdot i(\overline{k}+\overline{l}))(\frac{2\pi r}{|\overline{k}+\overline{l}|})^{\frac{3}{2}}J_{\frac{3}{2}}(r|\overline{k}+\overline{l}|)e^{i(k-l)ct}$$

$= ((\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi r}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(r|\bar{k}+\bar{l}|)})^{\frac{1}{2}}(P_1(\frac{1}{r|\bar{k}+\bar{l}|})sin(r|\bar{k}+\bar{l}|)$

$-Q_0(\frac{1}{r|\bar{k}+\bar{l}|})cos(r|\bar{k}+\bar{l}|))e^{i(k-l)ct}$

$= ((\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi r}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(r|\bar{k}+\bar{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{r|\bar{k}+\bar{l}|}sin(r|\bar{k}+\bar{l}|)$

$-Q_{0,0}cos(r|\bar{k}+\bar{l}|))e^{i(k-l)ct}$

$= ((\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\bar{k}+\bar{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{|\bar{k}+\bar{l}|}sin(r|\bar{k}+\bar{l}|)e^{i(k-l)ct}$

$-((\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\bar{k}+\bar{l}|)})^{\frac{1}{2}}Q_{0,0}rcos(r|\bar{k}+\bar{l}|))e^{i(k-l)ct}$ $(*)$

By $(*)$, we have that;

$lim_{r\rightarrow\infty}P(r) = \frac{1}{(2\pi)^3}lim_{r\rightarrow\infty}\int_{\mathcal{R}^6}((\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\bar{k}+\bar{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{|\bar{k}+\bar{l}|}$

$sin(r|\bar{k}+\bar{l}|)e^{i(k-l)ct}d\bar{k}d\bar{l}$

$-\frac{1}{(2\pi)^3}lim_{r\rightarrow\infty}\int_{\mathcal{R}^6}((\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\bar{k}+\bar{l}|)})^{\frac{1}{2}}Q_{0,0}$

$rcos(r|\bar{k}+\bar{l}|))e^{i(k-l)ct}d\bar{k}d\bar{l}$

Let $g(\bar{k},\bar{l},t) = \frac{1}{(2\pi)^3}(\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l}))(\frac{2\pi}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\bar{k}+\bar{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{|\bar{k}+\bar{l}|}e^{i(k-l)ct}$

and $h(\bar{k},\bar{l},t) = -\frac{1}{(2\pi)^3}(\bar{b}(\bar{k}) \times \bar{d}_1(\bar{l})) \cdot i(\bar{k}+\bar{l})(\frac{2\pi}{|\bar{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\bar{k}+\bar{l}|)})^{\frac{1}{2}}Q_{0,0}e^{i(k-l)ct}$
$(***)$

We want to compute;

$= lim_{r\rightarrow\infty}\int_{\mathcal{R}^6} g(\bar{k},\bar{l},t)d\bar{k}sin(r|\bar{k}+\bar{l}|)d\bar{l}$

$+lim_{r\rightarrow\infty}r\int_{\mathcal{R}^6} h(\bar{k},\bar{l},t)d\bar{k}cos(r|\bar{k}+\bar{l}|)d\bar{l}$

and show it is zero. Then $lim_{r\rightarrow\infty}P(r,t)$ will be this limit plus 3 other similar terms going to zero, which gives the result.

From $(***)$, we have that;

$g(\bar{k},\bar{l},t) = \frac{iP_{1,1}}{2\pi^2}(\bar{b}(\bar{k}) \times \bar{d}'(\bar{l})) \cdot \frac{\bar{u}(\bar{k},\bar{l})}{|\bar{k}+\bar{l}|^2}e^{i(k-l)ct}$

where $\overline{u}(\overline{k},\bar{l})$ is a unit vector, so that, using Fubini's Theorem, and a change of variables $\overline{k}' = \overline{k} + \bar{l}$, we have;

$\int_{\mathcal{R}^6}(g(\overline{k},\bar{l},t)e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$ (P)

$= \int_{\mathcal{R}^6} \frac{iP_{1,1}}{2\pi^2}(\bar{b}(\overline{k}) \times \overline{d}'(\bar{l})) \centerdot \frac{\overline{u}(\overline{k},\bar{l})}{|\overline{k}+\bar{l}|^2} e^{i(k-l)ct} e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$

$= \int_{\mathcal{R}^6} \frac{\overline{\phi}(\overline{k},\bar{l},t)}{|\overline{k}+\bar{l}|^2} e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}^3} \frac{\phi(\overline{k},\bar{l},t)}{|\overline{k}+\bar{l}|^2} e^{i(r|\overline{k}+\bar{l}|)}d\overline{k})d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}^3} \frac{\phi(\overline{k}'-\bar{l},\bar{l},t)}{|\overline{k}'|^2} e^{i(r|\overline{k}'|)}d\overline{k}')d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}^3} \frac{\phi(\overline{k}-\bar{l},\bar{l},t)}{|\overline{k}|^2} e^{i(r|\overline{k}|)}d\overline{k})d\bar{l}$

where $\phi(\overline{k},\bar{l},t) = \frac{iP_{1,1}}{2\pi^2}(\bar{b}(\overline{k}) \times \overline{d}'(\bar{l})) \centerdot \overline{u}(\overline{k},\bar{l})e^{i(k-l)ct}$

It follows, switching to polars coordinates;

$k_1 = Rsin(\theta)cos(\phi)$, $k_2 = Rsin(\theta)sin(\phi)$, $k_3 = Rcos(\theta)$

that;

$\int_{\mathcal{R}^6}(g(\overline{k},\bar{l},t)e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}d\overline{k}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} \frac{q(R,\theta,\phi,t,\bar{l})}{R^2} e^{irR}R^2 sin(\theta)dRd\theta)d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} q(R,\theta,\phi,t,\bar{l})e^{irR}sin(\theta)dRd\theta)d\bar{l}$ (2)

where $q(R,\theta,\phi,t,\bar{l}) = \phi(\overline{k}-\bar{l},\bar{l},t)$.

From $(***)$ again, we have that;

$h(\overline{k},\bar{l},t) = \frac{-iQ_{0,0}}{2\pi^2}(\bar{b}(\overline{k}) \times \overline{d}'(\bar{l})) \centerdot \frac{\overline{u}(\overline{k},\bar{l})}{|\overline{k}+\bar{l}|} e^{i(k-l)ct}$

where $\overline{u}(\overline{k},\bar{l})$ is a unit vector, so that, using Fubini's Theorem, and a change of variables $\overline{k}' = \overline{k} + \bar{l}$, we have;

$\int_{\mathcal{R}^6}(h(\overline{k},\bar{l},t)e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$ (P')

$= \int_{\mathcal{R}^6} \frac{-iQ_{0,0}}{2\pi^2}(\overline{b}(\overline{k}) \times \overline{d}'(\bar{l})) \cdot \frac{\overline{u}(\overline{k},\bar{l})}{|\overline{k}+\bar{l}|} e^{i(k-l)ct} e^{i(r|\overline{k}+\bar{l}|)} d\overline{k} d\bar{l}$

$= \int_{\mathcal{R}^6} \frac{\overline{\theta}(\overline{k},\bar{l},t)}{|\overline{k}+\bar{l}|} e^{i(r|\overline{k}+\bar{l}|)} d\overline{k} d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}^3} \frac{\theta(\overline{k},\bar{l},t)}{|\overline{k}+\bar{l}|} e^{i(r|\overline{k}+\bar{l}|)} d\overline{k}) d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}^3} \frac{\theta(\overline{k}'-\bar{l},\bar{l},t)}{|\overline{k}'|} e^{i(r|\overline{k}'|)} d\overline{k}') d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}^3} \frac{\theta(\overline{k}-\bar{l},\bar{l},t)}{|\overline{k}|} e^{i(r|\overline{k}|)} d\overline{k}) d\bar{l}$

where $\theta(\overline{k},\bar{l},t) = \frac{-iQ_{0,0}}{2\pi^2}(\overline{b}(\overline{k}) \times \overline{d}'(\bar{l})) \cdot \overline{u}(\overline{k},\bar{l}) e^{i(k-l)ct}$

It follows, switching to polars coordinates;

$k_1 = Rsin(\theta)cos(\phi)$, $k_2 = Rsin(\theta)sin(\phi)$, $k_3 = Rcos(\theta)$

that;

$\int_{\mathcal{R}^6}(h(\overline{k},\bar{l},t) e^{i(r|\overline{k}+\bar{l}|)} d\overline{k} d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}} \int_{0\leq\theta<\pi} \int_{0\leq\phi\leq 2\pi} \frac{p(R,\theta,\phi,t,\bar{l})}{R} e^{irR} R^2 sin(\theta) dR d\theta) d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}} \int_{0\leq\theta<\pi} \int_{0\leq\phi\leq 2\pi} p(R,\theta,\phi,t,\bar{l}) e^{irR} R sin(\theta) dR d\theta) d\bar{l}$ (3)

where $p(R,\theta,\phi,t,\bar{l}) = \theta(\overline{k}-\bar{l},\bar{l},t)$.

We follow through the calculation for $(P),(P')$ but we also need the corresponding results for;

$\int_{\mathcal{R}^6}(g(\overline{k},\bar{l},t) e^{-i(r|\overline{k}+\bar{l}|)} d\overline{k} d\bar{l}$ (P'')
$\int_{\mathcal{R}^6}(h(\overline{k},\bar{l},t) e^{-i(r|\overline{k}+\bar{l}|)} d\overline{k} d\bar{l}$ (P''')

and we can use the fact that;

$sin(r|\overline{k}+\bar{l}|) = \frac{1}{2i}(e^{i(r|\overline{k}+\bar{l}|)} - e^{-i(r|\overline{k}+\bar{l}|)})$

$cos(r|\overline{k}+\bar{l}|) = \frac{1}{2}(e^{i(r|\overline{k}+\bar{l}|)} + e^{-i(r|\overline{k}+\bar{l}|)})$

We leave the details to the reader. Write $\overline{b}(\overline{k}) = \overline{b}_1(\overline{k}) + i\overline{b}_2(\overline{k})$, $\overline{d}_1(\bar{l}) = \overline{d}_{1,1}(\bar{l}) + i\overline{d}_{1,2}(\bar{l})$

where;

$\overline{b}_1(\overline{k}) = \frac{1}{2}Re(\mathcal{F}((\overline{E}+\overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)}) + \frac{1}{2kc}Im(\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$\overline{b}_2(\overline{k}) = \frac{1}{2}Im(\mathcal{F}((\overline{E}+\overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)}) - \frac{1}{2kc}Re(\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$\overline{d}_1'(\bar{l}) = \frac{1}{2}Re(\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\bar{l},0)}) - \frac{1}{2lc}Im(\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\overline{x},0)})|_{(\bar{l},0)})$

$\overline{d}_2'(\bar{l}) = \frac{1}{2}Im(\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\bar{l},0)}) + \frac{1}{2lc}Re(\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\overline{x},0)})|_{(\bar{l},0)})$

We have that;

$q(R,\theta,\phi,t,\bar{l})$

$= \frac{iP_{1,1}}{2\pi^2}[(\overline{b}_{1,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,1}(\bar{l}) - \overline{b}_{2,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,2}(\bar{l}))$

$\centerdot\overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)$

$-\frac{P_{1,1}}{2\pi^2}[(\overline{b}_{2,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,1}(\bar{l}) + \overline{b}_{1,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,2}(\bar{l}))$

$\centerdot\overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)$ (1)

and, similarly;

$p(R,\theta,\phi,t,\bar{l})$

$= \frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{1,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,1}(\bar{l}) - \overline{b}_{2,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,2}(\bar{l}))$

$\centerdot\overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)$

$+\frac{Q_{0,0}}{2\pi^2}[(\overline{b}_{2,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,1}(\bar{l}) + \overline{b}_{1,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,2}(\bar{l}))$

$\centerdot\overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)$ (4)

where $\overline{b}_{1,\bar{l}}(\overline{k}) = \overline{b}_1(\overline{k}-\bar{l})$, $\overline{b}_{2,\bar{l}}(\overline{k}) = \overline{b}_2(\overline{k}-\bar{l})$, $\overline{u}_{\bar{l}}(\overline{k},\bar{l}) = \overline{u}(\overline{k}-\bar{l},\bar{l})$, $\mu(\overline{k},\bar{l},t) = e^{i(|\overline{k}-\bar{l}|-|\bar{l}|)ct}$

and, from $(1),(2)$, we have that;

$\int_{\mathcal{R}^6} g(\overline{k},\bar{l},t)e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{1,\bar{l}}(R,\theta,\phi)\times\bar{d}_{1,1}(\bar{l})-\bar{b}_{2,\bar{l}}(R,\theta,\phi)$

$\times\bar{d}_{1,2}(\bar{l}))\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)-\frac{P_{1,1}}{2\pi^2}[(\bar{b}_{2,\bar{l}}(R,\theta,\phi)\times\bar{d}_{1,1}(\bar{l})+\bar{b}_{1,\bar{l}}(R,\theta,\phi)$

$\times\bar{d}_{1,2}(\bar{l}))\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\bar{l}$ (E)

and, from $(3),(4)$;

$\int_{\mathcal{R}^6}h(\bar{k},\bar{l},t)e^{i(r|\bar{k}+\bar{l}|)}d\bar{k}d\bar{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{-iQ_{0,0}}{2\pi^2}[(\bar{b}_{1,\bar{l}}(R,\theta,\phi)\times\bar{d}_{1,1}(\bar{l})-\bar{b}_{2,\bar{l}}(R,\theta,\phi)$

$\times\bar{d}_{1,2}(\bar{l}))\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)+\frac{Q_{0,0}}{2\pi^2}[(\bar{b}_{2,\bar{l}}(R,\theta,\phi)\times\bar{d}_{1,1}(\bar{l})+\bar{b}_{1,\bar{l}}(R,\theta,\phi)$

$\times\bar{d}_{1,2}(\bar{l}))\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}Rsin(\theta)dRd\theta d\phi)d\bar{l}$ (F)

Write $\bar{b}_1(\bar{k})=\bar{b}_{11}(\bar{k})+\frac{\bar{b}_{12}(\bar{k})}{k}$, $\bar{d}_{1,1}(\bar{l})=\bar{d}_{1,1,1}(\bar{l})+\frac{\bar{d}_{1,1,2}(\bar{l})}{l}$

Then;

$\bar{b}_{1,\bar{l}}(\bar{k})=\bar{b}_1(\bar{k}-\bar{l})=\bar{b}_{11}(\bar{k}-\bar{l})+\frac{\bar{b}_{12}(\bar{k}-\bar{l})}{|\bar{k}-\bar{l}|}$

and;

$\bar{b}_{1,\bar{l}}(R,\theta,\phi)=\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$

where $\bar{b}_{11,\bar{l}}(\bar{k})=\bar{b}_{11}(\bar{k}-\bar{l})$ and $\bar{b}_{12,\bar{l}}(\bar{k})=\bar{b}_{12}(\bar{k}-\bar{l})$

Again, it is sufficient to consider the first term in (E). We have that;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[\bar{b}_{1,\bar{l}}(R,\theta,\phi)\times\bar{d}_{1,1}(\bar{l}))\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}sin(\theta)dRd\theta d\phi)$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})$

$\times(\bar{d}_{1,1,1}(\bar{l})+\frac{\bar{d}_{1,1,2}(\bar{l})}{l})]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\bar{l}$

Similarly, we consider the first term in (F), and, we have that;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{1,\bar{l}}(R,\theta,\phi)\times\overline{d}_{1,1}(\bar{l})).\overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}Rsin(\theta)$

$dRd\theta d\phi)d\bar{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\overline{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})$

$\times(\overline{d}_{1,1,1}(\bar{l})+\frac{\overline{d}_{1,1,2}(\bar{l})}{l})].\overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}Rsin(\theta)dRd\theta d\phi)d\bar{l}$

From ($\sharp$), we have that the real and imaginary components of;

$\{\mathcal{F}((\overline{B}_0)|_{(\bar{x},0)})|_{(\bar{l},0)},\mathcal{F}((\overline{E}+\overline{E}_0)|_{(\bar{x},0)})|_{(\bar{l},0)},\mathcal{F}((\frac{\partial\overline{B}_0}{\partial t}|_{(\bar{x},0)})|_{(\bar{l},0)},\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t})|_{(\bar{x},0)})|_{(\bar{l},0)}\}$

decay faster than $\frac{1}{|l|^4}$, but strengthening the definition of simple to infinitely differentiable if necessary, and adapting the proof, we can assume the decay rate is faster than $\frac{1}{|l|^6}$. It follows that the components of;

$\{\overline{b}_{11,\bar{l}}(\bar{k})\times\overline{d}_{1,1,1}(\bar{l}),\frac{\overline{b}_{11,\bar{l}}(\bar{k})\times\overline{d}_{1,1,2}(\bar{l})}{l},\frac{\overline{b}_{12,\bar{l}}(\bar{k})\times\overline{d}_{1,1,1}(\bar{l})}{|\bar{k}-\bar{l}|},\frac{\overline{b}_{12,\bar{l}}(\bar{k})\times\overline{d}_{1,1,2}(\bar{l})}{|\bar{k}-\bar{l}|l}\}$

decay faster than $\frac{1}{|\bar{k}|^6|\bar{l}|^6|\bar{k}-\bar{l}|}$, and, as $\overline{u}_{\bar{l}}(\bar{k},\bar{l})$ is a unit vector, $|\nu(\bar{k},\bar{l},t)|=|\mu(R,\theta,\phi,\bar{l},t)|=1$, $|sin(\theta(\bar{k}))|\leq 1$, so do the components of;

$\{[(\overline{b}_{11,\bar{l}}(\bar{k})\times\overline{d}_{1,1,1}(\bar{l})).\overline{u}_{\bar{l}}(\bar{k},\bar{l})]\nu(\bar{k},\bar{l},t)sin(\theta(\bar{k})),[(\frac{\overline{b}_{11,\bar{l}}(\bar{k})\times\overline{d}_{1,1,2}(\bar{l})}{l}).\overline{u}_{\bar{l}}(\bar{k},\bar{l}))]\nu(\bar{k},\bar{l},t)sin(\theta(\bar{k})),$

$[(\frac{\overline{b}_{12,\bar{l}}(\bar{k})\times\overline{d}_{1,1,1}(\bar{l})}{|\bar{k}-\bar{l}|}).\overline{u}_{\bar{l}}(\bar{k},\bar{l})]\nu(\bar{k},\bar{l},t)sin(\theta(\bar{k})),[(\frac{\overline{b}_{12,\bar{l}}(\bar{k})\times\overline{d}_{1,1,2}(\bar{l})}{|\bar{k}-\bar{l}|l}).\overline{u}_{\bar{l}}(\bar{k},\bar{l})]\nu(\bar{k},\bar{l},t)sin(\theta(\bar{k}))\}$

Noting that, for $C\in\mathcal{R}_{>0}$, $D\in\mathcal{R}_{>0}$ and fixed $\bar{l}\in\mathcal{R}^3$, $\bar{l}\neq\bar{0}$, without loss of generality, assuming that $D\leq|\bar{l}|$?;

$|\int_{|\bar{k}|>D}\frac{C}{|\bar{k}|^6||\bar{l}|^6|\bar{k}-\bar{l}|}|d\bar{k}$

$=|\int_{D<|\bar{k}|<|\bar{l}|+1}\frac{C}{|\bar{k}|^6||\bar{l}|^6|\bar{k}-\bar{l}|}|d\bar{k}+\int_{D>|\bar{l}|+1}\frac{C}{|\bar{k}|^6||\bar{l}|^6|\bar{k}-\bar{l}|}|d\bar{k}$

$\leq|\int_{D<|\bar{k}|<|\bar{l}|+1}\frac{C}{|\bar{k}|^6||\bar{l}|^6|\bar{k}-\bar{l}|}d\bar{k}|+|\int_{|\bar{k}|>|\bar{l}|+1>D}\frac{C}{|\bar{k}|^6||\bar{l}|^6|\bar{k}-\bar{l}|}d\bar{k}|$

$\leq\frac{C}{D^6|\bar{l}|^6}\int_{Ann(D,|\bar{l}|+1)}\frac{1}{|\bar{k}-\bar{l}|}d\bar{k}+\frac{1}{|\bar{l}|^6}\int_{|\bar{k}|>|\bar{l}|+1}\frac{C}{|\bar{k}|^6}d\bar{k}$

$=\frac{C}{D^6|\bar{l}|^6}\int_{Ann_{\bar{l}}(D,|\bar{l}|+1)}\frac{1}{|\bar{k}|}d\bar{k}+\frac{1}{|\bar{l}|^6}\int_0^\pi\int_{-\pi}^\pi\int_{|\bar{l}|+1}^\infty\frac{CR^2sin(\theta)}{R^6}dRd\theta d\phi$

$\leq\frac{C}{D^6|\bar{l}|^6}\int_{B(\bar{0},2|\bar{l}|+2D+1)}\frac{1}{|\bar{k}|}d\bar{k}+\frac{1}{|\bar{l}|^6}\int_0^\pi\int_{-\pi}^\pi\int_{|\bar{l}|+1}^\infty\frac{C}{R^4}dRd\theta d\phi$

$\leq \frac{2\pi^2 C}{D^6|\bar{l}|^6}\int_0^{2|\bar{l}|+2D+1}\frac{R^2}{R}dR+\frac{2\pi^2 C}{3(|\bar{l}|+1)^3|\bar{l}|^6}$

$\leq \frac{\pi^2 C(2|\bar{l}|+2D+1)^2}{D^6|\bar{l}|^6}+\frac{2\pi^2 C}{3D^3|\bar{l}|^6}$

It follows, that for fixed $r\in\mathcal{R}_{>0}$, sufficiently large, we have that;

$max(\int_{|\bar{k}|>r}\int_{|\bar{l}>r}|\alpha(\bar{k},\bar{l},t)|d\bar{k}d\bar{l},\int_{|\bar{k}|>r}\int_{|\bar{l}>r}|\beta(\bar{k},\bar{l},t)|d\bar{k}d\bar{l})$

$\leq \int_{|\bar{l}|>r}\frac{M}{|\bar{l}|^4 r^4}$

$\leq \frac{2\pi^2 M}{r.r^4}$

$=\frac{2\pi^2 M}{r^5}$

where $M\in\mathcal{R}_{>0}$, and;

$\alpha(\bar{k},\bar{l},t)=\alpha(R,\theta,\phi,\bar{l},t)=\frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})\times(\bar{d}_{1,1,1}(\bar{l})+\frac{\bar{d}_{1,1,2}(\bar{l})}{l})]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)sin(\theta)$

$\beta(\bar{k},\bar{l},t)=\beta(R,\theta,\phi,\bar{l},t)=\frac{-iQ_{0,0}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})\times(\bar{d}_{1,1,1}(\bar{l})+\frac{\bar{d}_{1,1,2}(\bar{l})}{l})]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)sin(\theta)$

We have that;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})$

$\times(\bar{d}_{1,1,1}(\bar{l})+\frac{\bar{d}_{1,1,2}(\bar{l})}{l})]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\bar{l}$

$=\int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi)d\bar{l}$

splits as four terms, the worst of which is;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$

$\times\frac{\bar{d}_{1,1,2}(\bar{l})}{l}]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\bar{l}$

$=\int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi)d\bar{l}$

Again, fix $\bar{l} \neq \bar{0}$, with $\theta \neq cos^{-1}(\frac{l_3}{l}) = \theta_{0,\bar{l}}$ and $\phi \neq tan^{-1}(\frac{l_2}{l_1}) = \phi_{0,\bar{l}}$. By the results of Lemmas 0.23,0.24 and 0.18, we can assume that the real and imaginary parts of $\alpha_4(R,\theta,\phi,\bar{l},t)$ are oscillatory, then as $lim_{R\rightarrow 0}\alpha_4(R,\theta,\phi,\bar{l},t) = M \in \mathcal{R}$, we can apply the result of Lemmas 0.15 and 0.8, and assume that;

$$|\int_{\mathcal{R}_{>0}} \alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|$$

$$\leq |\int_{\mathcal{R}_{>0}} Re(\alpha_4)(R,\theta,\phi,\bar{l},t)e^{irR}dR|+|\int_{\mathcal{R}_{>0}} Im(\alpha_4)(R,\theta,\phi,\bar{l},t)e^{irR}dR|$$

$$\leq \frac{2}{r}(\frac{n_{\bar{l},\theta,\phi,Re}||Re(\alpha_4)||_\infty}{\xi_{Re}} + \frac{D_{\bar{l},\theta,\phi,Re}}{n_{\bar{l},\theta,\phi}\xi_{Re}})$$

$$+\frac{2}{r}(\frac{n_{\bar{l},\theta,\phi,Im}||Im(\alpha_4)||_\infty}{\xi_{Im}} + \frac{D_{\bar{l},\theta,\phi,Im}}{n_{\bar{l},\theta,\phi}\xi_{Im}})$$

so that, for $l > 1$;

$$|\int_{\mathcal{R}_{>0}} \alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|$$

$$\leq \frac{2}{r}(\frac{4\sqrt{3}l||Re(\alpha_4)||_\infty}{\xi_{Re}} + \frac{C2^{\frac{5}{2}}|\frac{\bar{d}_{1,1,2}(\bar{l})}{l}|}{4\sqrt{3}l\xi_{Re}})$$

$$+\frac{2}{r}(\frac{4\sqrt{3}l||Im(\alpha_4)||_\infty}{\xi_{Im}} + \frac{C2^{\frac{5}{2}}|\frac{\bar{d}_{1,1,2}(\bar{l})}{l}|}{4\sqrt{3}l\xi_{Im}})$$

$$\leq \frac{2}{r\xi}(4\sqrt{3}l(||Re(\alpha_4)||_\infty + ||Im(\alpha_4)||_\infty) + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}_{1,1,2}(\bar{l})}{l}|}{4\sqrt{3}l})$$

$$\leq \frac{2}{r\xi}(4\sqrt{6}l||\alpha_4||_\infty + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}_{1,1,2}(\bar{l})}{l}|}{4\sqrt{3}l})$$

and, similarly, for $0 < l \leq 1$;

$$|\int_{\mathcal{R}_{>0}} \alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|$$

$$\leq \frac{2}{r\xi}(4\sqrt{6}||\alpha_4||_\infty + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}_{1,1,2}(\bar{l})}{l}|}{4\sqrt{3}}) \ (D)$$

for sufficiently large $r \in \mathcal{R}_{>0}$, where $\xi_{Re} > 0, \xi_{Im} > 0$ are constants independent of $\bar{l},\theta,\phi$, $\xi = min(\xi_{Re},\xi_{Im}) > 0$, $\{D_{\bar{l},\theta,\phi,Re}, D_{\bar{l},\theta,\phi,Im}\}$ are the decay rates for the real and imaginary components of $\alpha_4(R,\theta,\phi,\bar{l},t)$. The constant ξ can be chosen independently of the parameters $\{\theta,\phi,\bar{l}\}$, see Lemma 0.19. We have that;

$||\alpha_4||_\infty = |\frac{iP_{1,1}}{2\pi^2}[\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}\times\frac{\bar{d}'_{12}(\bar{l})}{l}].\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})sin(\theta)|$

$\leq \frac{P_{1,1}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)sin(\theta)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$= |\frac{P_{1,1}}{2\pi^2}\frac{\bar{b}_{12,\bar{l}}(\bar{k})}{k^2|\bar{k}-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

where;

$\frac{P_{1,1}}{2\pi^2}\frac{\bar{b}_{12,\bar{l}}(\bar{k})}{k^2|\bar{k}-\bar{l}|} = \frac{P_{1,1}}{2\pi^2}\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)sin(\theta)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$

Fix $\kappa > 0$, then, as, for fixed $\bar{l} \neq \bar{0}$, $\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{k^2|\bar{k}-\bar{l}|} \in L^1(\mathcal{R}^3)$, we can choose $\theta_{0,\bar{l},\kappa_1} < \theta_{0,\bar{l}} < \theta_{0,\bar{l},\kappa_2}$, $\phi_{0,\bar{l},\kappa_1} < \phi_{0,\bar{l}} < \phi_{0,\bar{l},\kappa_2}$, such that;

$|\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{k^2|\bar{k}-\bar{l}|}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi| \leq \kappa'$

Then;

$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq |\int_{\mathcal{R}_{>0}}\int_{([0,\pi)\times[0,2\pi)\setminus[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}]\times[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}])}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$+|\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq |\int_{\mathcal{R}_{>0}}\int_{V_{\bar{l},\kappa_1,\kappa_2}}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi| + \kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$\leq \int_{V_{\bar{l},\kappa_1,\kappa_2}}(|\int_{\mathcal{R}_{>0}}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|)d\theta d\phi + \kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

Using (D), it follows that, for $l > 1$;

$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq 2\pi^2\frac{2}{r\xi}(4\sqrt{6}l||\alpha_4|_{V_{\bar{l},\kappa_1,\kappa_2}}||_\infty + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}l}) + \kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}l}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|_{V_{\bar{l},\kappa_1,\kappa_2}}||\frac{\bar{d}'_{12}(\bar{l})}{l}|+\frac{C2^{\frac{7}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}l})$

$+\kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

and, for $0 < l \leq 1$;

$$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$$

$$\leq 2\pi^2\frac{2}{r\xi}(4\sqrt{6}||\alpha_4|_{V_{\bar{l},\kappa_1,\kappa_2}}||_\infty + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}}) + \kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|_{V_{\bar{l},\kappa_1,\kappa_2}}||\frac{\bar{d}'_{12}(\bar{l})}{l}|+\frac{C2^{\frac{7}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}})$$

$$+\kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}| \ (H)$$

Fix $\delta > 0$ arbitrary, then we have that, for $l > \delta$, sufficiently small $0 < \kappa < min(\frac{\delta}{2}, \delta^2)$;

$$\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}\frac{P_{1,1}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)sin(\theta)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|dRd\theta d\phi$$

$$= \int_{W_{\bar{l},\kappa_1,\kappa_2}}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}-\bar{l}||\bar{k}|^2}$$

$$= \int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}$$

$$\leq \int_{B(\bar{0},\kappa)}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k} + \int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}\setminus B(\bar{0},\kappa)}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}$$

$$\leq \frac{P_{1,1}}{2\pi^2}||\frac{\bar{b}_{12}(\bar{k})}{|\bar{k}+\bar{l}|^2}||_{\infty,B(\bar{0},\kappa)}\int_{0<R<\kappa}\frac{1}{R}R^2|sin(\theta)|dRd\theta d\phi + \frac{P_{1,1}}{2\pi^2}\int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}\setminus B(\bar{0},\kappa)}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}$$

$$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2} + \frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}\int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}}|\frac{\bar{b}_{12}(\bar{k})}{|\bar{k}+\bar{l}|^2})|d\bar{k}$$

$$= \frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2} + \frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}\int_{(W_{\bar{l},\kappa_1,\kappa_2})}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{R^2}|R^2sin(\theta)dRd\theta d\phi$$

$$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2} + \frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}|\theta_{0,\bar{l},\kappa_2}-\theta_{0,\bar{l},\kappa_1}||\phi_{0,\bar{l},\kappa_2}-\phi_{0,\bar{l},\kappa_2}|_{S^1(1)}\int_{\mathcal{R}_{>0}}|\bar{b}_{12,\bar{l}}(R)|dR$$

$$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2} + \frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}|\theta_{0,\bar{l},\kappa_2}-\theta_{0,\bar{l},\kappa_1}||\phi_{0,\bar{l},\kappa_2}-\phi_{0,\bar{l},\kappa_2}|_{S^1(1)}K$$

$$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2} + \frac{P_{1,1}}{2\pi^2}\kappa$$

$$\leq \frac{2P_{1,1}}{\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\delta^2}{2} + \frac{P_{1,1}}{2\pi^2}\kappa = \kappa' \ (M)$$

for $|\theta_{0,\bar{l},\kappa_2}-\theta_{0,\bar{l},\kappa_1}| = |\phi_{0,\bar{l},\kappa_2}-\phi_{0,\bar{l},\kappa_1}|_{S^1(1)}$, $|\theta_{0,\bar{l},\kappa_2}-\theta_{0,\bar{l},\kappa_1}| \leq \frac{\kappa}{\sqrt{K}}$ (G)

where;

$W_{\bar{l},\kappa_1,\kappa_2} = ([\phi_{0,\bar{l},\kappa_1}, \phi_{0,\bar{l},\kappa_2}] \times [\phi_{0,\bar{l},\kappa_1}, \phi_{0,\bar{l},\kappa_2}] \times \mathcal{R}_{>0})$

$(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}} = \{\bar{k} : \bar{k} + \bar{l} \in W_{\bar{l},\kappa_1,\kappa_2}\}$

and, we can assume that $|\bar{b}_{12,\bar{l}}(R)|$ is independent of $\{\theta, \phi\}$, with $||\bar{b}_{12,\bar{l}}(R)||_{L^1(\mathcal{R}_{>0})} \leq K$, independently of $\bar{l}$, due to the decay.

In particularly, choosing $\theta_{0,\bar{l},\kappa_2} = \theta_{0,\bar{l}} + \frac{\kappa}{2\sqrt{K}}$, $\theta_{0,\bar{l},\kappa_1} = \theta_{0,\bar{l}} - \frac{\kappa}{2\sqrt{K}}$, $\phi_{0,\bar{l},\kappa_2} = \phi_{0,\bar{l}} + \frac{\kappa}{2\sqrt{K}}$, $\phi_{0,\bar{l},\kappa_1} = \phi_{0,\bar{l}} - \frac{\kappa}{2\sqrt{K}}$, we have that (G) holds and $d(\bar{l}, V_{\bar{l},\kappa_1,\kappa_2}) \geq lsin(\frac{\kappa}{2\sqrt{K}}) \geq \frac{l\kappa}{4\sqrt{K}}$, for sufficiently small κ. We then have that;

$$|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|_{V_{\bar{l},\kappa_1,\kappa_2}}| \leq \frac{4\sqrt{K}}{l\kappa}||\bar{b}_{12,\bar{l}}(R,\theta,\phi)||_\infty = \frac{4\sqrt{K}D}{l\kappa}$$

where $D \in \mathcal{R}_{>0}$, independent of $\bar{l}$. From $(H), (M)$, we obtain that, for $l > 1$;

$$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$$

$$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}l}{2\pi^2}(\frac{4\sqrt{K}D}{l\kappa})|\frac{\bar{d}'_{12}(\bar{l})}{l}| + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}l})$$

$$+\kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|\ (l > \delta)$$

and, for $0 < l \leq 1$;

$$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$$

$$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}}{2\pi^2}(\frac{4\sqrt{K}D}{l\kappa})|\frac{\bar{d}'_{12}(\bar{l})}{l}| + \frac{C2^{\frac{7}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}})$$

$$+\kappa'||\frac{\bar{d}'_{12}(\bar{l})}{l}|\ (l > \delta)$$

Using the fact that $\{\frac{|\bar{d}'_{12}(\bar{l})|}{l^2}, \frac{|\bar{d}'_{12}(\bar{l})|}{l}\} \subset L^1(\mathcal{R}^3)$, and integrating $g(\bar{k},\bar{l},t)e^{ir|\bar{k}+\bar{l}|}$ over $\mathcal{R}^3 \times B(\bar{0},\delta)$ separately, using Lemma 0.9, looking at all components, for sufficiently large $r \in \mathcal{R}_{>0}$, need uniformity in $\bar{l}$ version of Lemma 0.12, follows that,

$$|\int_{\mathcal{R}^6} g(\bar{k},\bar{l},t)e^{ir|\bar{k}+\bar{l}|}d\bar{k}d\bar{l}| \leq A\delta + \frac{F(\kappa)}{r} + H\kappa'$$

where $\{A, H\} \subset \mathcal{R}$. Follows that?(split again $Re(g), Im(g)$)

$|\int_{\mathcal{R}^6} g(\overline{k},\overline{l},t)sin(r|\overline{k}+\overline{l}|)d\overline{k}d\overline{l} \leq B\delta + \frac{T(\kappa)}{r} + S\kappa'$

for sufficiently large r, In particular as $\kappa' > 0, \delta > 0$ can be made arbitrarily small, and;

$|lim_{r\rightarrow\infty} \int_{\mathcal{R}^6} g(\overline{k},\overline{l},t)cos(r|\overline{k}+\overline{l}|)d\overline{k}d\overline{l}| < A\delta + H\kappa'$

$lim_{r\rightarrow\infty} \int_{\mathcal{R}^6} g(\overline{k},\overline{l},t)cos(r|\overline{k}+\overline{l}|)d\overline{k}d\overline{l} = 0$

so the no radiation condition holds.

Similarly, we have that;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} \frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{11,\overline{l}}(R,\theta,\phi)+\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})$

$\times(\overline{d}'_{11}(\overline{l})+\frac{\overline{d}'_{12}(\overline{l})}{l})] \centerdot \overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}Rsin(\theta)dRd\theta d\phi)d\overline{l}$

$= \int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} \beta(R,\theta,\phi,\overline{l},t)e^{irR}Rd\theta d\phi)d\overline{l}$

splits as four terms, the worst of which is;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} \frac{-iQ_{0,0}}{2\pi^2}[\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}$

$\times\frac{\overline{d}'_{12}(\overline{l})}{l}] \centerdot \overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)RdRd\theta d\phi)d\overline{l}$

$= \int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} \beta_4(R,\theta,\phi,\overline{l},t)e^{irR}RdRd\theta d\phi)d\overline{l}$

Again, fix $\overline{l} \neq \overline{0}$, with $\theta \neq cos^{-1}(\frac{l_3}{l}) = \theta_{0,\overline{l}}$ and $\phi \neq tan^{-1}(\frac{l_2}{l_1}) = \phi_{0,\overline{l}}$. By the result of Lemma 0.23, we can assume that the real and imaginary parts of $\frac{\partial R\beta_4(R,\theta,\phi,\overline{l},t)}{\partial R}$ are non-oscillatory when restricted to a finite interval $[0,L]$. Moreover, we have that $lim_{R\rightarrow 0}R\beta_4(R,\theta,\phi,\overline{l},t) = 0$ and $lim_{R\rightarrow 0}\frac{\partial R\beta_4(R,\theta,\phi,\overline{l},t)}{\partial R} = M \in \mathcal{R}$, both functions being of moderate decrease. We restrict the $\overline{l}$ parameter to an annulus $Ann(\epsilon,l_0) \subset \mathcal{R}^3$. Then, using integration by parts, see Lemma 0.16, we have that;

$|\int_{Ann(\epsilon,l_0)}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} \beta_4(R,\theta,\phi,\overline{l},t)e^{irR}RdRd\theta d\phi)d\overline{l}|$

$= |\int_{Ann(\epsilon,l_0)}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi,\theta\neq\theta_{0,\overline{l}}}\int_{0\leq\phi\leq 2\pi,\phi\neq\phi_{0,\overline{l}}} \beta_4(R,\theta,\phi,\overline{l},t)e^{irR}RdRd\theta d\phi)d\overline{l}|$

$= \frac{1}{r}|\int_{Ann(\epsilon,l_0)}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi,\theta\neq\theta_{0,\bar{l}}}\int_{0\leq\phi\leq 2\pi,\phi\neq\phi_{0,\bar{l}}}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi)d\bar{l}|$

(TUT)

We have that, for fixed $\bar{l}\in Ann(\epsilon,l_0)$, $\frac{\partial R\beta_4}{\partial R}\in L^1(\mathcal{R}^3)$, see calculation below and Lemma 0.9, so we can restrict the parameters $\{\theta,\phi\}$ to $|\theta-\theta_{0,\bar{l}}|\geq\delta_1|$, $|\phi-\phi_{0,\bar{l}}|\geq\delta_2$ such that, for arbitrary $\delta>0$, $\bar{l}\in Ann(\epsilon,l_0)$;

$|\int_{\mathcal{R}_{>0}}\int_{|\theta-\theta_{0,\bar{l}}|\leq\delta_1}\int_{|\phi-\phi_{0,\bar{l}}|\leq\delta_2}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|\leq\delta$

(SUS)

By the proof of Lemma 0.16, for arbitrary $\epsilon'>0$, there exists $C_{\epsilon'}\in\mathcal{R}_{>0}$, such that, uniformly in $\bar{l}\in Ann(\epsilon,l_0)$, $|\theta-\theta_{0,\bar{l}}|\geq\delta_1$, $|\phi-\phi_{0,\bar{l}}|\geq\delta_2$;

$|\int_{\mathcal{R}_{>0}}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dR|\leq\epsilon'+\frac{C_{\epsilon'}}{r}$

where $C_{\epsilon'}=2val_{\frac{\partial R\beta_4}{R}}||\frac{\partial R\beta_4}{\partial R}||_\infty$ restricted to the parameters, so that;

$|\int_{\mathcal{R}_{>0}}\int_{|\theta-\theta_{0,\bar{l}}|\geq\delta_1}\int_{|\phi-\phi_{0,\bar{l}}|\geq\delta_2}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq 2\pi^2(\epsilon'+\frac{C_{\epsilon'}}{r})$

and, therefore;

$|\int_{\mathcal{R}_{>0}}\int_{\theta\neq\theta_{0,\bar{l}}}\int_{\phi\neq\phi_{0,\bar{l}}}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|\leq\delta+2\pi^2(\epsilon'+\frac{C_{\epsilon'}}{r})$

It follows that;

$|\int_{Ann(\epsilon,l_0)}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi,\theta\neq\theta_{0,\bar{l}}}\int_{0\leq\phi\leq 2\pi,\phi\neq\phi_{0,\bar{l}}}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi)d\bar{l}|$

$\leq\frac{4\pi l_0^3}{3}(\delta+2\pi^2(\epsilon'+\frac{C_{\epsilon'}}{r}))$

and by (TUT), for sufficiently large r;

$|\int_{Ann(\epsilon,l_0)}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}RdRd\theta d\phi)d\bar{l}|$

$\leq\frac{4\pi l_0^3}{3}(\frac{\delta'}{r}+\frac{2\pi^2 C_{\epsilon'}}{r^2})$ (TUS)

where $\delta' > 0$ is arbitrary. We have that;

$$||\frac{\partial R\beta_4}{\partial R}||_\infty = ||\beta_4 + R\frac{\partial\beta_4}{\partial R}||_\infty$$

$$\leq ||\beta_4||_\infty + ||R\frac{\partial\beta_4}{\partial R}||_\infty$$

$$= |\frac{-iQ_{0,0}}{2\pi^2}[\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|} \times \frac{\bar{d}'_{12}(\bar{l})}{l}]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})sin(\theta)|$$

$$+|\frac{-iQ_{0,0}R}{2\pi^2}[\frac{\partial}{\partial R}(\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}) \times \frac{\bar{d}'_{12}(\bar{l})}{l}]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})sin(\theta)|$$

$$+|\frac{-iQ_{0,0}R}{2\pi^2}[\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|} \times \frac{\bar{d}'_{12}(\bar{l})}{l}]\centerdot\frac{\partial}{\partial R}(\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l}))sin(\theta)|$$

$$\leq \frac{Q_{0,0}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}|\frac{\partial}{\partial R}(\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}||\frac{\partial}{\partial R}(\frac{\bar{k}}{|\bar{k}|})|$$

$$= \frac{Q_{0,0}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}|\frac{\partial}{\partial R}(\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$\leq \frac{Q_{0,0}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}|\frac{\partial}{\partial R}(\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$= \frac{Q_{0,0}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}|\frac{\frac{\partial}{\partial R}(\bar{b}_{12,\bar{l}}(R,\theta,\phi))}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$$

$$+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)<(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l},\frac{\partial}{\partial R}((Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l})>}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|^3}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$\leq \frac{Q_{0,0}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}|\frac{\frac{\partial}{\partial R}(\bar{b}_{12,\bar{l}}(R,\theta,\phi))}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$$+\frac{Q_{0,0}R}{2\pi^2}\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)||((sin(\theta)cos(\phi),sin(\theta)sin(\phi),cos(\theta)))|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$$

$\leq \frac{Q_{0,0}}{2\pi^2}\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{Q_{0,0}R}{2\pi^2}\frac{|\frac{\partial}{\partial R}(\bar{b}_{12,\bar{l}}(R,\theta,\phi))|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{\sqrt{3}Q_{0,0}R}{2\pi^2}\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$ (F)

By (F), we see that changing to Cartesian coordinates, the above terms are of the form;

$\frac{C_1|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$, $\frac{C_2|\bar{c}_{12}(\bar{k},\bar{l})|}{|\bar{k}||\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$, $\frac{C_3|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

where $\{C_1,C_2,C_3\}\subset\mathcal{R}_{>0}$ and;

$|\bar{c}_{12}(\bar{k},\bar{l})|\leq|\bar{e}_{12,\bar{l}}(\bar{k})|$, $\bar{e}_{12,\bar{l},j}(\bar{k})=|(\frac{\partial\bar{b}_{12,\bar{l},j}}{\partial x},\frac{\partial\bar{b}_{12,\bar{l},j}}{\partial y},\frac{\partial\bar{b}_{12,\bar{l},j}}{\partial z})|$

By Lemma 0.9, we can see that for fixed $\bar{l}\neq\bar{0}$, $\frac{\partial R\beta_4}{\partial R}\in L^1(\mathcal{R}^3)$. Moreover, we can see that when we limit the parameters $\{\theta,\phi\}$ and use compactness, to obtain (SUS) above, we then have that with $|\theta-\theta_{0,\bar{l}}|\geq\delta_1|$, $|\phi-\phi_{0,\bar{l}}|\geq\delta_2$, that $|\bar{k}-\bar{l}|\geq lsin(\theta-\theta_{0,\bar{l}})\geq\frac{l\delta_1}{2}$, so that $||\frac{\partial R\beta_4}{\partial R}||_\infty$ is uniformly bounded on the restricted parameters. Using the same argument as in (TUT), (F), and the proof of Lemma 0.9;

$|\int_{B(\bar{0},l_0)^c}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}RdRd\theta d\phi)d\bar{l}|$

$=\frac{1}{r}|\int_{B(\bar{0},l_0)^c}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi,\theta\neq\theta_{0,\bar{l}}}\int_{0\leq\phi\leq 2\pi,\phi\neq\phi_{0,\bar{l}}}\frac{\partial R\beta_4}{\partial R}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi)d\bar{l}|$

$\leq\frac{1}{r}\int_{B(\bar{0},l_0)^c}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}|\frac{\partial R\beta_4}{\partial R}|_\infty dRd\theta d\phi)d\bar{l}$

$\leq\frac{1}{r}\int_{B(\bar{0},l_0)^c}\int_{\mathcal{R}^3}\frac{C_1|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|+\frac{C_2|\bar{c}_{12}(\bar{k},\bar{l})|}{|\bar{k}||\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|+\frac{C_3|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{k}d\bar{l}$

$\leq\frac{\epsilon_1}{r}$ (ABC)

where $\epsilon_1>0$ is arbitrary for $l_0(\epsilon_1)$ sufficiently large. Similarly;

$|\int_{B(\bar{0},\epsilon)}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}RdRd\theta d\phi)d\bar{l}|$

$\leq\frac{\epsilon_2}{r}$ (DEF)

where $\epsilon_2 > 0$ is arbitrary for $\epsilon(\epsilon_2)$ sufficiently small. It follows that, for sufficiently large r, combining $(TUS),(ABC),(DEF)$, that;

$$|\int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}RdRd\theta d\phi)d\bar{l}|$$

$$\leq \frac{4\pi l_0^3}{3}(\frac{\delta'}{r}+\frac{2\pi^2 C_{\epsilon'}}{r^2})+\frac{\epsilon_1+\epsilon_2}{r}\ (GHI)$$

It follows that;

$$|\int_{\mathcal{R}^6} h(\bar{k},\bar{l},t)e^{ir|\bar{k}+\bar{l}}d\bar{k}d\bar{l}| \leq \frac{4\pi l_0^3}{3}(\frac{\delta'}{r}+\frac{2\pi^2 C_{\epsilon'}}{r^2})+\frac{\epsilon_1+\epsilon_2}{r}$$

It follows that, splitting the calculation into real and imaginary components $Re(h), Im(h)$, that;

$$|\int_{\mathcal{R}^6} h(\bar{k},\bar{l},t)cos(r|\bar{k}+\bar{l}|)d\bar{k}d\bar{l} \leq \frac{4\pi l_0^3}{3}(\frac{\delta'}{r}+\frac{2\pi^2 C_{\epsilon'}}{r^2})+\frac{\epsilon_1+\epsilon_2}{r}$$

for sufficiently large r. In particular;

$$|lim_{r\rightarrow\infty} r\int_{\mathcal{R}^6} h(\bar{k},\bar{l},t)cos(r|\bar{k}+\bar{l}|)d\bar{k}d\bar{l}|$$

$$\leq lim_{r\rightarrow\infty}\frac{4\pi l_0^3}{3}(\delta'+\frac{2\pi^2 C_{\epsilon'}}{r})+\epsilon_1+\epsilon_2$$

$$= \frac{4\pi l_0^3}{3}\delta'+\epsilon_1+\epsilon_2$$

As δ and ϵ' in the proof can be made arbitrarily small relative to the choice of l_0, and $\{\epsilon_1,\epsilon_2\}$ were arbitrary, we must have that;

$$lim_{r\rightarrow\infty} r\int_{\mathcal{R}^6} h(\bar{k},\bar{l},t)cos(r|\bar{k}+\bar{l}|)d\bar{k}d\bar{l}=0$$

so the no radiation condition holds again.

□

Lemma 0.8. *We have that;*

$$|\alpha_4(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|,\ for\ R>4l\sqrt{3}, l>1$$

$$R>4\sqrt{3}, 0<l\leq 1$$

$$|Re(\alpha_4)(R,\theta,\phi,t,\bar{l}) \leq |\frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|,\ for\ R>4l\sqrt{3}, l>1$$

$R > 4\sqrt{3}, 0 < l \leq 1$

$|Im(\alpha_4)(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$, *for* $R > 4l\sqrt{3}, l > 1$

$R > 4\sqrt{3}, 0 < l \leq 1$

where $C \in \mathcal{R}_{>0}$

In particularly, the families $\{Re(\alpha_4)(R,\theta,\phi,t,\bar{l}) : \bar{l} \in \mathcal{R}^3, \bar{l} \neq \bar{0}, \theta \neq cos^{-1}(\frac{l_3}{l_1}), \phi \neq tan^{-1}(\frac{l_2}{l_1})\}$ *and* $\{Im(\alpha_4)(R,\theta,\phi,t,\bar{l}) : \bar{l} \in \mathcal{R}^3, \bar{l} \neq \bar{0}, \theta \neq cos^{-1}(\frac{l_3}{l_1}), \phi \neq tan^{-1}(\frac{l_2}{l_1})\}$ *are of moderate decrease* $n_{\bar{l},\theta,\phi}$*, with;*

$n_{\bar{l},\theta,\phi} = 4l\sqrt{3}$, $l > 1$

$n_{\bar{l},\theta,\phi} = 4\sqrt{3}$, $0 < l \leq 1$

and $D_{\bar{l},\theta,\phi} = C2^{\frac{5}{2}}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

Proof. We have that;

$|\alpha_4| \leq |\frac{P_{1,1}}{2\pi^2}\frac{\bar{b}_{12,\bar{l}}(\bar{k})}{k^2|\bar{k}-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$|\bar{b}_{12,\bar{l}}(\bar{k})| \leq \frac{D}{|\bar{k}-\bar{l}|^4}$, $|\bar{k}-\bar{l}| > 0$ (change this)

where $D \in \mathcal{R}_{>0}$

so that;

$|\alpha_4(R,\theta,\phi,t,\bar{l})| \leq |\frac{\bar{d}'_{12}(\bar{l})}{l}|\frac{C}{|\bar{k}-\bar{l}|^5}$

$= C|\frac{\bar{d}'_{12}(\bar{l})}{l}|\frac{1}{[(Rsin(\theta)cos(\phi)-l_1)^2+(Rsin(\theta)sin(\phi)-l_2)^2+(Rcos(\theta)-l_3)^2]^{\frac{5}{2}}}$

$= \frac{C}{R^5}|\frac{\bar{d}'_{12}(\bar{l})}{l}|\frac{1}{[(sin(\theta)cos(\phi)-\frac{l_1}{R})^2+(sin(\theta)sin(\phi)-\frac{l_2}{R})^2+(cos(\theta)-\frac{l_3}{R})^2]^{\frac{5}{2}}}$

$= \frac{C}{R^5}|\frac{\bar{d}'_{12}(\bar{l})}{l}|\frac{1}{[1-\frac{2l_1sin(\theta)cos(\phi)}{R}-\frac{2l_2sin(\theta)sin(\phi)}{R}-\frac{2l_3cos(\theta)}{R}+\frac{l^2}{R^2}]^{\frac{5}{2}}}$

$= \frac{C}{R^5}|\frac{\bar{d}'_{12}(\bar{l})}{l}|\frac{1}{(1-x+\frac{l^2}{R^2})^{\frac{5}{2}}}$

where $C \in \mathcal{R}_{>0}$ and;

$|x| \leq \frac{2(|l_1|+|l_2|+|l_3|)}{R} \leq \frac{2l\sqrt{3}}{R} \leq \frac{1}{2}$, for $R > 4l\sqrt{3}$

so that;

$|\alpha_4(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^5}|\frac{\bar{d}'_{12}(\bar{l})}{l}| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}$ (for $R > 4l\sqrt{3}, l > 1$,

$R > 4\sqrt{3}, 0 < l \leq 1)$

In particularly;

$|Re(\alpha_4)(R,\theta,\phi,t,\bar{l})| \leq |\alpha_4(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

for $R > 4l\sqrt{3}, l > 1,\ R > 4\sqrt{3}, 0 < l \leq 1$

$|Im(\alpha_4)(R,\theta,\phi,t,\bar{l})| \leq |\alpha_4(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

for $R > 4l\sqrt{3}, l > 1,\ R > 4\sqrt{3}, 0 < l \leq 1$

□

Lemma 0.9. *We have that;*

$\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}| \in L^1(\mathcal{R}^6),\ \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}| \in L^1(\mathcal{R}^6)$

Proof. For the first claim, fix $\bar{l} \neq \bar{0}$, then;

$\frac{1}{|\bar{k}|^2}|_{B(\bar{l},\frac{l}{2})} \leq \frac{4}{l^2},\ \frac{1}{|\bar{k}-\bar{l}|}|_{\mathcal{R}^3 \setminus B(\bar{l},\frac{l}{2})} \leq \frac{2}{l}$

so that;

$\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|d\bar{k} = \int_{B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}d\bar{k} + \int_{\mathcal{R}^3 \setminus B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}d\bar{k}$

$\leq \frac{4}{l^2}\int_{B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}-\bar{l}|}d\bar{k} + \frac{2}{l}\int_{\mathcal{R}^3 \setminus B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$\leq \frac{4}{l^2}\int_{B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}-\bar{l}|}d\bar{k} + \frac{2}{l}\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$= \frac{4}{l^2}\int_{B(\bar{0},\frac{l}{2})} \frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}|}d\bar{k} + \frac{2}{l}\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$= \frac{4}{l^2}\int_0^{\frac{l}{2}}\int_{0 \leq \theta \leq \pi, -\pi \leq \phi \leq \pi} \frac{|\bar{b}_{12}(R,\theta,\phi)|}{R}R^2 sin(\theta)dRd\theta d\phi + \frac{2}{l}\int_{B(\bar{0},1)} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$+\int_{\mathcal{R}^3\setminus B(\bar{0},1)}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$\leq \frac{8\pi^2}{l^2}[\frac{R^2}{2}]_0^{\frac{l}{2}}+\frac{2}{l}\int_0^1\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi}\frac{|\bar{b}_{12}(R,\theta,\phi)|}{R^2}R^2sin(\theta)dRd\theta d\phi+\int_{\mathcal{R}^3\setminus B(\bar{0},1)}|\bar{b}_{12,\bar{l}}(\bar{k})|d\bar{k}$

$\leq \pi^2+\frac{4\pi^2}{l}[R]_0^1+C$

$=\pi^2+\frac{4\pi^2}{l}+C$

where $C=||\bar{b}_{12,\bar{l}}||_{L^1(\mathcal{R}^3)}$ is independent of $\bar{l}$. It follows that;

$\int_{\mathcal{R}^6}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{k}d\bar{l}\leq\int_{\mathcal{R}^3}(\pi^2+\frac{4\pi^2}{l}+C)|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{l}$

$=(\pi^2+C)\int_{\mathcal{R}^3}\frac{|\bar{d}'_{12}(\bar{l})|}{|\bar{l}|}d\bar{l}+4\pi^2\int_{\mathcal{R}^3}\frac{|\bar{d}'_{12}(\bar{l})|}{|\bar{l}|^2}d\bar{l}$

$\leq(\pi^2+C)(\int_{B(\bar{0},1)}\frac{|\bar{d}'_{12}(\bar{l})|}{|\bar{l}|}d\bar{l}+\int_{\mathcal{R}^3\setminus B(\bar{0},1)}|\bar{d}'_{12}(\bar{l})|d\bar{l})$

$+4\pi^2(\int_{B(\bar{0},1)}\frac{|\bar{d}'_{12}(\bar{l})|}{|\bar{l}|^2}d\bar{l}+\int_{\mathcal{R}^3\setminus B(\bar{0},1)}|\bar{d}'_{12}(\bar{l})|d\bar{l})$

$\leq(\pi^2+C)(\int_0^1\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi}||\bar{d}'_{12}(R,\theta,\phi)|Rsin(\theta)d\theta d\phi+D)$

$+4\pi^2(\int_0^1\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi}||\bar{d}'_{12}(R,\theta,\phi)|sin(\theta)d\theta d\phi+D)$

$\leq(\pi^2+C)(\pi^2+D)+4\pi^2(2\pi^2+D)$

$=9\pi^4+\pi^2C+5\pi^2D+CD$

where $D=||\bar{d}'_{12}||_{L^1(\mathcal{R}^3)}$

For the second claim, fix $\bar{l}\neq\bar{0}$, then, using the substitution $\bar{k}'=\bar{k}-\bar{l}$ and the previous proof, we obtain that;

$\int_{\mathcal{R}^3}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|d\bar{k}=\int_{\mathcal{R}^3}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}|^2|\bar{k}+\bar{l}|}|d\bar{k}\leq\pi^2+\frac{4\pi^2}{l}+C$

Following the above proof again, we have that;

$\int_{\mathcal{R}^6}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{k}d\bar{l}\leq\int_{\mathcal{R}^3}(\pi^2+\frac{4\pi^2}{l}+C)|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{l}$

$\leq 9\pi^4+\pi^2C+5\pi^2D+CD$

□

Definition 0.10. *We say that $f \in C(\mathcal{R})$ is of moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|^2}$ for $|x| > 1$. We say that $f \in C(\mathcal{R}_{>0})$ is of moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|^2}$ for $|x| > 1$. We say that $f \in C(\mathcal{R})$ is of moderate decrease n, if there exists a constant $D_n \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D_n}{|x|^2}$ for $|x| > n$. We say that $f \in C(\mathcal{R}_{>0})$ is of moderate decrease n if there exists a constant $D_n \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D_n}{|x|^2}$ for $|x| > n$. We say that $f \in C(\mathcal{R})$ is of very moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|}$ for $|x| > 1$. We say that $f \in C(\mathcal{R})$ is of very moderate decrease n if there exists a constant $D_n \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|}$ for $|x| > n$. We say that $f \in C(\mathcal{R}_{>0})$ is of very moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|}$ for $|x| > 1$. We say that $f \in C(\mathcal{R}_{>0})$ is of very moderate decrease n if there exists a constant $D_n \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D_n}{|x|}$ for $|x| > n$. We say that $f \in C(\mathcal{R})$ is non-oscillatory if there are finitely many points $\{y_i : 1 \leq i \leq n\} \subset \mathcal{R}$ for which $f|_{(y_i,y_{i+1})}$ is monotone, $1 \leq i \leq n-1$, and $f|_{(-\infty,y_1)}$ and $f|_{(y_n,\infty)}$ is monotone. We denote by $val(f)$ the minimum number of such points. We denote by $val_f(n)$, the minimum number of points on the interval $(0,n)$. We say that $f \in C(\mathcal{R}_{>0})$ is non-oscillatory if there are finitely many points $\{y_i : 1 \leq i \leq n\} \subset \mathcal{R}_{>0}$ for which $f|_{(y_i,y_{i+1})}$ is monotone, $1 \leq i \leq n-1$, and $f|_{(0,y_1)}$ and $f|_{(y_n,\infty)}$ is monotone. Similarily, we denote by $val(f)$ the minimum number. We say that $f \in C(\mathcal{R})$ is oscillatory if there exists an increasing sequence $\{y_i : i \in \mathcal{Z}\} \subset \mathcal{R}$, for which $f|_{(y_i,y_{i+1})}$ is monotone, $i \in \mathcal{Z}$, and there exists $\delta > 0$, with $y_{i+1} - y_i > \delta$, for $i \in \mathcal{Z}$. We say that $f \in C(\mathcal{R}_{>0})$ is oscillatory if there exists a sequence $\{y_i : i \in \mathcal{N}\} \subset \mathcal{R}$, for which $f|_{(0,y_1)}$ is monotone, and $f|_{(y_i,y_{i+1})}$ is monotone, $i \in \mathcal{N}$, and there exists $\delta > 0$, with $y_1 > \delta$ and $y_{i+1} - y_i > \delta$, for $i \in \mathcal{N}$.*

Lemma 0.11. *Let $f \in C(\mathcal{R})$ and $\frac{df}{dx} \in C(\mathcal{R})$ be of moderate decrease, with $\frac{df}{dx}$ non-oscillatory, then defining the Fourier transform by;*

$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{R}} f(x) e^{-ikx} dx$$

we have that, there exists a constant $C \in \mathcal{R}_{>0}$, such that;

$$|\mathcal{F}(f)(k)| \leq \frac{C}{|k|^2}$$

for sufficiently large k. *Let* $f \in C(\mathcal{R})$ *and* $\frac{df}{dx} \in C(\mathcal{R})$ *be of moderate decrease, with* $\frac{df}{dx}$ *oscillatory, then, similarly;*

we have that, there exists a constant $C \in \mathcal{R}_{>0}$, *such that;*

$|\mathcal{F}(f)(k)| \leq \frac{C}{|k|^2}$

for sufficiently large k.

The same result holds in the two claims, replacing moderate decrease with moderate decrease n.

Let $f \in C(\mathcal{R})$ *be analytic, with* f *and* $\frac{df}{dx}$ *of moderate decrease, then given* $\epsilon > 0$, *there exists* E_ϵ *such that, for sufficiently large* k*;*

$\mathcal{F}(f)(k) \leq \frac{E_\epsilon}{|k|^2} + \frac{\epsilon}{|k|}$

where $E_\epsilon = 2val_{\frac{df}{dx}}([-L_\epsilon, L_\epsilon])||\frac{df}{dx}||_\infty$, $L_\epsilon \in \mathcal{R}_{>0}$.

The same result holds with the assumption that $\frac{df}{dx}$ *is just continuous or when* $\frac{df}{dx} \in C(\mathcal{R}_{\neq 0})$ *and* $\frac{df}{dx} \in L^p(\mathcal{R})$, *for some* $p > 1$.

Proof. As f is of moderate decrease, we have that $f \in L^1(\mathcal{R})$ and $lim_{|x|\rightarrow\infty} f(x) = 0$ Similarly, $\frac{df}{dx} \in L^1(\mathcal{R})$ and $\frac{df}{dx}$ is continuous. We have, using integration by parts, that;

$\mathcal{F}(\frac{df}{dx})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{R}} \frac{df}{dx}(y) e^{-iky} dy$

$= [f(y)e^{-iky}]_{-\infty}^{\infty} + ik \int_{\mathcal{R}} f(y) e^{-iky} dy$

$= ik \int_{\mathcal{R}} f(y) e^{-iky} dy$

$= ik\mathcal{F}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}$, $(\dagger)$

As $\frac{df}{dx}$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_\epsilon \in \mathcal{N}$ such that;

$|\mathcal{F}(\frac{df}{dx})(k)-\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N_\epsilon}^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|<\epsilon\ (*)$

As $\frac{df}{dx}|_{-N_\epsilon,N_\epsilon}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [10], using underflow, we can find $\{D_\epsilon,E_\epsilon\}\subset\mathcal{R}_{>0}$, such that, for all $|k|>D_\epsilon$, we have that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N_\epsilon}^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|<\frac{E_\epsilon}{|k|},\ (**)$

It is easy to see from the proof, that $\{D_\epsilon,E_\epsilon\}$ can be chosen uniformly in ϵ. Then, from $(*),(**)$, and the triangle inequality, we obtain that, for $|k|>D_\epsilon$;

$|\mathcal{F}(\frac{df}{dx})(k)|$

$\leq|\mathcal{F}(\frac{df}{dx})(k)-\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N_\epsilon}^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|+|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N_\epsilon}^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|$

$<\epsilon+\frac{E_\epsilon}{|k|}$

so that, as $\{D_\epsilon,E_\epsilon\}$ were uniform and ϵ was arbitrary, we obtain that;

$|\mathcal{F}(\frac{df}{dx})(k)|<\frac{E}{|k|}$, for $|k|>D$

and, from (†), for $|k|>D$, that;

$|\mathcal{F}(f)(k)|\leq\frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}<\frac{E}{|k|^2}$

For the next claim, we can follow the proof of the second claim in Lemma 0.13. The next claim is a simple adaptation of the first two claims.

For the penultimate claim, we can follow the above proof up to (†) to obtain that;

$|\mathcal{F}(f)(k)|\leq\frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}\ |k|>1\ (AA)$

As $\frac{df}{dx}$ is of moderate decrease, we can find $L_\epsilon\in\mathcal{R}_{>0}$ such that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{|y|>L_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|<\epsilon\ (BB)$

As $\frac{df}{dx}|_{[-L_\epsilon,L_\epsilon]}$ is analytic, $\frac{d^2f}{dx^2}|_{[-L_\epsilon,L_\epsilon]}$ has finitely many zeroes, in particularly $\frac{df}{dx}|_{[-L_\epsilon,L_\epsilon]}$ is non-oscillatory. Using the proof above, we have that, for sufficiently large k;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-L_\epsilon}^{L_\epsilon}\frac{df}{dx}(y)e^{-iky}dy| < \frac{E_\epsilon}{|k|}$ (CC)

where $E_\epsilon = val_{\frac{df}{dx}}([-L_\epsilon, L_\epsilon])||\frac{df}{dx}||_\infty$. It follows that, from $(BB), (CC)$, that;

$|\mathcal{F}(\frac{df}{dx})| \leq \frac{E_\epsilon}{|k|} + \epsilon$ (DD)

It follows, combining $(AA), (DD)$, that, for sufficiently large k;

$|\mathcal{F}(f)(k)| \leq \frac{\frac{E_\epsilon}{|k|}+\epsilon}{|k|}$

$= \frac{E_\epsilon}{|k|^2} + \frac{\epsilon}{|k|}$

as required.

For the final claim, we can follow the above proof up to $(\dagger)$ again to obtain that;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}$ $|k| > 1$ (AAB)

As $\frac{df}{dx} \in L^p(\mathcal{R})$, by Holder's inequality, we have that, for $\delta > 0$;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\delta}^{\delta}\frac{df}{dx}(y)e^{-iky}dy|$

$\leq \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\delta}^{\delta}|\frac{df}{dx}(y)|dy$

$\leq ||\frac{df}{dx}|_{(-\delta,\delta)}||_p||1_{-\delta,\delta}||_{\frac{p}{p-1}}$

$\leq C(2\delta)^{\frac{p}{p-1}}$

$\leq \frac{\epsilon}{3}$ (CDC)

for δ sufficiently small, $\epsilon > 0$ arbitrary. As $\frac{df}{dx}$ is of moderate decrease, we can find $L_\epsilon \in \mathcal{R}_{>0}$ such that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{|y|>L_\epsilon}\frac{df}{dx}(y)e^{-iky}dy| < \frac{\epsilon}{3}$ (BBB)

As $\frac{df}{dx}$ is continuous on $[-L_\epsilon, -\delta] \cup [\delta, L_\epsilon]$, we can, using the Stone-Weierstrass approximation theorem, find a polynomial $p_{\delta,\epsilon}$ such that

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{df}{dx}-\frac{1}{(2\pi)^{\frac{1}{2}}}p_{\delta,\epsilon}|<\frac{\epsilon}{6L_{\epsilon}}$, so that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\delta<|y|<L_{\epsilon}}\frac{df}{dx}(y)e^{-iky}dy|$

$\leq\frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{\delta<|y|<L_{\epsilon}}|\frac{df}{dx}-p_{\delta,\epsilon}(y)|dy|+\frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{\delta<|y|<L_{\epsilon}}p_{\delta,\epsilon}(y)e^{-iky}dy|$

$\leq\frac{2L_{\epsilon}\epsilon}{6L_{\epsilon}}+\frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{\delta<|y|<L_{\epsilon}}p_{\delta,\epsilon}(y)e^{-iky}dy|$

$\leq\frac{\epsilon}{3}+\frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{\delta<|y|<L_{\epsilon}}p_{\delta,\epsilon}(y)e^{-iky}dy|$ (CCD)

We have that $p_{\delta,\epsilon}$ is analytic on an open neighborhood of $\delta<|y|<L_{\epsilon}$, so that $\frac{dp_{\delta,\epsilon}}{dx}|_{\delta<|y|<L_{\epsilon}}$ has finitely many zeroes, in particularly $\frac{dp_{\delta,\epsilon}}{dx}|_{\delta<|y|<L_{\epsilon}}$ is non-oscillatory. Using the proof above, we have that, for sufficiently large k;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\delta<|y|<L_{\epsilon}}p_{\delta,\epsilon}(y)e^{-iky}dy|<\frac{E_{\epsilon}}{|k|}$ (CCB)

where;

$E_{\epsilon}=2val_{p_{\delta,\epsilon}}(\delta<|y|<L_{\epsilon})||p_{\delta,\epsilon}|_{\delta<|y|<L_{\epsilon}}||_{\infty}$

It follows from $(CCB),(CCD),(BBB),(CDC)$, that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{df}{dx}(y)e^{-iky}dy|$

$\leq|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\delta}^{\delta}\frac{df}{dx}(y)e^{-iky}dy|+|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\delta<|y|<L_{\epsilon}}\frac{df}{dx}(y)e^{-iky}dy|+|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{|y|>L_{\epsilon}}\frac{df}{dx}(y)e^{-iky}dy|$

$<3\frac{\epsilon}{3}+\frac{E_{\epsilon}}{|k|}$

$<\epsilon+\frac{E_{\epsilon}}{|k|}$

By (AAB), we then have that, for $|k|$ sufficiently large;

$|\mathcal{F}(f)(k)|\leq\frac{\epsilon}{|k|}+\frac{E_{\epsilon}}{|k|^2}$

□

Lemma 0.12. *Let $f\in C(\mathcal{R}_{>0})$ be of moderate decrease, with f non-oscillatory, and $lim_{x\rightarrow 0}f(x)=M$, with $M\in\mathcal{R}$, then defining the half Fourier transform $\mathcal{G}$, by;*

$\mathcal{G}(f)(k) = \int_0^\infty f(x)e^{-ikx}dx$

we have that, there exists a constant $E \in \mathcal{R}_{>0}$*, such that;*

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|}$

for sufficiently large $|k|$*. Moreover, we can choose;*

$E = 2||f||_\infty val(f)$

Let $f \in C(\mathcal{R}_{>0})$ *be of moderate decrease, with* f *oscillatory, and* $lim_{x\rightarrow 0}f(x) = M$*, with* $M \in \mathcal{R}$*, then, similarly;*

we have that, there exists a constant $E \in \mathcal{R}_{>0}$*, such that;*

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|}$

for sufficiently large $|k|$*. Moreover, we can choose* $E = \frac{(4||f||_\infty + D)}{\delta}$*, where* D *and* δ *are given in Definition 0.10.*

The first claim is the same, replacing moderate decrease with moderate decrease n*. The second claim is the same, replacing moderate decrease with moderate decrease* n*, with the modification that we can choose* $E = \frac{2n||f||_\infty}{\delta} + \frac{2D_n}{n\delta}$*. We can also choose;*

$E = 2val_f(n)||f||_\infty + \frac{2D_n}{n\delta}$

where δ *is the spacing in the interval* (n, ∞)*.*

Let $f \in C(\mathcal{R}_{>0})$ *be of moderate decrease, with* f *analytic and* $lim_{x\rightarrow 0}f(x) = M$*. Then for all* $\epsilon > 0$*, we have that there exists* E_ϵ *such that, for sufficiently large* k*;*

$|\mathcal{G}(f)(k)| \leq \frac{E_\epsilon}{|k|} + \epsilon$

where $E_\epsilon = 2val_f(L_\epsilon)||f||_\infty$ *and* $L_\epsilon \in \mathcal{R}_{>0}$*.*

Let $f \in C(\mathcal{R}_{>0}$ *be of moderate decrease, with* $lim_{x\rightarrow 0}f(x) = M$*, then for all* $\epsilon > 0$*, we have that there exists* E_ϵ*, such that for sufficiently large* k*;*

$|\mathcal{G}(f)(k)| \leq \frac{E_\epsilon}{|k|} + \epsilon$

Proof. As f is of moderate decrease and $lim_{x\rightarrow 0}f(x) = M$, we have that $f \in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty}f(x) = 0$.

As f is of moderate decrease, for any $\epsilon > 0$, we can find $N_\epsilon \in \mathcal{N}$ such that;

$|\mathcal{G}(f)(k) - \int_0^{N_\epsilon} f(y)e^{-iky}dy| < \epsilon\ (*)$

As $f|_{0,N_\epsilon}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [10], using underflow, we can find $\{D_\epsilon, E_\epsilon\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\int_0^{N_\epsilon} f(y)e^{-iky}dy| < \frac{E_\epsilon}{|k|}, (**)$

It is easy to see from the proof, that $\{D_\epsilon, E_\epsilon\}$ can be chosen uniformly in ϵ, Splitting the calculation into real and imaginary components, it is straightfoward to see that it is possible to choose E_ϵ with $E_\epsilon = 2||f||_\infty val(f)$, noting that the infinitesimal correction existing after the use of underflow, drops out after taking the standard part. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k| > D_\epsilon$;

$|\mathcal{G}(f)(k)|$

$\leq |\mathcal{G}(f)(k) - \int_0^{N_\epsilon} f(y)e^{-iky}dy| + |\int_0^{N_\epsilon} f(y)e^{-iky}dy|$

$< \epsilon + \frac{E_\epsilon}{|k|}$

so that, as $\{D_{\epsilon,\rho}, E_\epsilon\}$ were uniform and ϵ was arbitrary, we obtain that;

$|\mathcal{G}(f)(k)| < \frac{E}{|k|}$, for sufficiently large $|k|$

For the second claim, after choosing $N \in \mathcal{N}$, we have that $f|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta}$ monotone intervals. As in $(**)$, and inspection of the proof in [10], we get;

$|\int_0^N fe^{-iky}dy| < \frac{E_N}{|k|}$

for sufficiently large $|k|$, where $E_N = \frac{2NC}{\delta}$ and $C = max_{x\in\mathcal{R}_{>0}}|f|$.

Choosing $N>1$, as f is of moderate decrease, we can assume that $|f|\leq\frac{D}{x^2}$, for $x>N$. Then, using the proof in [10] again, the definition of oscillatory, and noting that ${}^*\sum_{y_i^*>N}\frac{D}{y_i^2}\simeq\sum_{y_i>N}\frac{D}{y_i^2}$, we have that, for sufficiently large $|k|$;

$|\int_N^\infty fe^{-iky}dy|<(\frac{2}{|k|}\sum_{y_i>N}\frac{D}{y_i^2})$

$\leq(\frac{2}{|k|}\sum_{n\in\mathcal{Z}_{\geq 0}}\frac{D}{(y_{i_0}+n\delta)^2})$

$\leq\frac{2D}{\delta|k|}\int_{y_{i_0}}^\infty\frac{dx}{x^2}$

$=\frac{2D}{\delta|k|y_{i_0}}$

$\leq\frac{2D}{\delta|k|N}$

where $y_{i_0}\geq N$ and $y_{i_0}\leq y_i$, for all $y_i\geq N$. It follows that;

$|\mathcal{G}(f)(k)|=|\int_0^N fe^{-iky}dy+\int_N^\infty fe^{-iky}dy|$

$\leq|\int_0^N fe^{-iky}dy|+|\int_N^\infty fe^{-iky}dy|$

$\leq\frac{E_N}{|k|}+\frac{2D}{\delta|k|N}$

$\leq\frac{2}{|k|}(\frac{NC}{\delta}+\frac{D}{\delta N})$

It follows, using (†), that;

$|\mathcal{G}(f)(k)\leq\frac{E}{|k|}$

where $E=2(\frac{NC}{\delta}+\frac{D}{\delta N})$

In particular, choosing $N=2$, we can take;

$E=2(\frac{2C}{\delta}+\frac{D}{2\delta})=\frac{(4C+D)}{\delta}=\frac{(4||f||_\infty+D)}{\delta}$

For the next claim, the modification for the first part is the same. In the second part, choose $N\geq n$, rather than $N>1$ in the proof, and replace D with D_n, to get $E=2(\frac{NC}{\delta}+\frac{D_n}{\delta N})$, then, taking $N=n$, we obtain $E=2(\frac{nC}{\delta}+\frac{D_n}{\delta n})$. For the next claim, replace the count of $\frac{N}{\delta}$ monotone intervals for $f|_{(0,n)}$ with $val_f(n)$.

For the penultimate claim, we have that, as f is of moderate decrease;

$|\int_n^\infty f(y)e^{-iky}dy| \leq \int_n^\infty \frac{D}{y^2}dy$

$= [\frac{-D}{y}]_n^\infty$

$= \frac{D}{n}$

$\leq \epsilon$

for $n \geq \frac{D}{\epsilon}$. In particularly, if we choose $L_\epsilon = \frac{D}{\epsilon}$, then as f is analytic, $f'|_{[0,L_\epsilon]}$ has finitely many zeros, so $f|_{[0,L_\epsilon]}$ is non-oscillatory. By the proof of Lemma 9 in [10] again, we can find $E_\epsilon \in \mathcal{R}_{>0}$, with $E_\epsilon = 2val_f(L_\epsilon)||f||_\infty$, such that;

$|\int_0^{L_\epsilon} f(y)e^{-iky}dy| < \frac{E_\epsilon}{|k|}$

for sufficiently large k. It follows that;

$|\mathcal{G}(f)(k)| = |\int_0^{L_\epsilon} f(y)e^{-iky}dy| + |\int_{L_\epsilon}^\infty f(y)e^{-iky}dy|$

$\leq \frac{E_\epsilon}{|k|} + \epsilon$

as required.

For the final claim, as above, using the fact that f is of moderate decrease, we can find n such that;

$|\int_n^\infty f(y)e^{-iky}dy| < \frac{\epsilon}{2}$ (C)

Using the Stone-Weierstrass approximation theorem, we can find an analytic function $f_{n,\delta}$ such that $|f - f_{n,\delta}| < \delta$ on $[0, n]$ and;

$|\int_0^n f(y)e^{-iky}dy| \leq \int_0^n |f(y) - f_{n,\delta}(y)|dy + |\int_0^n f_{n,\delta}(y)e^{-iky}dy|$

$\leq n\delta + |\int_0^n f_{n,\delta}(y)e^{-iky}dy|$ (D)

Using the methods above, we can find $E_{\epsilon,\delta}$ such that, for sufficiently large k;

$|\int_0^n f_{n,\delta}(y)e^{-iky}dy| \leq \frac{E_{\epsilon,\delta}}{|k|}$ (E)

Choosing $\delta = \frac{\epsilon}{2n}$, we have from $(D),(E)$, that;

$|\int_0^n f(y)e^{-iky}dy| \leq \frac{\epsilon}{2} + \frac{E_{\epsilon,\frac{\epsilon}{2}}}{|k|}$

and from (C), that;

$|\mathcal{G}(f)(y)| \leq \epsilon + \frac{E_\epsilon}{|k|}$

where $E_\epsilon = 2val_{f_n}(n)||f_{n,\delta}|_{[0,n]}||_\infty$ for the choice of analytic function $f_{n,\delta}$.

□

Lemma 0.13. *Let $f \in C(\mathcal{R}_{>0})$ and $\frac{df}{dx} \in C(\mathcal{R}_{>0})$ be of moderate decrease, with $\frac{df}{dx}$ non-oscillatory, and $lim_{x\rightarrow 0}f(x) = 0$, $lim_{x\rightarrow 0}\frac{df}{dx}(x) = M$, with $M \in \mathcal{R}$, then defining the half Fourier transform $\mathcal{G}$, by;*

$\mathcal{G}(f)(k) = \int_0^\infty f(x)e^{-ikx}dx$

we have that, there exists a constant $E \in \mathcal{R}_{>0}$, such that;

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|^2}$

for sufficiently large k. Moreover, we can choose $E = 2||\frac{df}{dx}||_\infty val(\frac{df}{dx})$

Let $f \in C(\mathcal{R}_{>0})$ and $\frac{df}{dx} \in C(\mathcal{R}_{>0})$ be of moderate decrease, with $\frac{df}{dx}$ oscillatory, and $lim_{x\rightarrow 0}f(x) = 0$, $lim_{x\rightarrow 0}\frac{df}{dx}(x) = M$, with $M \in \mathcal{R}$, then, similarly;

we have that, there exists a constant $E \in \mathcal{R}_{>0}$, such that;

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|^2}$

for sufficiently large k, Moreover, we can choose $E = \frac{(4||\frac{df}{dx}||_\infty + D)}{\delta}$.

The first claim is the same, replacing moderate decrease with moderate decrease n. The second claim is the same, replacing moderate decrease with moderate decrease n, with the modification that we can *choose* $E = \frac{2n||\frac{df}{dx}||_\infty}{\delta} + \frac{2D_n}{n\delta}$.

Let $f \in C(\mathcal{R}_{\geq 0})$ *be analytic, with* $\{f, \frac{df}{dx}\}$ *of moderate decrease, such that* $lim_{x\rightarrow 0} f = 0$ *and* $lim_{x\rightarrow 0}\frac{df}{dx} = M$*, then, for* $\epsilon > 0$*, there exists* $E_\epsilon \in \mathcal{R}_{>0}$ *such that, for sufficiently large* k*;*

$|\mathcal{G}(f)(k)| \leq \frac{E_\epsilon}{|k|^2} + \frac{\epsilon}{|k|}$

where $E_\epsilon = 2val_{\frac{df}{dx}}(L_\epsilon)||\frac{df}{dx}||_\infty$, $L_\epsilon \in \mathcal{R}_{>0}$

The same claim holds with just $f \in C^1(\mathcal{R}_{>0})$ *instead of analytic.*

Proof. As f is of moderate decrease and $lim_{x\rightarrow 0} f(x) = 0$, we have that $f \in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty} f(x) = 0$. Similarly, $\frac{df}{dx} \in L^1(\mathcal{R}_{>0})$ and $\frac{df}{dx}$ is continuous. We have, using integration by parts, that;

$\mathcal{G}(\frac{df}{dx})(k) = \int_0^\infty \frac{df}{dx}(y)e^{-iky}dy$

$= [f(y)e^{-iky}]_0^\infty + ik\int_0^\infty f(y)e^{-iky}dy$

$= ik\int_0^\infty f(y)e^{-iky}dy$

$= ik\mathcal{G}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|}$, $(\dagger)$

As $\frac{df}{dx}$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_\epsilon \in \mathcal{N}$ such that;

$|\mathcal{G}(\frac{df}{dx})(k) - \int_0^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy| < \epsilon$ $(*)$

As $\frac{df}{dx}|_{0,N_\epsilon}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [10], using underflow, we can find $\{D_\epsilon, E_\epsilon\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\int_0^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy| < \frac{E_\epsilon}{|k|}$, $(**)$

It is easy to see from the proof, that $\{D_\epsilon, E_\epsilon\}$ can be chosen uniformly in ϵ. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k|>D_\epsilon$;

$|\mathcal{G}(\frac{df}{dx})(k)|$

$\leq |\mathcal{G}(\frac{df}{dx})(k)-\int_0^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|+|\int_0^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy|$

$<\epsilon+\frac{E_\epsilon}{|k|}$

so that, as $\{D_\epsilon, E_\epsilon\}$ were uniform and ϵ was arbitrary, we obtain that;

$|\mathcal{G}(\frac{df}{dx})(k)|<\frac{E}{|k|}$, for $|k|>D$

and, from (†), for $|k|>D$, that;

$|\mathcal{G}(f)(k)|\leq\frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|}<\frac{E}{|k|^2}$

The choice of E is the same as in the proof of Lemma 0.12. For the second claim, the proof up to (†) is the same. After choosing $N\in\mathcal{N}$, we have that $\frac{df}{dx}|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta}$ monotone intervals. As in $(**)$, and inspection of the proof in [10], we get;

$|\int_0^N\frac{df}{dx}e^{-iky}dy|<\frac{E_N}{|k|}$

where $E_N\leq\frac{2NC}{\delta}$ and $C=max_{x\in\mathcal{R}_{>0}}|\frac{df}{dx}|$.

Choosing $N>1$, as $\frac{df}{dx}$ is of moderate decrease, we can assume that $|\frac{df}{dx}|\leq\frac{D}{x^2}$, for $x>N$. Then, using the proof in [10] again, and the definition of oscillatory, we have that, for sufficiently large $|k|$;

$|\int_N^\infty\frac{df}{dx}e^{-iky}dy|<(\frac{2}{|k|}\sum_{y_i>N}\frac{D}{y_i^2})$

$\leq(\frac{2}{|k|}\sum_{n\in\mathcal{Z}_{\geq 0}}\frac{D}{(y_{i_0}+n\delta)^2})$

$\leq\frac{2D}{\delta|k|}\int_{y_{i_0}}^\infty\frac{dx}{x^2}$

$=\frac{2D}{\delta|k|y_{i_0}}$

$\leq\frac{2D}{\delta|k|N}$

where $y_{i_0}\geq N$ and $y_{i_0}\leq y_i$, for all $y_i\geq N$. It follows that;

$|\mathcal{G}(\frac{df}{dx})(k)| = |\int_0^N \frac{df}{dx}e^{-iky}dy + \int_N^\infty \frac{df}{dx}e^{-iky}dy|$

$\leq |\int_0^N \frac{df}{dx}e^{-iky}dy| + |\int_N^\infty \frac{df}{dx}e^{-iky}dy|$

$\leq \frac{E_N}{|k|} + \frac{2D}{\delta|k|N}$

$\leq \frac{2}{|k|}(\frac{NC}{\delta} + \frac{D}{\delta N})$

It follows, using (†), that;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|} < \frac{E_N}{|k|^2}$

where $E_N = 2(\frac{NC}{\delta} + \frac{D}{\delta N})$

As in Lemma 0.12, we can choose E as in the final claim of the two parts.

For the next claim, the modification for the first part is the same. In the second part, choose $N \geq n$, rather than $N > 1$ in the proof, and replace D with D_n, to get $E_N = 2(\frac{NC}{\delta} + \frac{D_n}{\delta N})$, then, taking $N = n$, we obtain $E = 2(\frac{nC}{\delta} + \frac{D_n}{\delta n})$.

The proof of the penultimate claim is similar to that of Lemma 0.11, we can use the proof up to (†) of this Lemma, to obtain, for $|k| > 1$;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|}$, (AA)

As in the proof of Lemma 0.11, we can find $\{E_\epsilon, L_\epsilon\}$ such that, for sufficiently large k;

$|\mathcal{G}(\frac{df}{dx})| \leq \frac{E_\epsilon}{|k|} + \epsilon$ (BB)

where $E_\epsilon = 2val_{\frac{df}{dx}}(L_\epsilon)||\frac{df}{dx}||_\infty$.

Combining (AA) and (BB), we obtain that, for sufficiently large k;

$|\mathcal{G}(f)(k)| \leq \frac{E_\epsilon}{|k|^2} + \frac{\epsilon}{|k|}$

For the final claim, when $f \in C^1(\mathcal{R}_{>0})$, we use (AA) again. Then, we use the Stone-Weierstrass approximation theorem, to find a polynomial $p_{\frac{\epsilon}{2}}$ such that $|\frac{df}{dx} - p_{\frac{\epsilon}{2}}| \leq \delta$ on $[0, L_\epsilon]$, and $L_\epsilon\delta \leq \frac{\epsilon}{2}$. As $p'_{\frac{\epsilon}{2}}$ has finitely many zeros on $[0, L_\epsilon]$, we have that, for sufficiently large k;

$|\int_0^{L_\epsilon} \frac{df}{dx}(y)e^{-iky}dy| \leq \frac{\epsilon}{2} + |\int_0^{L_\epsilon} p_{\frac{\epsilon}{2}} e^{-iky}dy|$

$\leq \frac{\epsilon}{2} + \frac{C_\epsilon}{|k|}$

so that, for sufficiently large k;

$|\int_0^\infty \frac{df}{dx}(y)e^{-iky}dy| \leq |\int_{L_\epsilon}^\infty \frac{df}{dx}(y)e^{-iky}dy| + \frac{\epsilon}{2} + \frac{C_\epsilon}{|k|}$

$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{C_\epsilon}{|k|}$

$\leq \epsilon + \frac{C_\epsilon}{|k|}$

Then;

$|\mathcal{G}(f)(k)| \leq \frac{|\int_0^\infty \frac{df}{dx}(y)e^{-iky}dy|}{|k|}$

$\leq \frac{\epsilon}{|k|} + \frac{C_\epsilon}{|k|^2}$

□

Definition 0.14. *We say that a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$, with $f_{\overline{v}} \in C(\mathcal{R}_{>0})$ and $V \subset \mathcal{R}^n$ open, is of moderate decrease if there exists constants $D_{\overline{v}} \in \mathcal{R}_{>0}$ with $|f_{\overline{v}}(x)| \leq \frac{D_{\overline{v}}}{|x|^2}$ for $|x| > 1$. We say that a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$, with $f_{\overline{v}} \in C(\mathcal{R}_{>0})$ and $V \subset \mathcal{R}^n$ open, is of moderate decrease $n_{\overline{v}}$ if there exists constants $D_{\overline{v}} \in \mathcal{R}_{>0}$ with $|f_{\overline{v}}(x)| \leq \frac{D_{\overline{v}}}{|x|^2}$ for $|x| > n_{\overline{v}}$, where $n : V \rightarrow \mathcal{R}_{>0}$ is continuous. We say that the family $\{f_{\overline{v}} : \overline{v} \in V\}$ is non-oscillatory or uniformly non-oscillatory if there are finitely many points $\{y_{i,\overline{v}} : 1 \leq i \leq n\} \subset \mathcal{R}$ for which $f_{\overline{v}}|_{(y_{i,\overline{v}}, y_{i+1,\overline{v}})}$ is monotone, $1 \leq i \leq n-1$, and $f|_{(-\infty, y_{1,\overline{v}})}$ and $f|_{(y_{n,\overline{v}}, \infty)}$ is monotone, the number of points being independent of $\overline{v}$. We denote by $val(W)$ the minimum number of such points. We denote by $val(W|_{[0,n]}$ the minimum number restricted to a finite interval $[0, n]$. We say that a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$, with $f_{\overline{v}} \in C(\mathcal{R}_{>0})$ is oscillatory if there exists a sequence $\{y_{i,\overline{v}} : i \in \mathcal{N}\} \subset \mathcal{R}$, for which $f|_{(0, y_{1,\overline{v}})}$ is monotone, and $f|_{(y_{i,\overline{v}}, y_{i+1,\overline{v}})}$ is monotone, $i \in \mathcal{N}$, and there exists $\delta_{\overline{v}} > 0$, with $y_1 > \delta_{\overline{v}}$ and $y_{i+1} - y_i > \delta_{\overline{v}}$, for $i \in \mathcal{N}$. We say that the family is uniformly non-oscillatory if the constants $\delta_{\overline{v}}$ can be chosen independently of $\overline{v}$, that is a single $\delta > 0$ works for each $f_{\overline{v}}$. We call a family W excellent if it is oscillatory with the property that there exist $n_{\overline{v}}$ such that $val_{f_{\overline{v}}}(n_{\overline{v}}) = val(W)$ is independent of $\overline{v}$ and the* *spacing of* $f_{\overline{v}}|_{(n_{\overline{v}},\infty)}$ *may also be chosen independently of* $\overline{v}$*. We denote by* $||W||_\infty$ *the best uniform bound for* $||f_{\overline{v}}||_\infty$*, if it exists.*

Lemma 0.15. *Let a family* $W=\{f_{\overline{v}}:\overline{v}\in V\}$ *be of moderate decrease, with* W *non-oscillatory, and* $lim_{x\rightarrow 0}f_{\overline{v}}(x)=M_{\overline{v}}$*, with* $M_{\overline{v}}\in\mathcal{R}$*, then we have that, there exists constants* $E_{\overline{v}}\in\mathcal{R}_{>0}$*, such that;*

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{E_{\overline{v}}}{|k|}$

for sufficiently large $|k|$*, independent of* $\overline{v}$*. Moreover, we can choose;*

$E_{\overline{v}}=2||f_{\overline{v}}||_\infty val(W)$

Let a family $W=\{f_{\overline{v}}:\overline{v}\in V\}$ *be of moderate decrease and oscillatory, and* $lim_{x\rightarrow 0}f_{\overline{v}}(x)=M_{\overline{v}}$*, with* $M_{\overline{v}}\in\mathcal{R}$*, then, similarly;*

we have that, there exists constants $E_{\overline{v}}\in\mathcal{R}_{>0}$*, such that;*

$|\mathcal{G}(f)(k)|\leq\frac{E_{\overline{v}}}{|k|}$

for sufficiently large $|k|$*. Moreover, we can choose*

$E_{\overline{v}}=\frac{(4||f_{\overline{v}}||_\infty+D_{\overline{v}})}{\delta_{\overline{v}}}$

where $D_{\overline{v}}$ *and* $\delta_{\overline{v}}$ *are given in Definition 0.14.*

The first claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$*. The second claim is the same, replacing moderate decrease with moderate decrease* $n_{\overline{v}}$*, with the modification that we can choose* $E_{\overline{v}}=\frac{2n_{\overline{v}}||f_{\overline{v}}||_\infty}{\delta_{\overline{v}}}+\frac{2D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}}$*. We can also choose;*

$E_{\overline{v}}=2val_{f_{\overline{v}}}(n_{\overline{v}})||f_{\overline{v}}||_\infty+\frac{2D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}}$

If the family is excellent, we can take;

$E_{\overline{v}}=2val(W)||f_{\overline{v}}||_\infty+\frac{2D_{\overline{v}}}{n_{\overline{v}}\delta}$

Let a family W *be of moderate decrease and analytic, such that* V *is closed and bounded,* $lim_{x\rightarrow 0}f_{\overline{v}}=M_{\overline{v}}$*. Then, given* $\epsilon>0$*, we can choose* $E_\epsilon\in\mathcal{R}_{>0}$ *independent of* $\overline{v}$ *such that, for sufficiently large* k*;*

$|\mathcal{G}(f)(k)| \leq \frac{E_\epsilon}{|k|} + \epsilon$

with $E_\epsilon = 2val(W|_{[0,L_\epsilon]}||W||_\infty$, $f \in W$.

The same claim holds replacing analytic in the family W by continuous.

Proof. As each $f_{\overline{v}}$ is of moderate decrease and $lim_{x\rightarrow 0} f_{\overline{v}}(x) = M_{\overline{v}}$, we have that each $f_{\overline{v}} \in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty} f_{\overline{v}}(x) = 0$.

As each $f_{\overline{v}}$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_{\epsilon,\overline{v}} \in \mathcal{N}$ such that;

$|\mathcal{G}(f_{\overline{v}})(k) - \int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy| < \epsilon \ (*)$

As each $f_{\overline{v}}|_{0,N_{\epsilon,\overline{v}}}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [10], quantifying over the nonstandard parameter space ${}^*V$, linking the parameters with $N_{\epsilon,\overline{v}}$, and using underflow again, we can find $\{D_\epsilon, E_{\epsilon,\overline{v}}\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy| < \frac{E_{\epsilon,\overline{v}}}{|k|}$, $(**)$

It is easy to see from the proof, that $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ can be chosen uniformly in ϵ, as the number of monotone intervals in the interval $(0, N_{\epsilon,\overline{v}})$ is always bounded by $val(W)$. Splitting the calculation into real and imaginary components, it is again straightfoward to see that it is possible to choose $E_{\epsilon,\overline{v}}$ with $E_{\epsilon,\overline{v}} = 2||f_{\overline{v}}||_\infty val(W)$. Again, note that the infinitesimal correction existing after the use of underflow, drops out after taking the standard part, for each $f_{\overline{v}}$. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k| > D_\epsilon$;

$|\mathcal{G}(f_{\overline{v}})(k)|$

$\leq |\mathcal{G}(f_{\overline{v}})(k) - \int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy| + |\int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy|$

$< \epsilon + \frac{E_{\epsilon,\overline{v}}}{|k|}$

so that, as $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ were uniform and ϵ was arbitrary, we obtain that;

$|\mathcal{G}(f_{\overline{v}})(k)| < \frac{E_{\overline{v}}}{|k|}$, for sufficiently large $|k|$, independently of $\overline{v}$.

For the second claim, after choosing $N\in\mathcal{N}$, we have that each $f_{\overline{v}}|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta_{\overline{v}}}$ monotone intervals. As in $(**)$, and inspection of the proof in [10], we get;

$|\int_0^N f_{\overline{v}}e^{-iky}dy|<\frac{E_N}{|k|}$

for sufficiently large $|k|$, independent of $\overline{v}$, where $E_N=\frac{2NC_{\overline{v}}}{\delta_{\overline{v}}}$ and $C_{\overline{v}}=max_{x\in\mathcal{R}_{>0}}|f_{\overline{v}}|$.

Choosing $N>1$, as each $f_{\overline{v}}$ is of moderate decrease, we can assume that $|f_{\overline{v}}|\leq\frac{D_{\overline{v}}}{x^2}$, for $x>N$. Then, using the proof in [10] again, and the definition of oscillatory, we have that, for sufficiently large $|k|$, independent of $\overline{v}$;

$|\int_N^\infty f_{\overline{v}}e^{-iky}dy|<(\frac{2}{|k|}\sum_{y_{i,\overline{v}}>N}\frac{D_{\overline{v}}}{y_{i,\overline{v}}^2})$

$\leq(\frac{2}{|k|}\sum_{n\in\mathcal{Z}_{\geq 0}}\frac{D_{\overline{v}}}{(y_{i_0,\overline{v}}+n\delta_{\overline{v}})^2})$

$\leq\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|}\int_{y_{i_0},\overline{v}}^\infty\frac{dx}{x^2}$

$=\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|y_{i_0,\overline{v}}}$

$\leq\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

where $y_{i_0,\overline{v}}\geq N$ and $y_{i_0,\overline{v}}\leq y_{i,\overline{v}}$, for all $y_{i,\overline{v}}\geq N$. It follows that;

$|\mathcal{G}(f_{\overline{v}})(k)|=|\int_0^N f_{\overline{v}}e^{-iky}dy+\int_N^\infty f_{\overline{v}}e^{-iky}dy|$

$\leq|\int_0^N f_{\overline{v}}e^{-iky}dy|+|\int_N^\infty f_{\overline{v}}e^{-iky}dy|$

$\leq\frac{E_N}{|k|}+\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

$\leq\frac{2}{|k|}(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{\delta_{\overline{v}}N})$

It follows, using (†), that;

$|\mathcal{G}(f_{\overline{v}})(k)\leq\frac{E_N}{|k|}$

where $E_N=2(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{N\delta_{\overline{v}}})$

In particular, choosing $N=2$, we can take;

$E=E_2=2(\frac{2C_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{2\delta_{\overline{v}}})=\frac{(4C_{\overline{v}}+D_{\overline{v}})}{\delta_{\overline{v}}}=\frac{(4||f_{\overline{v}}||_\infty+D_{\overline{v}})}{\delta_{\overline{v}}}$

For the next claim, the modification for the first part is the same. In the second part, choose $N\geq n_{\overline{v}}$, rather than $N>1$ in the proof, then, taking $N=n_{\overline{v}}$, we obtain $E=E_{n_{\overline{v}}}=2(\frac{n_{\overline{v}}C_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}})$

For the next two claims, replace the $\frac{n_{\overline{v}}}{\delta_{\overline{v}}}$ monotone intervals on $(0,n_{\overline{v}})$ with $val_{f_{\overline{v}}}(n_{\overline{v}})$. Then note that in an excellent family, we can replace $val_{f_{\overline{v}}}(n_{\overline{v}})$ by $val(W)$ and $\delta_{\overline{v}}$ by δ.

For the penultimate claim, as V is compact we have, by continuity, that $sup_{\overline{v}\in V}|||f_{\overline{v}}|x^2||_\infty=D$ exists, so that uniformly, $|f|\leq\frac{D}{x^2}$, for $|x|>1$. It follows that;

$|\int_n^\infty f_{\overline{v}}e^{-iky}(y)dy|$

$\leq\int_n^\infty\frac{D}{x^2}dx$

$\leq\frac{D}{n}$

$\leq\epsilon$

for $n\geq\frac{D}{\epsilon}$, uniformly in $\overline{v}\in V$. In particular, choosing $L_\epsilon=\frac{D}{\epsilon}$, we have that there exists a uniform bound $val(W|_{[0,L_\epsilon]})$ for the number of zeros of $f'_{\overline{v}}|_{[0,L_\epsilon]}$. By continuity and the fact that V is compact, we can find a uniform bound $||W||_\infty$ for $||f_{\overline{v}}||_\infty$. It follows, as in Lemma 0.12, taking care to quantify over w, when using the underflow argument, that, for sufficiently large k;

$|\int_0^{L_\epsilon}f_{\overline{v}}(y)e^{-iky}dy|\leq\frac{E_\epsilon}{|k|}$

where $E_\epsilon=2val(W|_{[0,L_\epsilon]})||W||_\infty$

As in Lemma 0.12, we have that;

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{E_\epsilon}{|k|}+\epsilon$

for sufficiently large k, $\overline{v}\in V$, as required.

For the final claim, we use the last proof to find L_ϵ with;

$|\int_{L_\epsilon}^{\infty} f_{\overline{v}} e^{-iky}(y)dy|$

$\leq \frac{\epsilon}{2}$ (FF)

uniformly in $\overline{v}$. As the family is continuous, and V is closed and bounded, we can find a polynomial $p(x,\overline{v})$ such that;

$|f(x,\overline{v}) - p(x,\overline{v})| < \delta$

for $\overline{v} \in V$ and $x \in [0, L_\epsilon]$. By the usual compactness argument, there exists a uniform bound in the number of zeros of $p'_{\overline{v}}$ restricted to $[0, L_\epsilon]$, and a uniform bound for $||p_{\overline{v}}||_\infty$ on $[0, L_\epsilon$. By the same argument as above, we have that;

$|\int_0^{L_\epsilon} p_{\overline{v}}(y) e^{-iky} dy| \leq \frac{E_\epsilon}{|k|}$

where $E_\epsilon = 2val(p_{\overline{v}}|_{[0,L_\epsilon]})||p_{\overline{v}}||_\infty$

Choosing $\delta \leq \frac{\epsilon}{2L_\epsilon}$, we have that;

$|\int_0^{L_\epsilon} f_{\overline{v}}(y) e^{-iky} dy| \leq \frac{E_\epsilon}{|k|} + \frac{\epsilon}{2}$

so that, using (FF);

$|G(f_{\overline{v}})| \leq \frac{E_\epsilon}{|k|} + \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$\leq \frac{E_\epsilon}{|k|} + \epsilon$

as required.

□

Lemma 0.16. *Let a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$ be of moderate decrease such that the family $W' = \{\frac{df}{dx}_{\overline{v}} : \overline{v} \in V\}$ is of moderate decrease and non-oscillatory, with $lim_{x\rightarrow 0} f_{\overline{v}}(x) = 0$, $lim_{x\rightarrow 0}\frac{df_{\overline{v}}}{dx}(x) = M_{\overline{v}}$, with $M_{\overline{v}} \in \mathcal{R}$, for $\overline{v} \in V$, then we have that, there exists constants $E_{\overline{v}} \in \mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{E_{\overline{v}}}{|k|^2}$

for sufficiently large k, independent of $\overline{v}$. Moreover, we can choose $E_{\overline{v}} = 2||\frac{df_{\overline{v}}}{dx}||_\infty val(W')$

Let the families $W=\{f_{\overline{v}}:\overline{v}\in V\}$ and $W'=\{\frac{df}{dx}_{\overline{v}}:\overline{v}\in V\}$ be of moderate decrease with W' oscillatory as well, with $lim_{x\rightarrow 0}f_{\overline{v}}(x)=0$, $lim_{x\rightarrow 0}\frac{df_{\overline{v}}}{dx}(x)=M_{\overline{v}}$, with $M_{\overline{v}}\in\mathcal{R}$, then, similarly, we have that, there exists constants $E_{\overline{v}}\in\mathcal{R}_{>0}$, such that;

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{E_{\overline{v}}}{|k|^2}$

for sufficiently large k, independent of $\overline{v}$. Moreover, we can choose;

$E_{\overline{v}}=\frac{(4||\frac{df_{\overline{v}}}{dx}||_\infty+D_{\overline{v}})}{\delta_{\overline{v}}}$

where $D_{\overline{v}}$ and $\delta_{\overline{v}}$ are given in Definition 0.14.

The first claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$. The second claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$, with the modification that we can choose $E_{\overline{v}}=\frac{2n_{\overline{v}}||\frac{df_{\overline{v}}}{dx}||_\infty}{\delta_{\overline{v}}}+\frac{2D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}}$.

Let the families $W=\{f_{\overline{v}}:\overline{v}\in V\}$ and $W'=\{\frac{df}{dx}_{\overline{v}}:\overline{v}\in V\}$ be of moderate decrease and analytic, such that V is closed and bounded, $lim_{x\rightarrow 0}f_{\overline{v}}=0$, $lim_{x\rightarrow 0}\frac{df_{\overline{v}}}{dx}(x)=M_{\overline{v}}$, with $M_{\overline{v}}\in\mathcal{R}$. Then, given $\epsilon>0$, we can choose $E_\epsilon\in\mathcal{R}_{>0}$ independent of $\overline{v}$ such that, for sufficiently large k;

$|\mathcal{G}(f)(k)|\leq\frac{E_\epsilon}{|k|^2}+\frac{\epsilon}{|k|}$

with $E_\epsilon=2val(W'|_{[0,L_\epsilon]}||W'||_\infty$, $\frac{df}{dx}\in W'$.

The same claim holds replacing analytic in the families W,W' by continuous.

Proof. As each $f_{\overline{v}}$ is of moderate decrease and $lim_{x\rightarrow 0}f_{\overline{v}}(x)=0$, we have that each $f_{\overline{v}}\in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty}f_{\overline{v}}(x)=0$. Similarly, each $\frac{df_{\overline{v}}}{dx}\in L^1(\mathcal{R}_{>0})$ and each $\frac{df_{\overline{v}}}{dx}$ is continuous. We have, using integration by parts, that;

$\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)=\int_0^\infty\frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy$

$=[f_{\overline{v}}(y)e^{-iky}]_0^\infty+ik\int_0^\infty f_{\overline{v}}(y)e^{-iky}dy$

$= ik\int_0^\infty f_{\overline{v}}(y)e^{-iky}dy$

$= ik\mathcal{G}(f_{\overline{v}})(k)$

so that, for $|k|>1$;

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|}{|k|}$, $(\dagger)$

As $\frac{df_{\overline{v}}}{dx}$ is of moderate decrease, for any $\epsilon>0$, we can find $N_{\epsilon,\overline{v}} \in \mathcal{N}$ such that;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k) - \int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy| < \epsilon$ $(*)$

As $\frac{df_{\overline{v}}}{dx}|_{0,N_{\epsilon,\overline{v}}}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [10], using underflow and quantifying over the nonstandard parameter space again, linked to the parameters $N_{\epsilon,\overline{v}}$, we can find $\{D_\epsilon, E_{\epsilon,\overline{v}}\} \subset \mathcal{R}_{>0}$, such that, for all $|k|>D_\epsilon$, we have that;

$|\int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy| < \frac{E_{\epsilon,\overline{v}}}{|k|}$, $(**)$

Again, as in the proof of Lemma 0.15, $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ can be chosen uniformly in ϵ. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k|>D_\epsilon$;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|$

$\leq |\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k) - \int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy| + |\int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy|$

$< \epsilon + \frac{E_{\epsilon,\overline{v}}}{|k|}$

so that, as $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ were uniform and ϵ was arbitrary, we obtain that;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)| < \frac{E_{\overline{v}}}{|k|}$, for $|k|>D$, independent of $\overline{v}$

and, from $(\dagger)$, for $|k|>D$, that;

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|}{|k|} < \frac{E_{\overline{v}}}{|k|^2}$

where the choice of $E_{\overline{v}}$ is the same as in the proof of Lemma 0.15. For the second claim, the proof up to $(\dagger)$ is the same. After choosing

$N \in \mathcal{N}$, we have that each $\frac{df_{\overline{v}}}{dx}|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta_{\overline{v}}}$ monotone intervals. As in $(**)$, and inspection of the proof in [10], we get;

$|\int_0^N \frac{df_{\overline{v}}}{dx} e^{-iky} dy| < \frac{E_N}{|k|}$

where $E_N \leq \frac{2NC_{\overline{v}}}{\delta_{\overline{v}}}$ and $C_{\overline{v}} = max_{x \in \mathcal{R}_{>0}} |\frac{df_{\overline{v}}}{dx}|$.

Choosing $N > 1$, as $\frac{df_{\overline{v}}}{dx}$ is of moderate decrease, we can assume that $|\frac{df_{\overline{v}}}{dx}| \leq \frac{D_{\overline{v}}}{x^2}$, for $x > N$. Then, using the proof in [10] again, and the definition of oscillatory, we have that, for sufficiently large $|k|$, independent of $\overline{v}$;

$|\int_N^\infty \frac{df_{\overline{v}}}{dx} e^{-iky} dy| < (\frac{2}{|k|} \sum_{y_{i,\overline{v}} > N} \frac{D_{\overline{v}}}{y_{i,\overline{v}}^2})$

$\leq (\frac{2}{|k|} \sum_{n \in \mathcal{Z}_{\geq 0}} \frac{D_{\overline{v}}}{(y_{i_0,\overline{v}} + n\delta_{\overline{v}})^2})$

$\leq \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|} \int_{y_{i_0,\overline{v}}}^\infty \frac{dx}{x^2}$

$= \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k| y_{i_0,\overline{v}}}$

$\leq \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

where $y_{i_0,\overline{v}} \geq N$ and $y_{i_0,\overline{v}} \leq y_{i,\overline{v}}$, for all $y_{i,\overline{v}} \geq N$. It follows that;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)| = |\int_0^N \frac{df}{dx} e^{-iky} dy + \int_N^\infty \frac{df_{\overline{v}}}{dx} e^{-iky} dy|$

$\leq |\int_0^N \frac{df_{\overline{v}}}{dx} e^{-iky} dy| + |\int_N^\infty \frac{df_{\overline{v}}}{dx} e^{-iky} dy|$

$\leq \frac{E_N}{|k|} + \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

$\leq \frac{2}{|k|}(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{\delta_{\overline{v}}N})$

It follows, using (†), that;

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|}{|k|} < \frac{E_{\overline{v}}}{|k|^2}$

where $E_{\overline{v}} = 2(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{\delta_{\overline{v}}N})$

As in Lemma 0.15, we can choose $E_{\overline{v}}$ as in the final claim of the two parts.

For the penultimate claim, the modification for the first part is the same. In the second part, choose $N \geq n_{\overline{v}}$, rather than $N > 1$ in the proof, then, taking $N = n_{\overline{v}}$, we obtain $E_{\overline{v}} = 2(\frac{n_{\overline{v}}C_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}})$

For the final claims, we can use integration by parts, uniformly in $\overline{v}$, together with the fact that $f_{\overline{v}}$ is of moderate decrease and $lim_{x\rightarrow 0} f_{\overline{v}} = 0$, to show that, for $|k| > 1$;

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})|}{|k|}$ (HH)

Then use the proof in Lemma 0.15 to find L_ϵ with;

$|\int_{L_\epsilon}^{\infty} \frac{df}{dx}_{\overline{v}}(y)e^{-iky}dy|$

$\leq \epsilon$ (GG)

uniformly in $\overline{v}$. As the family W' is analytic, and V is closed and bounded, by the usual compactness argument, there exists a uniform bound in the number of zeros of $f''_{\overline{v}}$ restricted to $[0, L_\epsilon]$, and a uniform bound for $||\frac{df}{dx}||_\infty$ on $[0, L_\epsilon$. By the same argument as above, we have that;

$|\int_0^{L_\epsilon} \frac{df}{dx}_{\overline{v}}(y)e^{-iky}dy| \leq \frac{E_\epsilon}{|k|}$

where $E_\epsilon = 2val(f'_{\overline{v}}|_{[0,L_\epsilon]})||f'_{\overline{v}}||_\infty$

Using (GG), we have that, for sufficiently large k;

$|\int_0^{\infty} \frac{df}{dx}_{\overline{v}}(y)e^{-iky}dy| \leq \frac{E_\epsilon}{|k|} + \epsilon$

so that, using (HH);

$|G(f_{\overline{v}})| \leq \frac{E_\epsilon}{|k|^2} + \frac{\epsilon}{|k|}$

as required.

If we replace analytic by continuous, then $(GG), (HH)$ are the same, with $\frac{\epsilon}{2}$ replacing ϵ. We use the Stone-Weierstrass approximation theorem to find a family of polynomials $p(y, \overline{v})$ with;

$|p(y,\overline{v})-\frac{df_{\overline{v}}}{dx}(y)|\leq\delta$ on $[0,L_\epsilon]$

and choose $\delta\leq\frac{\epsilon}{2L_\epsilon}$, so that;

$|\int_0^{L_\epsilon}\frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy|\leq\frac{\epsilon}{2}+|\int_0^{L_\epsilon}p_{\overline{v}}(y)e^{-iky}dy|+\frac{\epsilon}{2}$ (SS)

By the usual argument, for sufficiently large k;

$|\int_0^{L_\epsilon}p_{\overline{v}}(y)e^{-iky}dy|\leq\frac{E_\epsilon}{|k|}$ (MM)

where $E_\epsilon=2val_{p_{\overline{v}}}(L_\epsilon)||p_{\overline{v}}||_\infty$.

so that, using $(SS),(MM)$;

$|\int_0^{L_\epsilon}\frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy|\leq\frac{E_\epsilon}{|k|}+\frac{\epsilon}{2}$

and, using (GG)

$|\int_0^{\infty}\frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy|\leq\frac{E_\epsilon}{|k|}+\frac{\epsilon}{2}+\frac{\epsilon}{2}$

$=\frac{E_\epsilon}{|k|}+\epsilon$

Applying (HH), we then have that, for sufficiently large k, uniformly in $\overline{v}$;

$|\mathcal{G}(f_{\overline{v}})|=|\int_0^{\infty}f_{\overline{v}}(y)e^{-iky}dy|$

$\leq\frac{E_\epsilon}{|k|^2}+\frac{\epsilon}{|k|}$

as required.

□

Definition 0.17. *We call a power series* $\sum_{n=0}^{\infty}a_nx^n$ *strongly analytic if, for sufficiently large* n*;*

$|a_n|\leq\frac{C}{n!(n+1)!}$

for some $C\in\mathcal{R}_{>0}$

Lemma 0.18. *There is a one to one correspondence between functions* $f:S^1\rightarrow S^1$ *which are analytic on the circle and functions* $f:\mathcal{R}\rightarrow\mathcal{R}$

which are analytic and analytic at infinity. There is a one to one correspondence between functions $f : S^3 \rightarrow S^3$ which are analytic on the three sphere and functions $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ which are analytic and analytic at infinity. If $f : \mathcal{R} \rightarrow \mathcal{R}$ is analytic and analytic at infinity, then so are the derivatives $f^{(n)}$, for $n \geq 1$. Moreover, f is eventually monotone and non-zero, and eventually all the derivatives $\frac{f'}{f}^{(n)}$, $n \geq 0$ have finitely many zeroes and are monotone. If $w : \mathcal{R} \rightarrow \mathcal{R}$ is strongly analytic and has the property that $w(x)e^{x^2} \in L^1(\mathcal{R})$, then $\mathcal{F}(w)$ is analytic and analytic at infinity, where $\mathcal{F}$ is the Fourier transform.

Proof. For the first claim, if $g : S^1 \rightarrow S^1$ is analytic, then define $\Phi(g) : \mathcal{R} \rightarrow \mathcal{R}$ by;

$$\Phi(g)(x) = g(2tan^{-1}(\tfrac{1}{x}))$$

with the principal branch in the range $(0, \frac{\pi}{2})$ for $x > 0$ and the branch in the range $(\frac{\pi}{2}, \pi)$, for $x < 0$ and $tan^{-1}(\infty) = \frac{\pi}{2}$. We have that $\Phi(g)(\frac{1}{x}) = g(2tan^{-1}(x))$ and, for $0 < x < 1$;

$$tan^{-1}(x) = \int_0^x \frac{dy}{1+y^2}$$

$$= \int_0^x (\sum_{n=0}^{\infty}(-1)^n y^{2n})dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

and, for $-1 < x < 0$;

$$tan^{-1}(x) = \pi + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

so that;

$$g(2tan^{-1}(x)) = g(2\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}), \text{ for } 0 < x < 1$$

$$g(2tan^{-1}(x)) = g(2\pi + 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1})$$

$$= g(2\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}), \text{ for } -1 < x < 0$$

$$g(tan^{-1}(0)) = g(0)$$

and so $\Phi(g)(\frac{1}{x})$ is analytic for $|x| < 1$, and $\Phi(g)$ is analytic at infinity. We have that, for $|x| < 1$;

$tan^{-1}(\frac{1}{x}) = cot^{-1}(x) = \frac{\pi}{2} - tan^{-1}(x)$

so that, $g(2tan^{-1}(\frac{1}{x}))$ is analytic for $|x| < 1$. Similar proofs can be shown for $\frac{1}{2} < |x| < \infty$, by considering $tan^{-1}(x)$ for $0 < |x| < 2$, so that $\Phi(g)(x)$ is analytic. Conversely, given $h : \mathcal{R} \rightarrow \mathcal{R}$ analytic and analytic at infinity, we can define $\Phi^{-1}(h)(\theta) = h(\frac{1}{tan(\frac{\theta}{2})})$, $0 \leq \theta \leq 2\pi$, and it is similarly checked that $\Phi^{-1}(h) : S^1 \rightarrow S^1$ is analytic. For the second claim, we can use the three spherical coordinates;

$x_0 = cos(\alpha)$

$x_1 = sin(\alpha)cos(\beta)$

$x_2 = sin(\alpha)sin(\beta)cos(\gamma)$

$x_3 = sin(\alpha)sin(\beta)sin(\gamma)$

$0 \leq \alpha, \beta \leq \pi$, $0 \leq \gamma \leq 2\pi$

with with the stereographic volume tangent to the three sphere at $(1, 0, 0, 0)$, Then, the point with coordinates (α, β, γ), for $\alpha \neq \frac{\pi}{2}$ corresponds to the point;

$\frac{1}{cos(\alpha)}(sin(\alpha)cos(\beta), sin(\alpha)sin(\beta)cos(\gamma), sin(\alpha)sin(\beta)sin(\gamma))$

$= (tan(\alpha)cos(\beta), tan(\alpha)sin(\beta)cos(\gamma), tan(\alpha)sin(\beta)sin(\gamma)$

and we can define;

$\Phi^{-1}(f)(\alpha, \beta, \gamma) = f(tan(\alpha)cos(\beta), tan(\alpha)sin(\beta)cos(\gamma), tan(\alpha)sin(\beta)sin(\gamma))$

If f is analytic and analytic at infinity, then it is easily checked that $\Phi^{-1}(f)$ has a unique extension to an analytic function on the three sphere, by taking limits on the closed subset $\alpha = \frac{\pi}{2}$. One needs to check that Φ is invertible, the details are left to the reader.

The third claim is well known. If f is analytic at infinity, then $f(\frac{1}{x}) = g(x)$ for g analytic in a neighborhood $B_\epsilon(0)$. We have that, for $x \neq 0$;

$f'(\frac{1}{x}) = lim_{h\rightarrow 0}\frac{f(\frac{1}{x+h})-f(\frac{1}{x})}{\frac{1}{x+h}-\frac{1}{x}}$

$= lim_{h\rightarrow 0}\frac{g(x+h)-g(x)}{\frac{1}{x+h}-\frac{1}{x}}$

$= lim_{h\rightarrow 0}\frac{g(x+h)-g(x)}{-\frac{h}{(x+h)x}}$

$= -g'(x)x^2$

which is analytic in the neighborhood $B_\epsilon(0)$, so that f' is analytic at infinity. It follows that f' has finitely many zeroes on $\mathcal{R}$ and f is eventually monotone and non-zero, as f is analytic. We can therefore define $\frac{f'}{f}$ eventually, $\frac{f'}{f}$ has finitely many zeroes as f' is analytic at infinity. We compute;

$\frac{f'}{f}' = \frac{f''f-f'^2}{f^2}$

which again has finitely many zeroes eventually, as $f''f - f'^2$ is analytic at infinity. It follows that $\frac{f'}{f}$ is eventually monotone. It is easy to see that, for $n \geq 0$, $(\frac{f'}{f})^{(n)} = \frac{p(f,f',\ldots f^{(n+1)})}{f^{2n}}$, where $p(x_0, x_1, \ldots, x_{n+1})$ is a polynomial. Clearly, then, $p(f, f', \ldots f^{(n+1)})$ is analytic at infinity, so has finitely many zeroes eventually and then $\frac{f'}{f}^{(n-1)}$ is eventually monotone, for $n \geq 1$.

For the fifth claim, observe that $w(x)e^{|x|} \leq w(x)e^{x^2}$, for $|x| \geq 1$, so that $\mathcal{F}(w)(y)$ is analytic on $\mathcal{R}$, by the Paley-Wiener theorem, as it has an analytic continuation $G(z)$ to the strip $|Im(z)| < 1$. Define;

$H(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{-ix}{z}}dx \ (z \neq 0)$

$H(0) = 0$

H is well defined as if $z = a + ib$, $z \neq 0$;

$H(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{-ix}{a+ib}}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{-ix(a-ib)}{a^2+b^2}}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{bx}{a^2+b^2}}e^{\frac{-ixa}{a^2+b^2}}dx \ (*)$

We have that $|w(x)e^{\frac{bx}{a^2+b^2}}| \leq |w(x)e^{x^2}|$, for $|x| > \frac{|b|}{a^2+b^2}$, so that $w(x)e^{\frac{bx}{a^2+b^2}} \in L^1(\mathcal{R})$ and the integral $(*)$ exists. We have that;

$$H'(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{-ix}{z}}\frac{ix}{z^2}dx \ (z \neq 0)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{bx}{a^2+b^2}}\frac{ix}{z^2}e^{\frac{-ixa}{a^2+b^2}}dx \ (**)$$

Similarly, $|w(x)e^{\frac{bx}{a^2+b^2}}\frac{ix}{z^2}| \leq |\frac{w(x)e^{x^2}}{z^2}|$, for $|x| \geq max(2, \frac{2|b|}{a^2+b^2})$, so that $w(x)e^{\frac{bx}{a^2+b^2}}\frac{ix}{z^2} \in L^1(\mathcal{R})$, and the integral in $(**)$ exists again. It follows that $H(z)$ is analytic except possibly at 0, in particularly it defines a real analytic function on $B_\epsilon(0) \setminus \{0\}$. We have that
$lim_{h\rightarrow 0, h\in\mathcal{R}} H(h) = 0$

by the Riemann-Lebesgue lemma, so that H is continuous on $B_\epsilon(0)$. If w is analytic, we can assume that $w = \sum_{n=0}^{\infty} a_n x^n$, for $x \in \mathcal{R}$, so that, using the DCT;

$$H(h) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x)e^{\frac{-ix}{h}}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sum_{n=0}^{\infty} a_n x^n)e^{\frac{-ix}{h}}dx$$

$$= \frac{1}{\sqrt{2\pi}}lim_{R(h)\rightarrow\infty}\int_{-R(h)}^{R(h)}(\sum_{n=0}^{\infty} a_n x^n)e^{\frac{-ix}{h}}dx$$

$$= \frac{1}{\sqrt{2\pi}}lim_{R(h)\rightarrow\infty}\sum_{n=0}^{\infty} a_n \int_{-R(h)}^{R(h)} x^n)e^{\frac{-ix}{h}}dx$$

$$= \frac{1}{\sqrt{2\pi}}lim_{R(h)\rightarrow\infty}\sum_{n=0}^{\infty} a_n I_{n,R(h)}$$

where, using integration by parts, for $n \geq 1$;

$$I_{n,R(h)} = \int_{-R(h)}^{R(h)} x^n e^{\frac{-ix}{h}}dx$$

$$= [ihx^n e^{\frac{-ix}{h}}]_{-R(h)}^{R(h)} - \int_{-R(h)}^{R(h)} nih\int_{-R(h)}^{R(h)} x^{n-1}e^{\frac{-ix}{h}}dx$$

$$= ih(R(h)^n e^{\frac{-iR(h)}{h}} - (-R(h))^n e^{\frac{iR(h)}{h}}) - nihI_{n-1,R(h)}$$

so that, for $n \geq 2$, n even;

$$I_{n,R(h)} = ihR(h)^n(-2isin(\frac{R(h)}{h})) - nihI_{n-1,R(h)}$$

for $n \geq 1$, n odd;

$I_{n,R(h)} = ihR(h)^n(2cos(\frac{R(h)}{h})) - nihI_{n-1,R(h)}$

$I_{0,R(h)} = \int_{-R(h)}^{R(h)} e^{\frac{-ix}{h}} dx = ih(-2isin(\frac{R(h)}{h})$

Let $R(h) = 2m\pi h$, so that $\frac{R(h)}{h} = 2m\pi$, where $m \in \mathcal{N}$ $sin(\frac{R(h)}{h}) = 0$, $cos(\frac{R(h)}{h}) = 1$. Then, for $n \geq 2$, n even;

$I_{n,m} = -nihI_{n-1,m}$

for $n \geq 1$, n odd;

$I_{n,m} = 2ih(2m\pi h)^n - nihI_{n-1,m}$

$I_{0,m} = 0$

It follows that, for n even, $n \geq 2$;

$I_{n,m} = -nihI_{n-1,m}$

$= -nih[2ih(2m\pi h)^{n-1} - (n-1)ihI_{n-2,m}]$

$= h^{n+1}(2n(2m\pi)^{n-1}) - h^2 n(n-1)I_{n-2,m}$

and, for n odd $n \geq 3$;

$I_{n,m} = 2ih(2m\pi h)^n - nihI_{n-1,m}$

$= 2ih(2m\pi h)^n - nih[-(n-1)ihI_{n-2,m}]$

$= 2ih(2m\pi h)^n - nih[-(n-1)ihI_{n-2,m}]$

$= h^{n+1}(2i(2m\pi)^n) - h^2(n(n-1)I_{n-2,m})$

with;

$I_{1,m} = 2ih(2m\pi h) = h^2(4im\pi)$

$I_{0,m} = 0$

We claim that, for $n \geq 1$, $I_{n,m} = c(n,m)h^{n+1}$, where $c(n,m)$ is a complex polynomial in the variables n, m. By induction on odd n, $n \geq 1$, it is true for $n = 1$, then, for $n \geq 3$;

$I_{n,m} = h^{n+1}(2i(2m\pi)^n) - h^2(n(n-1)I_{n-2,m})$

$= h^{n+1}(2i(2m\pi)^n) - h^2(n(n-1)c(n-2,m)h^{n-1})$

$= [2i(2m\pi)^n - n(n-1)c(n-2,m)]h^{n+1}$

$= c(n,m)h^{n+1}$

where $c(n,m) = 2i(2m\pi)^n - n(n-1)c(n-2,m)$ is a complex polynomial in the variables n, m again.

By induction on even n, $n \geq 2$, we have that;

$I_{2,m} = h^3(8m\pi)$

so true for $n = 2$ and;

$I_{n,m} = h^{n+1}(2n(2m\pi)^{n-1}) - h^2 n(n-1)c(n-2,m)h^{n-1}$

$= [2n(2m\pi)^{n-1} - n(n-1)c(n-2,m)]h^{n+1}$

$= c(n,m)h^{n+1}$

where $c(n,m) = 2n(2m\pi)^{n-1} - n(n-1)c(n-2,m)$ is a complex polynomial in the variables n, m again.

It follows that;

$H(h) = \frac{1}{\sqrt{2\pi}} lim_{m\rightarrow\infty} \sum_{n=0}^{\infty} a_n I_{n,m}$

$= \frac{1}{\sqrt{2\pi}} lim_{m\rightarrow\infty} \sum_{n=1}^{\infty} a_n I_{n,m}$

$= \frac{1}{\sqrt{2\pi}} lim_{m\rightarrow\infty} \sum_{n=1}^{\infty} a_n c(n,m)h^{n+1}$, $(*)$

For n odd, $n \geq 3$;

$c(n,m)=4im\pi n!+2i\sum_{k=0}^{\frac{n-3}{2}}(-1)^k\frac{n!}{(n-2k)!}(2m\pi)^{n-2k}$

and, for n even, $n\geq 4$;

$c(n,m)=-8m\pi n!+2\sum_{k=0}^{\frac{n-4}{2}}(-1)^k\frac{n!}{(n-1-2k)!}(2m\pi)^{n-1-2k}$

so that $|c(n,m)|\leq (2m\pi)^n(n+1)!$

and, with the assumption of strong analyticity;

$|a_nc(n,m)|\leq\frac{(2m\pi)^n}{n!}$

$|\sum_{n=1}^\infty a_nc_{n,m}h^n|\leq\sum_{n=1}^\infty\frac{(2m\pi|h|)^n}{n!}\leq e^{2m\pi|h|}-1$

so that the functions $f_m=\sum_{n=1}^\infty a_nc(n,m)h^{n+1}$ are analytic on $B(0,1)$

The convergence of f_m is pointwise, as for given $h\in B(0,1)$, $h\neq 0$ we have that;

$\mathcal{F}(w)(\frac{1}{h})=\frac{1}{\sqrt{2\pi}}lim_{m\rightarrow\infty}\int_{-2m\pi h}^{2m\pi h}w(x)e^{\frac{-ix}{h}}dx$

and $f_m(0)=0$, for $m\in\mathcal{N}$

We have that f_m are uniformly bounded, as for $h\in B(0,1)$, $h\neq 0$, we have that;

$|f_m(h)|=|\int_{-2m\pi h}^{2m\pi h}w(x)e^{\frac{-ix}{h}}dx|$

$\leq\int_{-2m\pi h}^{2m\pi h}|w(x)|dx$

$\leq ||w||_1$

By Montel's theorem the functions f_m are uniformly convergent, so the limit is analytic on $B(0,1)$ and w is analytic at infinity.

□

Lemma 0.19. *If $f:\mathcal{R}^3\rightarrow\mathcal{R}$ is analytic and analytic at infinity then so are the partial derivatives $\frac{\partial^{i+j+k}}{\partial x^i\partial y^j\partial z^k}$, $(i,j,k)\in\mathcal{Z}^3_{\geq 0}$. If $f:\mathcal{R}^3\rightarrow\mathcal{R}$ is analytic and analytic at infinity, then for $0\leq\theta\leq\pi$, $0\leq\phi<2\pi$,*

$R>0$, the polar representation $f(R,\theta,\phi)$ of $f(\overline{k})$ is analytic and analytic at infinity, uniformly in (θ,ϕ). Moreover $val(f_{\theta,\phi})$ is uniformly bounded. If $\bar{l}\in\mathcal{R}^3$, with $f(R,\theta,\phi,\bar{l})=f(\overline{k}-\bar{l})$, then again, $f(R,\theta,\phi,\bar{l})$ is analytic and analytic at infinity, uniformly in $(\theta,\phi,\bar{l})$, with $val(f_{\theta,\phi,\bar{l}})$ uniformly bounded. For $\theta\neq cos^{-1}(\frac{l_3}{l_1})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$, the polar representation of $g(\overline{k},\bar{l})=\frac{f(\overline{k}-\bar{l})}{|\overline{k}-\bar{l}|}$ is analytic at infinity, uniformly in $(\theta,\phi,\bar{l})$, with $val(g_{\theta,\phi,\bar{l}})$ uniformly bounded. The polar representation of the components of $u(\overline{k})=\frac{\overline{k}}{|\overline{k}|}$ are analytic, analytic at infinity, uniformly in (θ,ϕ), with $val(u_{\theta,\phi})$ uniformly bounded. If $\overline{h}(\bar{l})$ is analytic and analytic at infinity, and the components of $\overline{f}(\overline{k})$ are analytic, analytic at infinity, uniformly in (θ,ϕ), with $val(u_{\theta,\phi})$ uniformly bounded, then, the polar representation of;

$$[\frac{\overline{f}(\overline{k}-\bar{l})}{|\overline{k}-\bar{l}|}\times\frac{\overline{h}(\bar{l})}{|\bar{l}|}]\cdot u(\overline{k})$$

is analytic and analytic at infinity. Moreover $val(f_{\theta,\phi,\bar{l}})$ is uniformly bounded. In particularly, the family is non-oscillatory and excellent.

Proof. For the first claim, it is sufficient to prove that $\frac{\partial f}{\partial x}$ is analytic and analytic at infinity. The claim that $\frac{\partial f}{\partial x}$ is analytic is clear. Without loss of generality, suppose that;

$$f(\frac{\epsilon_1}{x},\frac{\epsilon_2}{y},\frac{\epsilon_3}{z})=g(x,y,z)$$

with g analytic, and $\epsilon_1\neq 0$, $\epsilon_2\neq 0$, $\epsilon_3\neq 0$, the general case can be proved by rotating coordinates. Then;

$$-\frac{\partial f}{\partial x}\frac{\epsilon_1}{x^2}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=\frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\frac{\epsilon_2}{y^2}+\frac{\partial f}{\partial z}=\frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}-\frac{\partial f}{\partial z}\frac{\epsilon_3}{z^2}=\frac{\partial g}{\partial z}$$

and;

$$\overline{M}(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})=(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z})$$

with $(\overline{M})_{11}=-\frac{\epsilon_1}{x^2}$, $(\overline{M})_{22}=-\frac{\epsilon_2}{y^2}$, $(\overline{M})_{33}=-\frac{\epsilon_3}{z^2}$

$(\overline{M})_{ij}=1$, $1\leq i<j\leq 3$.

so that;

$(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})=\overline{M}^{-1}(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z})$

with $\{\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z}\}$ analytic in a neighborhood of $(0,0,0)$. We have that;

$\overline{M}^{-1}=\frac{1}{det(M)}adj(\overline{M})^t$

$=\frac{x^2y^2z^2}{-\epsilon_1\epsilon_2\epsilon_3+2x^2y^2z^2+\epsilon_1y^2z^2+\epsilon_2x^2z^2+\epsilon_3x^2y^2}\overline{N}$

where;

$\overline{N}_{11}=\frac{\epsilon_2\epsilon_3}{y^2z^2}$

$\overline{N}_{22}=\frac{\epsilon_1\epsilon_3}{x^2z^2}$

$\overline{N}_{33}=\frac{\epsilon_1\epsilon_2}{x^2y^2}$

$\overline{N}_{12}=\overline{N}_{21}=\frac{\epsilon_3}{z^2}+1$

$\overline{N}_{13}=\overline{N}_{31}=\frac{\epsilon_2}{y^2}+1$

$\overline{N}_{23}=\overline{N}_{32}=\frac{\epsilon_1}{x^2}+1$

Clearing denominators, and using Newton's expansion, with $\epsilon_1\epsilon_2\epsilon_3\neq 0$, it is clear that the components of $\overline{M}^{-1}$ are locally analytic at $(0,0,0)$, so that $\{\frac{\partial f}{\partial x}|_{(\frac{\epsilon_1}{x},\frac{\epsilon_2}{y},\frac{\epsilon_3}{z})},\frac{\partial f}{\partial y}|_{(\frac{\epsilon_1}{x},\frac{\epsilon_2}{y},\frac{\epsilon_3}{z})},\frac{\partial f}{\partial z}|_{(\frac{\epsilon_1}{x},\frac{\epsilon_2}{y},\frac{\epsilon_3}{z})}\}$ are locally analytic at $(0,0,0)$ In particularly, $\frac{\partial f}{\partial x}$ is analytic at infinity.

For the second claim, if f is analytic, with power series expansion at $\overline{0}$ given by $\sum_{i,j,k\geq 0}a_{ijk}x^iy^jz^k$, then making the substitutions $x=Rsin(\theta)cos(\phi)$, $y=Rsin(\theta)sin(\phi)$, $z=Rcos(\theta)$, we obtain;

$\sum_{i,j,k\geq 0}a_{ijk}(Rsin(\theta)cos(\phi))^i(Rsin(\theta)sin(\phi))^j(Rcos(\theta))^k$

$=\sum_{i,j,k\geq 0}a_{ijk}sin^{i+j}(\theta)sin^j(\phi)cos^j(\phi)cos^k(\theta)R^{i+j+k}$

$= \sum_{l\geq 0} b_{l,\theta,\phi}R^l$

where, for $l \geq 0$, $b_{l,\theta,\phi} = \sum_{i+j+k=l} a_{ijk}sin^{i+j}(\theta)sin^j(\phi)cos^j(\phi)cos^k(\theta)$, so that $f(R,\theta,\phi)$ is analytic, uniformly in $\{\theta,\phi\}$ around $\overline{0}$. If (R_0,θ_0,ϕ_0) is fixed, $x_0 = R_0sin(\theta_0)cos(\phi_0)$, $y_0 = R_0sin(\theta_0)sin(\phi_0)$, $z_0 = R_0cos(\theta_0)$, with local power series expansion;

$\sum_{i,j,k\geq 0} a_{ijk}(x-x_0)^i(y-y_0)^j(z-z_0)^k$

making the substitutions again, we obtain;

$\sum_{i,j,k\geq 0} a_{ijk}(Rsin(\theta_0)cos(\phi_0) - R_0sin(\theta_0)cos(\phi_0))^i(Rsin(\theta_0)sin(\phi_0)$

$-R_0sin(\theta_0)sin(\phi_0))^j(Rcos(\theta_0) - R_0cos(\theta_0))^k$

$= \sum_{i,j,k\geq 0} a_{ijk}sin^{i+j}(\theta_0)sin^j(\phi_0)cos^k(\phi_0)cos^k(\theta_0)(R-R_0)^{i+j+k}$

$= \sum_{l\geq 0} b_{l,\theta_0,\phi_0}(R-R_0)^l$

where $b_l = \sum_{i+j+k=l} a_{ijk}sin^{i+j}(\theta_0)sin^j(\phi_0)cos^j(\phi_0)cos^k(\theta_0)$

so that f_{θ_0,ϕ_0} is locally analytic around R_0. Similarly, if f is analytic at ∞ with $(x_0,y_0,z_0) \in P^2(\mathcal{R})$ and power series expansion;

$\sum_{i,j,k\geq 0} a_{ijk}x^iy^jz^k$

for $f(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})$, then, letting $x_0 = sin(\theta_0)cos(\phi_0)$, $y_0 = sin(\theta_0)sin(\phi_0)$, $z_0 = cos(\theta_0)$, we obtain an expansion for;

$f(\frac{sin(\theta_0)cos(\phi_0)}{x},\frac{sin(\theta_0)sin(\phi_0)}{y},\frac{cos(\theta_0)}{z})$

so that;

$f_{\theta_0,\phi_0}(\frac{1}{R}) = f(\frac{sin(\theta_0)cos(\phi_0)}{R},\frac{sin(\theta_0)sin(\phi_0)}{R},\frac{cos(\theta_0)}{R})$

$= \sum_{i,j,k\geq 0} a_{ijk}R^{i+j+k}$

$= \sum_{l\geq 0} b_lR^l$

where $b_l = \sum_{i+j+k=l} a_{ijk}sin^{i+j}(\theta_0)sin^j(\phi_0)cos^j(\phi_0)cos^k(\theta_0)$

For the uniformity claim, let $Z \subset P^3(\mathcal{R})$ be the zero locus of $f(x,y,z) = 0$ and define a relation $R \subset P^3(\mathcal{R}) \times P^2(\mathcal{R})$ by;

$R(x,l)$ iff $x \in l \cap Z$

where l is a line passing through the origin of $\mathcal{R}^3$. Considering the projection $pr : P^3(\mathcal{R}) \times P^2(\mathcal{R}) \rightarrow P^2(\mathcal{R})$, restricted to R. By the previous result, that $f_{\theta,\phi}$ is analytic and analytic at infinity, and Lemma 0.46, we have that $pr|_R$ is a finite cover. Moreover R is closed and locally analytic in $P^3(\mathcal{R}) \times P^2(\mathcal{R})$ of dimension 2. In particularly, R is compact. Let $W \subset \mathcal{R}^3 \times P^2(\mathcal{R})$ be the variety defined by $W(y,l)$ iff $y \in l$, so W has dimension 3, and let $\overline{W}$ be its closure in $P^3(\mathcal{R}) \times P^2(\mathcal{R})$. Then pr factors through $\overline{W}$. By real Weierstrass preparation, see [1], we can present the local power series $S(X,Y,Z) = \sum_{i,j,k \geq 0} a_{ijk} X^i Y^j Z^k$ defining $R \subset \overline{W}$ at $(0,0,0)$ in the form $S(X,Y,Z) = G(X,Y,Z)H(X,Y,Z)$, with $G(0,0,0) \neq 0$ and $H(X,Y,Z) = Z^d + c_1(X,Y)Z^{d-1} + \ldots c_d(X,Y)$, where $c_i(X,Y) \in \mathcal{R}[[X,Y]]$, for $1 \leq i \leq d$, $c_i(0,0) = 0$, and $d = ord(S(0,0,Z))$. Then on the set $G \neq 0$, we have the cover has degree at most d, and, by compactness, we can find a finite sequence of open sets $U_1, \ldots U_r$, for which $pr|U_i$ has degree d_i, so that the total degree of the cover is at most $r max_{1 \leq i \leq r} d_i$, (Q).

By the proof of the above, we have that f is analytic at infinity, in the coordinates (R,θ,ϕ), in the sense that;

$$f(\tfrac{1}{R},\theta,\phi) = g(R,\theta,\phi)$$

where g is analytic in (R,θ,ϕ) in a neighborhood of $(0,\theta_0,\phi_0$ for fixed (θ_0,ϕ_0). Differentiating, we obtain that;

$$-\frac{1}{R^2}\frac{\partial f}{\partial R} = \frac{\partial g}{\partial R}$$

so that;

$$\frac{\partial f}{\partial R}|_{(\frac{1}{R},\theta,\phi)} = -R^2\frac{\partial g}{\partial R}$$

where $\frac{\partial g}{\partial R}$ and $-R^2\frac{\partial g}{\partial R}$ are analytic in (R,θ,ϕ) in a neighborhood of $(0,\theta_0,\phi_0$ for fixed (θ_0,ϕ_0), so that $\frac{\partial f}{\partial R}$ is analytic at infinity, in the coordinates (R,θ,ϕ). The proof above also shows that f is analytic, in the coordinates (R,θ,ϕ), and so is $\frac{\partial f}{\partial R}$. Let;

$Cone_{\theta_0} = \{(x,y,z) : x = rsin(\theta)cos(\phi), y = rsin(\theta)sin(\phi), z = rcos(\theta), r \in \mathcal{R}, 0 \leq \theta \leq \theta_0, 0 \leq \phi \leq 2\pi\}$

so that $\frac{\partial f}{\partial R}$ is analytic on the cone and at infinity, or in the restricted coordinates. Let Z_{θ_0} be the zero set of $\frac{\partial f}{\partial R}$ on the compactification of the coordinates $P^1(\mathcal{R}) \times S^1(\mathcal{R}) \times [0,\theta_0]$. Then we can follow through the argument of (Q), to obtain a bound on the cardinality of zeros of $\frac{\partial f}{\partial R}$ for $0 \leq \theta \leq \theta_0$, $0 \leq \phi \leq 2\pi$. We can carry out a similar argument for $Cone_{\theta_0,\pi}$, letting $\theta_0 \leq \theta \leq \pi$. In particularly $val(f)$ is bounded as $\{\theta,\phi\}$ varies. For the next claim, we have that;

$f(R,\theta,\phi,\bar{l}) = f(Rsin(\theta)cos(\phi)-l_1, Rsin(\theta)sin(\phi)-l_2, Rcos(\theta)-l_3)$

so at the coordinate (R_0,θ_0,ϕ_0);

$f(R,\theta_0,\phi_0,\bar{l}) = f(Rsin(\theta_0)cos(\phi_0)-R_0sin(\theta_0)cos(\phi_0)+R_0sin(\theta_0)cos(\phi_0)-l_1,$

$Rsin(\theta_0)sin(\phi_0)-R_0sin(\theta_0)cos(\phi_0)+R_0sin(\theta_0)cos(\phi_0)-l_2, Rcos(\theta_0)$

$-R_0cos(\theta_0)+R_0cos(\theta_0-l_3)$

$= \sum_{i,j,k\geq 0} a_{ijk,\bar{l}}(Rsin(\theta_0)cos(\phi_0)-R_0sin(\theta_0)cos(\phi_0))^i(Rsin(\theta_0)sin(\phi_0)-$

$R_0sin(\theta_0)cos(\phi_0))^j(Rcos(\theta_0)-R_0cos(\theta_0))^k$

where $\sum_{i,j,k\geq 0} a_{ijk,\bar{l}}(x-m_{1,0}+l_1)^i(y-m_{2,0}+l_2)^j(z-m_{3,0}+l_3)^k$

is the analytic expansion for f around $\overline{m}_0-\bar{l}$, with;

$\overline{m}_0 = (R_0sin(\theta_0)cos(\phi_0), R_0sin(\theta_0)cos(\phi_0), R_0cos(\theta_0))$

For the next claim, it is easily shown that f is analytic and analytic at infinity precisely when f extends to a real analytic map $\overline{f} : P^3(\mathcal{R}) \rightarrow \mathcal{R}$. The components of the translation map $T_{\bar{l}}; \mathcal{R}^3 \rightarrow \mathcal{R}^3$ defined by $T(\overline{k}) = \overline{k}-\bar{l}$, for $\bar{l} \in \mathcal{R}^3$ are analytic. $T_{\bar{l}}$ extends to an analytic map of $P(\mathcal{R}^3)$, given in coordinates (x,y,z,w) by;

$T_{\bar{l}}([x:y:z:w]) = ([x-wl_1, y-wl_2, z-wl_3, w])$

On the chart $x \neq 0$, the map $T_{\bar{l}}$ is given by;

$(y, z, w) \mapsto [1 : y : z : w] \mapsto [1 - wl_1, y - wl_2, z - wl_3, w]$

$\mapsto (\frac{y-wl_2}{1-wl_1}, \frac{z-wl_3}{1-wl_1}, \frac{w}{1-wl_1})$

which, without loss of generality, assuming $l_1 \neq 0, l_2 \neq 0, l_3 \neq 0$, is analytic for $|w| < min(\frac{1}{|l_1}, \frac{1}{|l_2|}, \frac{1}{|l_3|})$, by Newton's theorem, in particular at $w = 0$. The same is true for the charts $y \neq 0$ and $z \neq 0$. It follows the components of $T_{\bar{l}}$ extend to analytic maps of $P^3(\overline{R})$. By composition, we then know that $f \circ T_{\bar{l}}$ is real analytic on $P^3(\mathcal{R})$, so that $f_{\bar{l}}$ is analytic and analytic at infinity. Repeating the argument above, it follows that $f_{\theta,\phi,\bar{l}}$ is analytic and analytic at infinity, uniformly in $\{\theta, \phi, \bar{l}\}$. For the uniformity claim, we replace $P^2(\mathcal{R})$ in the argument above by the Grassmanian of lines $Gr(P^3(\mathcal{R}))$, let $Z \subset P^3(\mathcal{R})$ be the zero locus of $f(x, y, z) = 0$ again and define the relation $R \subset P^3(\mathcal{R}) \times Gr(P^3(\mathcal{R}))$ by;

$R(x, l)$ iff $x \in l \cap Z$

where l is a line in $P^3(\mathcal{R})$. Consider the projection $pr : P^3(\mathcal{R}) \times Gr(P^3(\mathcal{R})) \rightarrow Gr(P^3(\mathcal{R}))$, restricted to R. By the previous result, that $f_{\theta,\phi,\bar{l}}$ is analytic and analytic at infinity, and Lemma 0.46 again, we have that $pr|_R$ is a finite cover. Moreover R is closed and locally analytic in $P^3(\mathcal{R}) \times Gr(P^3(\mathcal{R}))$ of dimension 4. In particularly, R is compact. Let $W \subset P(\mathcal{R}^3) \times Gr(P^3(\mathcal{R}))$ be the variety defined by $W(y, l)$ iff $y \in l$, so W has dimension 5, and let $\overline{W}$ be its closure in $P(\mathcal{R}^3) \times Gr(P^3(\mathcal{R}))$. Then pr factors through $\overline{W}$. By real Weierstrass preparation, see [1], we can present the local power series $S(X, Y, Z, U, V) = \sum_{i,j,k,l,m \geq 0} a_{ijklm} X^i Y^j Z^k U^l V^m$ defining $R \subset \overline{W}$ at $(0, 0, 0, 0, 0)$ in the form $S(X, Y, Z, U, V) = G(X, Y, Z, U, V)H(X, Y, Z, U, V)$, with
$G(0, 0, 0, 0, 0) \neq 0$ and $H(X, Y, Z, U, V) = V^d + c_1(X, Y, Z, U)V^{d-1} + \ldots c_d(X, Y, Z, U)$, where $c_d(X, Y, U, V) \in \mathcal{R}[[X, Y, U, V]]$, for $1 \leq i \leq d$, $c_i(0, 0, 0, 0) = 0$, and $d = ord(S(0, 0, 0, 0, V))$. Then on the set $G \neq 0$, we have the cover has degree at most d, and, by compactness, we can find a finite sequence of open sets $U_1, \ldots U_r$, for which $pr|U_i$ has degree d_i, so that the total degree of the cover is at most $rmax_{1 \leq i \leq r} d_i$, (QQ). For the next claim, we can again show that $\frac{\partial f}{\partial R}$ is analytic on $Cone_{\theta_0} \times \mathcal{R}^3$ in the variables $(r, \theta, \phi, \bar{l})$. We can embed $Cone_{\theta_0} \times \mathcal{R}^3$ into $P^1(\mathcal{R}) \times [0, \theta_0] \times S^1 \times P(\mathcal{R}^3)$ into $P^3(\mathcal{R}) \times P^3(\mathcal{R})$ via the maps;

$(r,\theta,\phi,\bar{l})\mapsto([r:1],\theta,\phi,[l_1:l_2:l_3:1])$

$([u,v],\theta,\phi,[l_1:l_2:l_3:l_4])\mapsto([ul_4sin(\theta)cos(\phi)-l_1v:ul_4sin(\theta)sin(\phi)$

$-l_2v:ul_4cos(\theta)-l_3v;l_4v],[l_1:l_2:l_3:l_4])$ (E)

We have that;

$\frac{\partial f}{\partial R}=\frac{\partial f}{\partial x}\frac{x}{R}+\frac{\partial f}{\partial y}\frac{y}{R}+\frac{\partial f}{\partial z}\frac{z}{R}$

By the first claim, $\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\}$ are analytic at infinity in $P^3(\mathcal{R})$. We have that $\frac{sign(xyz)x}{R}$ extends to a map on $P^3(\mathcal{R})$ by;

$[x:y:z:w]\mapsto\frac{sign(xyz)x}{R}$

as, for $t\in\mathcal{R}_{\neq 0}$;

$\frac{sign((tx)(ty)(tz))tx}{((tx)^2+(ty)^2+(tz)^2)^{\frac{1}{2}}}=\frac{sign(t)sign(xyz)tx}{|t|R}$
$=\frac{sign(xyz)x}{R}$

On the chart $x\neq 0$;

$[1:y,z,w]\mapsto\frac{sign(yz)}{(1+y^2+z^2)^{\frac{1}{2}}}$

we have that, using Newton's theorem, the map is analytic on the connected components of the region $y\neq 0$, $z\neq 0$. By considering the map;

$[x:y:z:w]\mapsto\frac{sign((x-y)(y-z)(z-x))x}{R}$

we can similarly prove local analyticity on the connected components of the region $x\neq 0$, $y\neq 1$, $y\neq z$, $x\neq z$, until we obtain analyticity everywhere except $x=y=z=0$. The same considerations apply to;

$\frac{sign(xyz)y}{R}$, $\frac{sign(xyz)z}{R}$, $\frac{sign((x-y)(y-z)(z-x))y}{R}$ and $\frac{sign((x-y)(y-z)(z-x))z}{R}$

so that as;

$sign(xyz)\frac{\partial f}{R}=\frac{\partial f}{\partial x}\frac{sign(xyz)x}{R}+\frac{\partial f}{\partial y}\frac{sign(xyz)y}{R}+\frac{\partial f}{\partial z}\frac{sign(xyz)z}{R}$

we can obtain generically analytic extensions of;

$\{sign(xyz)\frac{\partial f}{\partial R}, sign((x-y)(y-z)(z-x))\frac{\partial f}{\partial R}\}$

to $P^3(\mathcal{R})$. We can define a map $\Theta_1 : P^3(\mathcal{R}) \times P^3(\mathcal{R}) \rightarrow \mathcal{R}$ by;

$\Theta_1([x:y:z:w],[l_1:l_2:l_3:l_4]) = sign(xyz)\frac{\partial f}{\partial R}([x:y:z:w])$

and a map $\Theta_2 : P^3(\mathcal{R}) \times P^3(\mathcal{R}) \rightarrow \mathcal{R}$ by;

$\Theta_2([x:y:z:w],[l_1:l_2:l_3:l_4]) = sign((x-y)(y-z)(z-x))$

$\frac{\partial f}{\partial R}([x:y:z:w])$

and compose $\{\Theta_1,\Theta_2\}$ with the map in (E) to obtain analytic projective extensions of $\{sign(xyz)\frac{\partial f}{\partial R}, sign((x-y)(y-z)(z-x))\frac{\partial f}{\partial R}\}$ on the cone $Cone_{\theta_0}$ in the variables $\{\theta,\phi,\bar{l}\}$. Fibring the extension over $P^3(\mathcal{R})$, and using an argument similar to Q, applying Weierstrass preparation to the generically analytic sets $sign(xyz)\frac{\partial f}{\partial R} = 0$ and $sign((x-y)(y-z)(z-x))\frac{\partial f}{\partial R} = 0$, we can obtain generic uniformity in the cardinality of the zeros of $sign(xyz)\frac{\partial f}{\partial R}(\theta,\phi,\bar{l})$ and $sign((x-y)(y-z)(z-x))\frac{\partial f}{\partial R}$. As $\frac{\partial f}{\partial R}$ differs from $\frac{sign(xyz)\partial f}{\partial R}$ and $\frac{sign((x-y)(y-z)(z-x))\partial f}{\partial R}$, possibly by a minus sign, on the connected regions, by the covering property, we obtain uniformity in the cardinality of zeros of $\frac{\partial f}{\partial R}$. We then obtain uniformity in $val(f_{\theta,\phi,\bar{l}})$. We have that, with the restrictions on $(\theta,\phi,\bar{l})$, that;

$\frac{f(\bar{k}-\bar{l})}{|\bar{k}-\bar{l}|} = 0$ iff $f(\bar{k}-\bar{l}) = 0$

so that $g(\bar{k},\bar{l})$ has a uniformly finite number of zeros in the polar representation $(R,\theta,\phi,\bar{l})$. We have that;

$\frac{\partial}{\partial R}(\frac{1}{|\bar{k}-\bar{l}|}) = -\frac{1}{2|\bar{k}-\bar{l}|^3}2(\bar{k}-\bar{l}) \centerdot \hat{\bar{k}} = -\frac{(\bar{k}-\bar{l})\centerdot\bar{k}}{|\bar{k}-\bar{l}|^3|\bar{k}|}$

so that;

$\frac{\partial g}{\partial R} = \frac{\frac{\partial f}{\partial R}}{|\bar{k}-\bar{l}|} - f\frac{(\bar{k}-\bar{l})\centerdot\bar{k}}{|\bar{k}-\bar{l}|^3|\bar{k}|}$ (U)

With the restriction on the parameters, this has zeros when;

$\frac{\partial f}{\partial R}|\overline{k}-\overline{l}|^2|\overline{k}| = f(\overline{k}-\overline{l})\centerdot\overline{k}$ $(U)'$

We have that $\frac{sign(xyz)}{|\overline{k}|}$ extends to $P^3(\mathcal{R})\setminus(0,0,0)$ by;

$[x:y:z:w]\mapsto\frac{sign(xyz)w}{R}$

as;

$[tx:ty:tz:tw]\mapsto\frac{sign((tx)(ty)(tz))tw}{|t|R}$

$=\frac{sign(t)tsign(xyz)w}{|t|R}$

$=\frac{sign(xyz)w}{R}$

The shift $T_{\overline{l}}(\overline{k})=\overline{k}-\overline{l}$, extends to $P^3(\mathcal{R})$ by;

$T_{\overline{l}}([x:y:z:w])=[x-wl_1,y-wl_2,z-wl_3,w]$

so that $\frac{sign((x-l_1)(y-l_2)(z-l_3))}{|\overline{k}-\overline{l}|}$ extends to $P^3(\mathcal{R})$. We have that $sign(xyz)\frac{\overline{k}}{|\overline{k}|}$: $\mathcal{R}^3\rightarrow S^2(\mathcal{R})$ extends to $P^3(\mathcal{R})$ by;

$[x:y:z:w]\mapsto\frac{sign(xyz)(x,y,z)}{|(x,y,z)|}$

as;

$[tx:ty:tz:tw]\mapsto\frac{sign(t)tsign(xyz)(x,y,z)}{|t||(x,y,z)|}$

$=\frac{sign(xyz)(x,y,z)}{|(x,y,z)|}$

Similarly, $sign(xyz)\frac{\overline{k}}{|\overline{k}|^3}:\mathcal{R}^3\rightarrow\mathcal{R}^3$ extends to $P^3(\mathcal{R})$ by;

$[x:y:z:w]\mapsto\frac{sign(xyz)(x,y,z)w^2}{|(x,y,z)|^3}$

as;

$[tx:ty:tz:tw]\mapsto\frac{sign(t)t^3sign(xyz)(x,y,z)w^2}{|t|^3|(x,y,z)|}$

$=\frac{sign(xyz)(x,y,z)w^2}{|(x,y,z)|^3}$

so that, as before, $sign((x-l_1)(y-l_2)(z-l_3))\frac{\overline{k}-\overline{l}}{|\overline{k}-\overline{l}|^3}$ extends. Taking the dot product;

$sign((x-l_1)(y-l_2)(z-l_3)xyz)\frac{\overline{k}-\overline{l}.\overline{k}}{|\overline{k}-\overline{l}|^3|\overline{k}|}$ (V)

extends to $P^3(\mathcal{R})$. It follows that, by $(U),(V)$;

$\frac{\partial g}{\partial R}sign((x-l_1)(y-l_2)(z-l_3))$

agrees with;

$= \frac{\frac{\partial f}{\partial R}sign((x-l_1)(y-l_2)(z-l_3))}{|\overline{k}-\overline{l}|} - \frac{f sign((x-l_1)(y-l_2)(z-l_3)xyz)(\overline{k}-\overline{l}).\overline{k}}{|\overline{k}-\overline{l}|^3|\overline{k}|}$ (X)

on the region $sign(xyz)>0$, and $\frac{\partial g}{\partial R}$ agrees with $\frac{\partial g}{\partial R}sign((x-l_1)(y-l_2)(z-l_3))$, on the region $sign((x-l_1)(y-l_2)(z-l_3))>0$, so that $\frac{\partial g}{\partial R}$ agrees with the term (X) on the region $V_{\overline{l}} = sign(xyz(x-l_1)(y-l_2)(z-l_3)>0$. By using the same trick, utilising $sign((x-y)(y-z)(z-x))$ instead of $sign(xyz)$, we can create a new term $(X)'$ such that $\frac{\partial g}{\partial R}$ agrees with the term $(X)'$ on $W_{\overline{l}} = sign((x-y)(y-z)(z-x)(x-y-l_1+l_2)(y-z-l_2+l_3)(z-x-l_3+l_1))$. Continuing in this way, using linear maps and hyperplane arrangements, it is clear we can create a finite number m of terms, and regions $W_{\overline{l},i}$, for $1\leq i\leq m$, such that, generically in $\overline{l}$, $\frac{\partial g}{\partial R}$ agrees with one of the terms on the region $W_{\overline{l},i}$ and $\mathcal{R}^3\setminus\overline{l} = \cup_{1\leq i\leq m}W_{\overline{l},i}$. Adding regions, if necessary, to cover the non generic locus if necessary, and noting that the fixed locus of linear maps is a point, we can assume that $\mathcal{R}^3\setminus\{\overline{l}\} = \cup_{1\leq i\leq m}W_{\overline{l},i}$, for all $\overline{l}\in\mathcal{R}^3$. Each of the terms, similar to (X) extends projectively, even at $\overline{l}$, and are generically analytic. By the covering property, we can use compactness and the Weierstrass preparation argument, to show that $\frac{\partial g}{\partial R}$ has finitely many zeros, uniformly in $\{\theta,\phi,\overline{l}\}$, in particular, for fixed $\{\theta,\phi,\overline{l}\}$ $g_{\theta,\phi,\overline{l}}$ is non oscillatory, with a uniform bound on the valency, $val(g_{\theta,\phi,\overline{l}})$. Note that $g_{\overline{l}}$ does actually extend to a map $\overline{g_{\overline{l}}}: P^3(\mathcal{R})\rightarrow P^1(\mathcal{R})$, defining it to be the point at infinity, in case $\overline{k}=\overline{l}$, but it is not analytic at $\overline{l}$. However, we can assume that the zero locus of $\frac{\partial g}{\partial R}$ does not include $\overline{l}$ in the extension, unless $f_{\overline{l}}(\overline{k})$ and $\frac{\partial f_{\overline{l}}}{\partial R}$ have high order zeros along $l_{\overline{l}}$ at $\overline{l}$. We can use the formulation $(U)'$ to resolve this case. As $f(\overline{k}-\overline{l})$ is analytic and analytic at infinity, the same argument proves that, with the restriction on fixed $\{\theta,\phi,\overline{l}\}$, $g_{\theta,\phi,\overline{l}}$ is analytic and analytic at infinity. This follows as the line l defined by $\{\theta,\phi\}$ will intersect the hyperplane at infinity, inside a region $W_{\bar{l},i}$ where the term is analytic, and we can use restriction of variables. The components of $u(\overline{k})$ are just $(sin(\theta)sin(\phi), sin(\theta)cos(\phi), cos(\theta))$, which are constant on any line defined by $\{\theta,\phi\}$, so trivially analytic and analytic at infinity, with the valency bounded. For the final claim, letting $\{h_1,h_2,h_3\}$ denote the components of $\frac{\overline{h}(\bar{l})}{|\bar{l}|}$, $\{f_1,f_2,f_3\}$ the components of $\frac{\overline{f}(\overline{k},\bar{l})}{|\overline{k}-\bar{l}|}$, $\{u_1,u_2,u_3\}$ the components of $\overline{u}$ we have that;

$$[\frac{\overline{f}(\overline{k}-\bar{l})}{|\overline{k}-\bar{l}|}\times\frac{\overline{h}}{|\bar{l}|}].u(\overline{k}) = u_1(\overline{k})f_2(\overline{k},\bar{l})h_3(\bar{l}) - u_1(\overline{k})f_3(\overline{k},\bar{l})h_2(\bar{l}) + u_2(\overline{k})f_3(\overline{k},\bar{l})h_1(\bar{l})$$

$$-u_2(\overline{k})f_1(\overline{k},\bar{l})h_3(\bar{l}) + u_3(\overline{k})f_1(\overline{k},\bar{l})h_2(\bar{l}) - u_3(\overline{k})f_2(\overline{k},\bar{l})h_1(\bar{l})$$

and we have proved the claims above for $\{f_1,f_2,f_3,u_1,u_2,u_3\}$. Considering the products f_iu_j, $1\leq i\leq j\leq 3$, it is clear the above argument can be extended to prove the corresponding analytic and analytic at infinity claims, with the uniform bound in valence. We are then left with a sum;

$$h_1(\bar{l})H_1(\overline{k},\bar{l}) + h_2(x)(\bar{l})H_2(\overline{k},\bar{l}) + h_3(\bar{l})H_3(\overline{k},\bar{l})\ (ZZ)$$

in which, for $\bar{l}\neq\overline{0}$, $\{h_1,h_2,h_3\}$ are rational functions without poles, and $\{H_1,H_2,H_3\}$ are uniformly analytic and analytic at infinity, with a valency bound, and the usual restrictions on $\{\theta,\phi,\bar{l}\}$. As the components of $\overline{h}$ are analytic and analytic at infinity, we can extend $\{h_1,h_2,h_3\}$ to $P^3(\mathcal{R})\setminus\{\overline{0}\}$, in the variables $\bar{l}$, so that they are analytic at infinity. Let $W\subset P^3(\mathcal{R})\times P^3(\mathcal{R})$ be the variety defined by;

$$W(\bar{l},\overline{k}) \text{ iff } \bar{l}\neq\overline{0} \text{ and } \overline{k}\in P^3(\mathcal{R})\setminus l_{\overline{0},\bar{l}}$$

where $l_{\overline{0},\bar{l}}$ is the line passing through $\overline{0}$ and $\bar{l}$. We then obtain a map;

$$\Theta : W\rightarrow\mathcal{R}$$

defined by (ZZ), and, as before, we can extend it to $\overline{\Theta} : P^3(\mathcal{R})\times P^3(\mathcal{R})\rightarrow P^1(\mathcal{R})$.

Generic analyticity gives the uniformity in analytic, analytic at infinity and valency claims, over the parameters $\{\theta,\phi,\bar{l}\}$, noting that the proof is the same, taking the derivative $\frac{\partial}{\partial R}$, which is linear in the coefficients h_i, $1\leq i\leq 3$. In particularly, the family, with the restriction on parameters is non-oscillatory and excellent.

□

Definition 0.20. *Given a closed interval* $[a,b]\subset\mathcal{R}$*, with* f *defining a real analytic function on an open neighborhood of* $[a,b]$*, we define the analytic degree of* f *to be the maximum number of intersections between the graph of* f *and a line* $y=cx+d$ *restricted to* (a,b)*.*

Lemma 0.21. *Definition 0.20 is well defined. If* f *has* n *inflexions in* $[a,b]$*, that is points* x_0 *for which* $f''(x_0)=0$*, then* f *has analytic degree at most* $n+2$*. For a non-oscillatory excellent family whose second derivative has valency* n*, restricted to* $[a,b]$*, the analytic degree of any function in the family is at most* $n+2$*.*

Proof. Suppose that f has $m\geq 3$ distinct intersections with a line l, which we order as $a\leq x_1<\ldots<x_m\leq b$. Considering (x_i,x_{i+1},x_{i+2}), with $1\leq i\leq m-2$, by the IVT, we can find points $\{y_{i,1},y_{i,2}\}$ with $x_i<y_i<x_{i+1}$ and $x_{i+1}<y_{i,2}<x_{i+2}$ such that $(f-l)'(y_{i,1})=(f-l)'(y_{i,2})=0$, that is $f'(y_{i,1})=f'(y_{i,2})=c$. Applying the IVT again, we can find z_i with $y_{i,1}<z_i<y_{i,2}$, such that $(f-l)''(z_i)=0$, that is $f''(z_i)=0$. In particularly, as the intervals $(y_{i,1},y_{i,2})$ and $(y_{i+1,1},y_{i+1,2})$ are disjoint, $z_i\neq z_{i+1}$ and we can find $m-2$ inflexions of f in the interval $[a,b]$. If f is analytic, so is f'', so, as $[a,b]$ is closed, f'' has finitely many m_0 zeroes on $[a,b]$, in particularly it can only have at most m_0+2 intersections with a line. It follows the definition 0.20 is well defined. The following claim is then clear.

□

Definition 0.22. *For* $\delta_1,\delta_2>0$*, we define the set* $W_{\delta_1,\delta_2,l_0}\subset[0,\pi]\times[-\pi,-\pi]\times\mathcal{R}^3$ *by;*

$$W_{\delta_1,\delta_2,l_0}(\theta,\phi,\bar{l})\ \textit{iff}\ |\theta-\theta_{\bar{l}}|\geq\delta_2,|\phi-\phi_{\bar{l}}|\geq\delta_2,\bar{l}\in Ann(\bar{0},\delta_1,l_0)$$

where $Ann(\bar{0},\delta_1,l_0)$ *is the closed annulus* $\delta_1\leq|\bar{l}|\leq l_0$*,* $\theta_{\bar{l}}=cos^{-1}(\frac{l_3}{l})$*,* $\phi_{\bar{l}}=tan^{-1}(\frac{l_2}{l_1})$

Lemma 0.23. *For fixed* $\bar{l}\in\mathcal{R}^3$*,* $t\in\mathcal{R}_{>0}$*, we have that the polar representation of* $e^{i(|\bar{k}-\bar{l}|)ct}$*,* $\bar{k}\in\mathcal{R}^3$*, is given by;*

$$e^{irct\nu(r,\theta,\phi,\bar{l})},\ r\in\mathcal{R}_{>0},\ 0\leq\theta<\pi,\ -\pi\leq\phi\leq\pi$$

where;

$lim_{r\rightarrow\infty}\nu(r,\theta,\phi,\bar{l})=1$

uniformly in $\{\theta,\phi\}$. *Moreover, for* $\theta\neq cos^{-1}(\frac{l_3}{l})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$, *the real and imaginary parts of* $e^{irct\nu(r,\theta,\phi,\bar{l})}$ *are oscillatory.*

If f *is analytic and analytic at infinity, of moderate decrease, then* $fcos(rct\nu(r,\theta,\phi,\bar{l}))$ *and* $fsin(rct\nu(r,\theta,\phi,\bar{l}))$, *for* $l\neq 0$, $\theta\neq cos^{-1}(\frac{l_3}{l})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$ *are non-oscillatory when restricted to any finite interval* $[0,L)$ *and have the property that, for any* $\epsilon>0$, *there exists* $L_{\epsilon,\theta,\phi\bar{l}}\in\mathcal{R}$ *such that;*

$|\int_{r>L_{\epsilon,\theta,\phi,\bar{l}}}fcos(rct\nu(r,\theta,\phi,\bar{l}))dr|<\epsilon$

Moreover, for the final family in Lemma 0.19 indexed by $(\theta,\phi,\bar{l})$ *with the above restriction, which is excellent and non-oscillatory, the families defined by* $f_{\theta,\phi,\bar{l}}cos(rct\nu(r,\theta,\phi,\bar{l}))$ *and* $f_{\theta,\phi,\bar{l}}sin(rct\nu(r,\theta,\phi,\bar{l}))$ *are also excellent and non oscillatory, restricted to a finite interval* $[0,L_{l_0}]$, *and when we restrict the parameters* $(\theta,\phi,\bar{l})$ *to* $W_{\delta_1,\delta_2,l_0}$.

Moreover, for any given $\epsilon>0$, *there exists* L, *uniform in* $(\theta,\phi,\bar{l})$, *with the restriction* $(\theta,\phi,\bar{l})\in W_{\delta_1,\delta_2,l_0}$, *such that;*

$|\int_{r>L}f_{\theta,\phi,\bar{l}}cos(rct\nu(r,\theta,\phi,\bar{l}))dr|<\epsilon$

The same results hold for $e^{i(|\bar{k}-\bar{l}|-|\bar{l}|)ct}$.

Proof. Making the substitution, $k_1=rsin(\theta)cos(\phi)$, $k_2=rsin(\theta)sin(\phi)$, $k_3=rcos(\theta)$, we obtain;

$e^{i(|\bar{k}-\bar{l})ct}=e^{i[(rsin(\theta)cos(\phi)-l_1)^2+(rsin(\theta)sin(\phi)-l_2)^2+(rcos(\theta)-l_3)^2]^{\frac{1}{2}}ct}$

$=e^{i(r^2-(2l_1rsin(\theta)cos(\phi)+2l_2rsin(\theta)sin(\phi)+2l_3rcos(\theta))+l^2)^{\frac{1}{2}}ct}$

$=e^{irct\nu(r,\theta,\phi,\bar{l})}$

where;

$\nu(r,\theta,\phi,\bar{l}) = (1 - \frac{1}{r}(2l_1 sin(\theta)cos(\phi) + 2l_2 sin(\theta)sin(\phi) + 2l_3 cos(\theta)) + \frac{l^2}{r^2})^{\frac{1}{2}}$

It is clear, as $|2l_1 sin(\theta)cos(\phi)+2l_2 sin(\theta)sin(\phi)+2l_3 cos(\theta)| \leq 2(|l_1|+|l_2|+|l_3|)$, that $lim_{r\rightarrow\infty}\nu(r,\theta,\phi,\bar{l}) = 1$, uniformly in $\{\theta,\phi\}$. For the next claim, we show that $cos(rct\nu(r,\theta,\phi,\bar{l}))$ is oscillatory, leaving the other case to the reader. We have that;

$\frac{\partial cos(rct\nu(r,\theta,\phi,\bar{l}))}{\partial r} = 0$

iff $-sin(rct\nu(r,\theta,\phi,\bar{l}))(ct\nu(r,\theta,\phi,\bar{l}) + rct\frac{\partial\nu(r,\theta,\phi,\bar{l})}{\partial r}) = 0$

iff $sin(rct\nu(r,\theta,\phi,\bar{l})) = 0$ or $ct\nu(r,\theta,\phi,\bar{l}) + rct\frac{\partial\nu(r,\theta,\phi,\bar{l})}{\partial r} = 0$

iff $rct\nu(r,\theta,\phi,\bar{l}) = \frac{\pi}{2} + n\pi$, $(n \in \mathcal{Z})$

or $ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3}) = 0$

where;

$\gamma(\theta,\phi,\bar{l}) = 2l_1 sin(\theta)cos(\phi) + 2l_2 sin(\theta)sin(\phi) + 2l_3 cos(\theta)$

We have;

$lim_{r\rightarrow\infty}[ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3})] = ct \neq 0$

so that, by continuity, the zeros of;

$ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3})$

are located in a compact interval $[0,K]$, for some $K \in \mathcal{R}_{>0}$. We have, for $r \neq 0$, that;

$\nu(r,\theta,\phi,\bar{l}) = 0$ iff $r\nu(r,\theta,\phi,\bar{l}) = 0$

iff $|\bar{k} - \bar{l}| = 0$

iff $\bar{k} = \bar{l}$

which implies that $\theta = cos^{-1}(\frac{l_3}{l})$, $\phi = tan^{-1}(\frac{l_2}{l_1})$. It follows that, with the assumption on $\{\theta, \phi\}$, we have that, for $r \neq 0$;

$ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3})$

is locally analytic, for $r \neq 0$. Clearing denominators, we have for $r \neq 0$, that;

$ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3}) = 0$

iff $ct\nu(r,\theta,\phi,\bar{l})r^3 + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(r\gamma(\theta,\phi,\bar{l}) - 2l^2) = 0$

which is an analytic relation, so it can only have a finite number of zeros located in the interval $[0, K]$, $(*)$. We have that $lim_{r\rightarrow\infty} rct\nu(r,\theta,\phi,\bar{l}) = \infty$ and $lim_{r\rightarrow 0} rct\nu(r,\theta,\phi,\bar{l}) = -ctl$, so, by the intermediate value theorem, we can find an infinite number of solutions to $rct\nu(r,\theta,\phi,\bar{l}) = \frac{\pi}{2} + n\pi$, $n \in \mathcal{Z}$, located in $\mathcal{R}_{>0}$. As;

$lim_{r\rightarrow\infty}[ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3})] = ct$

and;

$lim_{r\rightarrow 0}[ct\nu(r,\theta,\phi,\bar{l}) + \frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l}) - \frac{2l^2}{r^3})]$

$= lim_{r\rightarrow 0}\frac{\partial rct\nu(r,\theta,\phi,\bar{l})}{\partial r}$

$= lim_{r\rightarrow 0}\frac{\partial ct|\bar{k}(r,\theta,\phi)-\bar{l}|}{\partial r}$

is finite, we have that $\frac{\partial rct\nu(r,\theta,\phi,\bar{l})}{\partial r}$ is bounded by $M \in \mathcal{R}_{>0}$ on $\mathcal{R}_{>0}$.

Using the mean value theorem, if r_n is a solution to $rct\nu(r,\theta,\phi,\bar{l}) = \frac{\pi}{2} + n\pi$, and r_m is a solution to $rct\nu(r,\theta,\phi,\bar{l}) = \frac{\pi}{2} + m\pi$, then

$|r_n - r_m| \geq \frac{|(\frac{\pi}{2}+n\pi)-(\frac{\pi}{2}+m\pi)|}{M}$

$= \frac{|(n-m)|\pi}{M}$

$\geq \frac{\pi}{M}$, $(n \neq m)$ (A)

By the observation $(*)$, and the fact that;

$[ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})]$

is monotone on (K,∞), there can be at most a finite number $\{n_{i_1},\ldots,n_{i_p}\}$ for which there exist multiple solutions $r_{n,n_{i_j}}\in\mathcal{R}_{>0}$ to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n_i\pi$. Let Z denote the $\{r_i:i\in\mathcal{N}\}$ for which there exists a solution to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n\pi$, $n\in\mathcal{Z}$, and Z_0 the finite set consisting of solutions to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n_{i_j}\pi$, $1\leq j\leq p$ and the zeros on $[0,K]$, corresponding to $(*)$. Ordering $Z\cup Z_0$ as a set $\{r_i:i\in\mathcal{N}\}$, it is clear that $cos(rct\nu(r,\theta,\phi,\bar{l}))|_{(r_i,r_{i+1})}$ is monotone. Choosing $\delta=min(\frac{\pi}{M},d(Z\setminus Z_0,Z_0),Sep(Z_0))>0$, where $Sep(Z_0)=min(d(r,r'):\{r,r'\}\subset Z_0,r\neq r')$, we obtain the result that $cos(rct\nu(r,\theta,\phi,\bar{l}))$ is oscillatory.

For the next claim $(fcos(rct\nu(r,\theta,\phi,\bar{l}))'$ is analytic on $[0,L]$, so has finitely many zeros, in which case $fcos(rct\nu(r,\theta,\phi,\bar{l}))|_{[0,L]}$ is non-oscillatory. As f is of moderate decrease, there exists a constant $C\in\mathcal{R}_{>0}$ for which $|f|\leq\frac{C}{r^2}$, for $r>1$. It follows that;

$\int_{r>M}|fcos(rct\nu(r,\theta,\phi,\bar{l})|dr$

$\leq\int_{r>M}\frac{C}{r^2}dr$

$=\frac{C}{M}$

$<\epsilon$

for $M>\frac{C}{\epsilon}$

If there is no bound on the valency of zeros to $(f_{\theta,\phi,\bar{l}}cos(rct\nu(r,\theta,\phi,\bar{l})))'$ on $(0,L_{l_0})$, then we can find closed sets $V_{n+1}\subset V_n\subset W_{\delta_1,\delta_2,l_0}$, where;

$V_n=\{\bar{l}\in W_{\delta_1,\delta_2,l_0}:Card((f_{\theta,\phi,\bar{l}}cos(rct\nu(r,\theta,\phi,\bar{l})))'=0$

$\cap(0,L_{l_0}))\geq n\}$

such that $V_n\neq\emptyset$. As V_n is closed, it follows, we can find $\bar{l}_0\in\cap_{n\in\mathcal{N}}V_n$, for which $(f_{\theta,\phi,\bar{l}}cos(rct\nu(r,\theta,\phi,\bar{l})))'$ has infinite zeros on $[0,L_{l_0}]$, which contradicts the definition of analytic. It follows that the family restricted to $[0,L_{l_0}]$ is excellent and non-oscillatory, when we restrict the parameters in this way.

For the penultimate claim, we have with the restrictions on $\{\theta,\phi,\bar{l}\}$, that;

$|\overline{f}(\overline{k})|\leq\frac{C}{|\overline{k}|^4}$

$|\overline{h}(\bar{l})|\leq M$

$|\bar{l}|\geq\delta_0$

$|\frac{\overline{hl}}{|\bar{l}|}|\leq\frac{M}{\delta_0}$

$|u(\overline{k})|=1$

$|\overline{k}-\bar{l}|\geq lsin(\delta_2)\geq\delta_1 sin(\delta_2)$

$|\frac{\overline{f}(\overline{k}-\bar{l})}{|\overline{k}-\bar{l}|}|\leq\frac{C}{\delta_1 sin(\delta_2)|\overline{k}-\bar{l}|^4}$

$|[\frac{\overline{f}(\overline{k}-\bar{l})}{|\overline{k}-\bar{l}|}\times\frac{\overline{h}(\bar{l})}{|\bar{l}|}]\centerdot u(\overline{k})|$

$\leq|\frac{\overline{f}(\overline{k}-\bar{l})}{|\overline{k}-\bar{l}|}||\frac{\overline{h}(\bar{l})}{|\bar{l}|}|$

$\leq\frac{C}{\delta_1 sin(\delta_2)|\overline{k}-\bar{l}|^4}\frac{M}{\delta_0}$

$\leq\frac{C}{\delta_1 sin(\delta_2)(R-l_0)^4}\frac{M}{\delta_0}$, for $R\geq l_0$

so that;

$|\int_{r>L}f_{\theta,\phi,\bar{l}}cos(rct\nu(r,\theta,\phi,\bar{l}))dr|$

$\leq\int_{r>L}|f_{\theta,\phi,\bar{l}}|dr$

$\leq\int_{r>L}\frac{C}{\delta_1 sin(\delta_2)(R-l_0)^4}\frac{M}{\delta_0}dr$, for $L\geq l_0$

$=[\frac{C}{\delta_1 sin(\delta_2)(R-l_0)^3}\frac{M}{-3\delta_0}]_L^\infty$

$=\frac{C}{\delta_1 sin(\delta_2)(L-l_0)^3}\frac{M}{3\delta_0}$

$<\epsilon$

for $L-l_0\geq(\frac{MC}{3\delta_0\delta_1 sin(\delta_2)\epsilon})^{\frac{1}{3}}$

For the final claim, if we want to do the calculation for;

$e^{i(|\bar{k}-\bar{l}|-|\bar{l}|)ct} = e^{i(|\bar{k}-\bar{l}|)ct}e^{-i|\bar{l}|ct}$

we can absorb the constant term $e^{-i|\bar{l}|ct}$ into the term f of the hypotheses, which will not effect any of the conditions analytic, analytic at infinity or moderate decrease, see also Lemma 0.24.

□

Lemma 0.24. *With notation as in Lemmas 0.23 and 0.7, if;*

$\alpha(\bar{k},\bar{l},t) = \alpha(R,\theta,\phi,\bar{l},t) = \frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})\times (\bar{d}'_{11}(\bar{l})+\frac{\bar{d}'_{12}(\bar{l})}{l})]\cdot\overline{u_{\bar{l}}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)sin(\theta)$

and;

$\beta(\bar{k},\bar{l},t) = \beta(R,\theta,\phi,\bar{l},t) = \frac{-iQ_{0,0}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})\times (\bar{d}'_{11}(\bar{l})+\frac{\bar{d}'_{12}(\bar{l})}{l})]\cdot\overline{u_{\bar{l}}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)sin(\theta)$

then;

$\alpha(R,\theta,\phi,\bar{l},t) = \alpha_1(R,\theta,\phi,\bar{l},t)\mu(R,\theta,\phi,\bar{l},t) = e^{-ilct}\alpha_1(R,\theta,\phi,\bar{l},t)e^{iRct\nu(R,\theta,\phi,\bar{l})}$

$\beta(R,\theta,\phi,\bar{l},t) = \beta_1(R,\theta,\phi,\bar{l},t)\mu(R,\theta,\phi,\bar{l},t) = e^{-ilct}\beta_1(R,\theta,\phi,\bar{l},t)e^{iRct\nu(R,\theta,\phi,\bar{l})}$

For fixed $\bar{l}\neq\bar{0}$ *and* $\theta\neq cos^{-1}(\frac{l_3}{l})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$, *if the real and imaginary components of* $e^{-ilct}\alpha_1(R,\theta,\phi,\bar{l},t)$ *satisfy the conditions of Lemma 0.23, then the real and imaginary components of* α *are oscillatory. Similarly, if the real and imaginary components of;*

$\{e^{-ilct}\beta_1(R,\theta,\phi,\bar{l},t), e^{-ilct}R\frac{\partial\beta_1(R,\theta,\phi,\bar{l},t)}{\partial R}, icte^{-ilct}R\beta_1(R,\theta,\phi,\bar{l},t)(\nu(R,\theta,\phi,\bar{l})+R\frac{\partial\nu(R,\theta,\phi,\bar{l})}{\partial R})\}$

satisfy the conditions of Lemma 0.23, then the real and imaginary components of $\frac{\partial R\beta(R,\theta,\phi,\bar{l},t)}{\partial R}$ *are oscillatory.*

Proof. We have that;

$Re(\alpha) = Re(e^{-ilct}\alpha_1 e^{iRct\nu}) = Re(e^{-ilct}\alpha_1 cos(Rct\nu)) + Re(ie^{-ilct}\alpha_1 sin(Rct\nu))$

$= Re(e^{-ilct}\alpha_1)cos(Rct\nu) + Im(e^{-ilct}\alpha_1)sin(Rct\nu)$

$Im(\alpha) = Im(e^{-ilct}\alpha_1 e^{iRct\nu}) = Im(e^{-ilct}\alpha_1 cos(Rct\nu)) + Im(ie^{-ilct}\alpha_1 sin(Rct\nu))$

$= Im(e^{-ilct}\alpha_1)cos(Rct\nu) + Re(e^{-ilct}\alpha_1)sin(Rct\nu)$

so the first claim, follows from Lemma 0.23.

We also have that;

$Re(\frac{\partial(R\beta)}{\partial R}) = Re(\frac{\partial(Re^{-ilct}\beta_1 e^{iRct\nu})}{\partial R}) = Re(e^{-ilct}\beta_1 e^{iRct\nu}) + Re(R\frac{\partial(e^{-ilct}\beta_1 e^{iRct\nu})}{\partial R})$

$= Re(e^{-ilct}\beta_1 e^{iRct\nu}) + Re(e^{-ilct}R\frac{\partial\beta_1}{R}e^{iRct\nu}) + Re(icte^{-ilct}R\beta_1(\nu + R\frac{\partial\nu}{\partial R})e^{iRct\nu})$

$Im(\frac{\partial(R\beta)}{\partial R}) = Im(\frac{\partial(Re^{-ilct}\beta_1 e^{iRct\nu})}{\partial R}) = Re(e^{-ilct}\beta_1 e^{iRct\nu}) + Re(R\frac{\partial(e^{-ilct}\beta_1 e^{iRct\nu})}{\partial R})$

$= Im(e^{-ilct}\beta_1 e^{iRct\nu}) + Re(e^{-ilct}R\frac{\partial\beta_1}{R}e^{iRct\nu}) + Re(icte^{-ilct}R\beta_1(\nu + R\frac{\partial\nu}{\partial R})e^{iRct\nu})$

and the second claim follows, using the previous calculation and Lemma 0.23.

□

Definition 0.25. *We say that $f \in C(\mathcal{R} \setminus \{0\})$ is of moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|^2}$ for $|x| > 1$. We say that $f \in C(\mathcal{R} \setminus \{0\})$ is of very moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|}$ for $|x| > 1$.We say that $f \in C(\mathcal{R} \setminus \{0\})$ is non-oscillatory if there are finitely many points $\{y_i : 1 \leq i \leq n\} \subset \mathcal{R}$ for which $f|_{(y_i, y_{i+1})}$ is monotone, $1 \leq i \leq n-1$, and $f|_{(-\infty, y_1)}$ and $f|_{(y_n, \infty)}$ is monotone. We say that $f \in C(\mathcal{R} \setminus \{0\})$ is symmetrically asymptotic if f and $\frac{df}{dx}$ are of moderate decrease, $\frac{df}{dx}$ is non-oscillatory, $\{f, \frac{df}{dx}\} \subset L^1((-\epsilon, \epsilon))$, and for $\epsilon > 0$;*

$lim_{y \to 0-} f(y) = lim_{y \to 0+} f(y) = M$

and

$lim_{y \to 0-} \frac{df}{dx}(y) = -lim_{y \to 0+} \frac{df}{dx}(y) = L \ (*)$

where $L \in \{+\infty, -\infty\}$, $M \in \mathcal{R}$. We say that $f \in C(\mathcal{R} \setminus \{0\})$ is light symmetrically asymptotic if f and $\frac{df}{dx}$ are of very moderate decrease, f

and $\frac{df}{dx}$ are non-oscillatory, $\{f, \frac{df}{dx}\} \subset L^1((-\epsilon,\epsilon))$, and for $\epsilon > 0$, the condition $(*)$ holds.

Lemma 0.26. *Let f be symmetrically asymptotic, then we have that, for any $\delta > 0$, there exist constants $\{C_\delta, D_\delta\} \subset \mathcal{R}_{>0}$, such that;*

$|\mathcal{F}(f)(k)| \leq \frac{\delta}{|k|} + \frac{C_\delta}{|k|^2}$, *for* $|k| > D_\delta$

Proof. As f is symmetrically asymptotic, we have that $lim_{x\rightarrow 0-}f(x) = lim_{x\rightarrow 0+}f(x) = M$, where $M \in \mathcal{R}$. In either case, we can apply integration by parts, to obtain $(\dagger)$ in Lemma 0.11. The step $(*)$ follows from the fact that $\frac{df}{dx}$ is of moderate decrease. As $\frac{df}{dx}$ is non-oscillatory, we can find $x_0 < 0 < x_1$, with $\frac{df}{dx}|_{x_0,0}$ and $\frac{df}{dx}|_{0,x_0}$ monotone. In particular, for any $\delta > 0$, we can find $x_0 < y_0 < 0 < y_1 < x_1$ such that $\int_{(y_0,y_1)} |\frac{df}{dx}(y)|dy < \delta((2\pi)^{\frac{1}{2}})$ and $\frac{df}{dx}(y_0) = L_{1,0}$, $\frac{df}{dx}(y_1 = L_{2,0}$, with $\{L_{1,0}, L_{2,0}\} \subset \mathcal{R}$. Then;

$$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N_\epsilon}^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy - \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{(-N_\epsilon,y_0)\cup(y_1,N_\epsilon)} \frac{df}{dx}(y)e^{-iky}dy|$$
$$\leq \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{(y_0,y_1)} |\frac{df}{dx}(y)|dy$$
$$< \delta$$

Again, by the proof of Lemma 0.9 in [10], using underflow, we can find $\{D_{\epsilon,y_0,y_1}, E_{\epsilon,y_0,y_1}\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_{\epsilon,y_0,y_1}$, we have that;

$$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{(-N_\epsilon,y_0)\cup(y_1,N_\epsilon)} \frac{df}{dx}(y)e^{-iky}dy| < \frac{E_{\epsilon,y_0,y_1}}{|k|}, \ (**)$$

It is easy to see from the proof, that $\{D_{\epsilon,y_0,y_1}, E_{\epsilon,y_0,y_1}\}$ can be chosen uniformly in ϵ, so that using the triangle inequality again, we obtain;

$$|\mathcal{F}(\frac{df}{dx})(k)| \leq \epsilon + \delta + \frac{E_{\epsilon,y_0,y_1}}{|k|}$$

for $|k| > D_{\epsilon,y_0,y_1}$

As ϵ was arbitrary, and E_{ϵ,y_0,y_1} is uniform in ϵ, we obtain that;

$$|\mathcal{F}(\frac{df}{dx})(k)| \leq \delta + \frac{E_{y_0,y_1}}{|k|}$$

for $|k|>D_{y_0,y_1}$.

so that, using (†) again;

$|\mathcal{F}(f)(k)|\leq\frac{\delta}{|k|}+\frac{E_{y_0,y_1}}{|k|^2}$, (†)

$=\frac{\delta}{|k|}+\frac{C_\delta}{|k|^2}$

for $|k|>D_\delta$, where $C_\delta=E_{y_0,y_1}$ and $D_\delta=D_{y_0,y_1}$.

□

Lemma 0.27. *Let $f\in C(\mathcal{R})$ and $\frac{df}{dx}\in C(\mathcal{R})$ be of very moderate decrease, with f and $\frac{df}{dx}$ non-oscillatory, then defining the Fourier transform by;*

$\mathcal{F}(f)(k)=\frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)e^{-iky}dy\ (k\neq 0)$

$\mathcal{F}(\frac{df}{dx})(k)=\frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}\int_{-r}^{r}\frac{df}{dx}(y)e^{-iky}dy\ (k\neq 0)$

we have that $\mathcal{F}(f)$ and $\mathcal{F}(\frac{df}{dx})$ are bounded and there exists a constant $G\in\mathcal{R}_{>0}$, such that;

$|\mathcal{F}(f)(k)|\leq\frac{G}{|k|^2}$

for sufficiently large k.

Proof. As f is of very moderate decrease, we have that f is continuous and $lim_{|x|\rightarrow\infty}f(x)=0$. Similarly, $\frac{df}{dx}$ is continuous and $lim_{|x|\rightarrow\infty}\frac{df}{dx}=0$. As $lim_{|x|\rightarrow\infty}f(x)=0$, and f is non-oscillatory, we have that, for $k\neq 0$, the indefinite integral;

$lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)e^{-iky}dy$

$=lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)cos(ky)dy-ilim_{r\rightarrow\infty}\int_{-r}^{r}f(y)sin(ky)dy$

exists. As f is of very moderate decrease and non-oscillatory, we have that $|f(x)|\leq\frac{D}{|x|}$, for $|x|>E$, and monotone in the intervals $(-\infty,E)$ and (E,∞). Using the method of [10], letting $K=max(|f||_{[-E,E]})$, we have that;

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)cos(ky)dy| \leq 2KE + 2K\int_{E}^{E+\frac{\pi}{2|k|}}\frac{Dcos(|k|(y-E))}{y}dy$

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)sin(ky)dy| \leq 2KE + 2K\int_{E}^{E+\frac{\pi}{2|y|}}\frac{Dcos(|k|(y-E))}{y}dy$

so that;

$|lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)e^{-iky}dy| \leq 4KE + 4K\int_{E}^{E+\frac{\pi}{2|k|}}\frac{Dcos(|k|(y-E))}{y}dy$

$= 4KE + 4KD([\frac{-sin(|k|(y-E))}{|k|}]_{E}^{E+\frac{\pi}{2|k|}} - \int_{E}^{E+\frac{\pi}{2|k|}}\frac{sin(|k|(y-E))}{y^2}dy)$

$= 4KE + 4KD(\frac{1}{|k|(E+\frac{\pi}{2|k|})} - \int_{E}^{E+\frac{\pi}{2|k|}}\frac{sin(|k|(y-E))}{y^2}dy)$

$\leq 4KE + 4KD(\frac{1}{E|k|+\frac{\pi}{2}} + \int_{E}^{\infty}\frac{1}{y^2}dy)$

$\leq 4KE + 4KD(\frac{2}{\pi} + \frac{1}{E}) = N$

so that $\mathcal{F}(f)(k)$ and, similarly, $\mathcal{F}(\frac{df}{dx})(k)$ are bounded, for $k \neq 0$, ([1]). We have, using integration by parts, that;

$\mathcal{F}(\frac{df}{dx})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}\int_{-r}^{r}\frac{df}{dx}(y)e^{-iky}dy$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}([f(y)e^{-iky}]_{-r}^{r} + ik\int_{-r}^{r}f(y)e^{-iky}dy)$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}[f(y)e^{-iky}]_{-\infty}^{\infty} + ik\frac{1}{(2\pi)^{\frac{1}{2}}}lim_{r\rightarrow\infty}\int_{-r}^{r}f(y)e^{-iky}dy$

$= ik\mathcal{F}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}$, $(\dagger)$

As $\frac{df}{dx}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [10], using underflow, for $r \in \mathcal{R}_{>0}$, we can find $\{F_r, G_r\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > F_r$, we have that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-r}^{r}\frac{df}{dx}(y)e^{-iky}dy| < \frac{G_r}{|k|}$, $(**)$

[1] $\mathcal{F}(f)(k)$ and $\mathcal{F}(\frac{df}{dx})(k)$ are differentiable for $k \neq 0$, limit interchange?

It is easy to see from the proof, that $\{F_r, G_r\}$ can be chosen uniformly in r. Then, from $(**)$, we obtain that, for $|k| > F$;

$|\mathcal{F}(\frac{df}{dx})(k)| < \frac{G}{|k|}$, for $|k| > F$

and, from $(\dagger)$, for $|k| > max(F, 1)$, that;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|} < \frac{G}{|k|^2}$

□

Definition 0.28. *Let $f \in C^3(\mathcal{R})$, with f, f', f'' and f''' bounded, then we define an approximating sequence $\{f_m : m \in \mathcal{N}\}$ by the requirements;*

(i). $f_m \in C^2(\mathcal{R})$, for $m \in \mathcal{N}$.

(ii). $f_m|_{[-m,m]} = f|_{[-m,m]}$.

(iii). f_m is of uniform moderate decay, in the sense that there exists a constant $C \in \mathcal{R}_{>0}$, independent of m, with;

$|f_m(x)| \leq \frac{C}{|x|^2}$, for $x \in (-\infty, -m - \frac{1}{m}) \cup (m + \frac{1}{m}, \infty)$

(iv). There exists constants $\{D, E\} \subset \mathcal{R}_{>0}$, with $\int_{-m-\frac{1}{m}}^{m} |f_m(x)|dx \leq \frac{D}{m}$ and $\int_{m}^{m+\frac{1}{m}} |f_m(x)|dx \leq \frac{D}{m}$.

Lemma 0.29. *Let $f \in C(\mathcal{R})$ and $\frac{df}{dx} \in C(\mathcal{R})$ be of very moderate decrease, with f and $\frac{df}{dx}$ non-oscillatory. Let $\{f_m; m \in \mathcal{N}\}$ be an approximating sequence. Let $\mathcal{F}$ be the ordinary Fourier transform, defined for each f_m, then the sequence $\{\mathcal{F}(f_m) : m \in \mathcal{N}\}$ converges pointwise and uniformly to $\mathcal{F}(f)$ on $\mathcal{R} \setminus \{0\}$, where $\mathcal{F}(f)$ is defined in Lemma 0.27.*

Proof. For $g \in C(\mathcal{R})$ and $n \in \mathcal{N}$, define;

$\mathcal{F}_n(g)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-n}^{n} f(y)e^{-iky}dy$

For $k \in \mathcal{R} \setminus \{0\}$, $\{m, n\} \subset \mathcal{N}$, and $m \geq n$, $\epsilon > 0$,$\delta > 0$, we have;

$|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq |\mathcal{F}(f)(k) - \mathcal{F}_n(f)(k)| + |\mathcal{F}_m(f)(k) - \mathcal{F}_m(f_m)(k)|$

$+|\mathcal{F}_m(f_m)(k)-\mathcal{F}(f_m)(k)|$

$=|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+|\mathcal{F}_m(f_m)(k)-\mathcal{F}(f_m)(k)|$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\int_{-\infty}^{-m}|f_m(x)|dx+|+\int_m^\infty|f_m(x)|dx$

$=|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\int_{-\infty}^{-m-\frac{1}{m}}|f_m(x)dx+\int_{-m-\frac{1}{m}}^{-m}|f_m(x)|dx$

$+\int_m^{m+\frac{1}{m}}|f_m(x)dx+\int_{m+\frac{1}{m}}^\infty|f_m(x)|dx$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\frac{D+E}{m}+\int_{-\infty}^{-m-\frac{1}{m}}\frac{C}{x^2}dx+\int_{m+\frac{1}{m}}^\infty\frac{C}{x^2}dx$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\frac{D+E}{m}+\frac{2C}{m+\frac{1}{m}}$

$\leq|\mathcal{F}(f)(k)-\mathcal{F}_m(f)(k)|+\frac{2C+D+E}{m}$

$\leq\epsilon+\delta$, for $m\geq max(m(\epsilon),\frac{2C+D+E}{\delta})$. As $\epsilon>0$ and $\delta>0$ were arbitrary, we obtain the result.

□

Lemma 0.30. *If $m\in\mathcal{R}_{>0}$ is sufficiently large, $\{a_0,a_1,a_2\}\subset\mathcal{R}$, there exists $h\in\mathcal{R}[x]$ of degree 5, with the property that;*

$h(m)=a_0,\ h'(m)=a_1,\ h''(m)=a_2,\ (i)$

$h(m+\frac{1}{m})=h'(m+\frac{1}{m})=h''(m+\frac{1}{m})=0\ (ii)$

$|h_{[m,m+\frac{1}{m}]}|\leq C$

for some $C\in\mathcal{R}_{>0}$, independent of m sufficiently large, and, if $h'''(m)>0$, $h'''(x)|_{[m,m+\frac{1}{m}]}>0$, if $h'''(m)<0$, $h'''|_{[m,m+\frac{1}{m}]}<0$. In particularly;

$\int_m^{m+\frac{1}{m}}|h'''(x)|dx=|a_2|$

Proof. If $p(x)$ is any polynomial, we have that $h(x)=(x-(m+\frac{1}{m}))^3p(x)$ satisfies condition (ii). Then;

$h'(x)=3(x-(m+\frac{1}{m}))^2p(x)+(x-(m+\frac{1}{m}))^3p'(x)$

$h''(x)=6(x-(m+\frac{1}{m}))p(x)+6(x-(m+\frac{1}{m}))^2p'(x)+(x-(m+\frac{1}{m}))^3p''(x)$

$h'''(x) = 6p(x) + 18(x - (m + \frac{1}{m}))p'(x) + 9(x - (m + \frac{1}{m}))^2 p''(x)$

so we can satisfy (i), by requiring that;

(a). $-\frac{p(m)}{m^3} = a_0$

(b). $\frac{3p(m)}{m^2} - \frac{p'(m)}{m^3} = a_1$

(c). $\frac{-6p(m)}{m} + \frac{6p'(m)}{m^2} - \frac{p''(m)}{m^3} = a_2$

which has the solution;

$p(m) = -a_0 m^3$, $p'(m) = -3a_0 m^4 - a_1 m^3$, $p''(m) = -12a_0 m^5 - 6a_1 m^4 - a_2 m^3$

and can be satisfied by the polynomial;

$p(x) = \frac{1}{2}(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3)(x - m)^2$

$+(-3a_0 m^4 - a_1 m^3)(x - m) + (-a_0 m^3)$

$= \frac{1}{2}(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3)x^2 + (-m(-12a_0 m^5 - 6a_1 m^4 - a_2 m^3)$

$+(-3a_0 m^4 - a_1 m^3))x + (\frac{m^2}{2}(-12a0m^5 - 6a_1 m^4 - a_2 m^3)$

$-m(-3a_0 m^4 - a_1 m^3) - a_0 m^3)$

$= (-6a_0 m^5 - 3a_1 m^4 - \frac{a_2}{2} m^3)x^2 + (12a_0 m^6 + 6a_1 m^5 + a_2 m^4 - 3a_0 m^4$

$-a_1 m^3)x + (-6a_0 m^7 - 3a_1 m^6 - \frac{a_2}{2} m^5 + 3a_0 m^5 + a_1 m^4 - a_0 m^3)$

$= (-6a_0 m^5 - 3a_1 m^4 - \frac{a_2}{2} m^3)x^2 + (12a_0 m^6 + 6a_1 m^5 + (a_2 - 3a_0)m^4$

$-a_1 m^3)x + (-6a_0 m^7 - 3a_1 m^6 + (3a_0 - \frac{a_2}{2})m^5 + 3a_0 m^5 + a_1 m^4 - a_0 m^3)$

$= ax^2 + bx + c$ $(*)$

so that;

$h'''(x) = 6(ax^2 + bx + c) + 18(x - (m + \frac{1}{m}))(2ax + b) + 9(x - (m + \frac{1}{m}))^2 2a$

$= (60a)x^2+(24b-72a(m+\frac{1}{m}))x+(6c-18(m+\frac{1}{m})b+18a(m+\frac{1}{m})^2)$

and, using the computation $(*)$

$h'''(x) = (60(-6a_0m^5)+O(m^4))x^2+(24.12a_0m^6-72m(-6a_0m^5)$

$+O(m^5))x+(6.-6a_0m^7-18m(12a_0m^6)+18m^2(-6a_0m^5)+O(m^6))$

$= (-360a_0m^5+O(m^4))x^2+(740a_0m^6+O(m^5))x+$

$(-360a_0m^7+O(m^6))$

which, by the quadratic formula, has roots when;

$x = \frac{-740a_0m^6+\sqrt{740^2a_0^2m^{12}-4(-360a_0m^5)(-360a_0m^7)}}{2.-360a_0m^5}+O(1)$

$= \frac{740m}{720}+\frac{170m}{720}+O(1)$

$= \frac{19m}{24}+O(1)$ or $\frac{91m}{72}+O(1)$

We have that $m > \frac{19m}{24}$ and $m+\frac{1}{m} < \frac{91m}{72}$ iff $m > \sqrt{\frac{72}{19}}$, and, clearly, we can ignore the $O(1)$ term for m sufficiently large. In particularly, for sufficiently large m, $h'''(x)$ has no roots in the interval $[m, m+\frac{1}{m}]$, so $h'''|_{[m,m+\frac{1}{m}]} > 0$ or $h'''|_{[m,m+\frac{1}{m}]} < 0$.

We calculate that;

$|h_{[m,m+\frac{1}{m}]}| = |(x-(m+\frac{1}{m}))^3p(x)|_{[m,m+\frac{1}{m}]}|$

$\leq \frac{1}{m^3}|p(x)|_{[m,m+\frac{1}{m}]}$

$= \frac{1}{m^3}|[\frac{1}{2}(-12a_0m^5-6a_1m^4-a_2m^3)(x-m)^2$

$+(-3a_0m^4-a_1m^3)(x-m)+(-a_0m^3)]|_{[m,m+\frac{1}{m}]}$

$\leq \frac{1}{m^3}[\frac{1}{2}|-12a_0m^5-6a_1m^4-a_2m^3|\frac{1}{m^2}+|-3a_0m^4-a_1m^3|\frac{1}{m}+|-a_0m^3|]$

$\leq \frac{12|a_0|m^5+6|a_1|m^4+|a_2|m^3|}{m^5}+\frac{3|a_0|m^4+|a_1|m^3}{m^4}+\frac{|a_0|m^3}{m^3}$

$\leq 12|a_0|+6|a_1|+|a_2|+3|a_0|+|a_1|+|a_0|$ $(m > 1)$

$\leq 16|a_0| + 7|a_1| + |a_2|$

For the final claim, we have, as $h'''|_{[m,m+\frac{1}{m}]} > 0$ or $h'''|_{[m,m+\frac{1}{m}]} < 0$, that, using the fundamental theorem of calculus;

$\int_m^{m+\frac{1}{m}} |h'''(x)|dx = |\int_m^{m+\frac{1}{m}} h'''(x)dx|$

$= |h''(m+\frac{1}{m}) - h''(m)| = |-h''(m)| = |a_2|$

□

Lemma 0.31. *If $m \in \mathcal{R}_{>0}$, $\{a_0, a_1, a_2, a_3\} \subset \mathcal{R}$, there exists $h \in C^3(\mathcal{R})$, with the property that;*

$h(m) = a_0$, $h'(m) = a_1$, $h''(m) = a_2$, $h'''(m) = a_3$, (i)

$h(m+\frac{1}{m}) = h'(m+\frac{1}{m}) = h''(m+\frac{1}{m}) = h'''(m+\frac{1}{m}) = 0$ (ii)

$|h|_{[m,m+\frac{1}{m}]} \leq C$

where $C \in \mathcal{R}_{>0}$ is independent of $m > 1$, and, if $a_3 > 0$, $h'''(x)|_{[m,m+\frac{1}{m}]} \geq 0$, $a_3 < 0$, $h'''(x)|_{[m,m+\frac{1}{m}]} \leq 0$. In particularly;

$\int_m^{m+\frac{1}{m}} |h'''(x)|dx = |a_2|$

Proof. Let $g(x)$ be a polynomial, then it is clear that the polynomial $h_1(x) = (x-(m+\frac{1}{m})^n g(x)$, for $n \geq 4$, has the property (ii), that $h_1(m+\frac{1}{m}) = h_1'(m+\frac{1}{m}) = h_1''(m+\frac{1}{m}) = h_1'''(m+\frac{1}{m}) = 0$. The condition (i), then amounts to the equations;

$(i)'$ $\frac{g(m)}{(-1)^n m^n} = a_0$

$(ii)'$ $\frac{ng(m)}{(-1)^{n-1}m^{n-1}} + \frac{g'(m)}{(-1)^n m^n} = a_1$

$(iii)'$ $\frac{n(n-1)g(m)}{(-1)^{n-2}m^{n-2}} + \frac{2ng'(m)}{(-1)^{n-1}m^{n-1}} + \frac{g''(m)}{(-1)^n m^n} = a_2$

$(iv)'$ $\frac{n(n-1)(n-2)g(m)}{(-1)^{n-3}m^{n-3}} + \frac{3n(n-1)g'(m)}{(-1)^{n-2}m^{n-2}} + \frac{3ng''(m)}{(-1)^{n-1}m^{n-1}} + \frac{g'''(m)}{(-1)^n m^n} = a_3$

which we can solve, by requiring that;

$(i)''$ $g(m) = (-1)^n a_0 m^n$

$(ii)''$ $g'(m)=(-1)^n a_1 m^n+(-1)^n a_0 n m^{n+1}$

$(iii)''$ $g''(m)=(-1)^n a_2 m^n+2(-1)^n n a_1 m^{n+1}+(-1)^n n(n+1)a_0 m^{n+2}$

$(iv)''$ $g'''(m)=(-1)^n a_3 m^n+3n(-1)^n a_2 m^{n+1}+(-1)^n a_1 n(n+3)m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3}$ $(*)$

Let;

$g_1(x)=((-1)^n a_3 m^n+3n(-1)^n a_2 m^{n+1}+(-1)^n a_1 n(n+3)m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3})(x-m)^3+((-1)^n a_2 m^n+2(-1)^n n a_1 m^{n+1}$

$+(-1)^n n(n+1)a_0 m^{n+2})(x-m)^2+((-1)^n a_1 m^n+(-1)^n a_0 n m^{n+1})$

$(x-m)+((-1)^n a_0 m^n)$

Then $g_1(x)$ satisfies $(*)$, and so does any function of the form $g_2(x)+g_1(x)$ where;

$g_2(m)=g_2'(m)=g_2''(m)=g_2'''(m)=0$

provided $g_2\in C^3(\mathcal{R})$. In this case, if;

$h(x)=(x-(m+\frac{1}{m})^n(g_2(x)+g_1(x))$

then h satisfies $(i),(ii)$. We have that;

$|x-(m+\frac{1}{m})^n g_1(x)|_{[m,m+\frac{1}{m}]}\leq\frac{1}{m^n}(|g_2|_{[m,m+\frac{1}{m}]}+|g_1|_{[m,m+\frac{1}{m}]})$

$\leq\frac{1}{m^n}(|g_2|_{[m,m+\frac{1}{m}]}+\frac{1}{m^n}|((-1)^n a_3 m^n+3n(-1)^n a_2 m^{n+1}+(-1)^n a_1 n(n+3)m^{n+2}$

$+n(n+1)(n+2)(-1)^n a_0 m^{n+3})\frac{1}{m^3}+((-1)^n a_2 m^n+2(-1)^n n a_1 m^{n+1}$

$+(-1)^n n(n+1)a_0 m^{n+2})\frac{1}{m^2}+((-1)^n a_1 m^n+(-1)^n a_0 n m^{n+1})$

$\frac{1}{m}+((-1)^n a_0 m^n)|$

$= |((-1)^n a_3 m^n + 3n(-1)^n a_2 m^{n+1} + (-1)^n a_1 n(n+3) m^{n+2}$

$+ n(n+1)(n+2)(-1)^n a_0 m^{n+3})\frac{1}{m^{n+3}} + ((-1)^n a_2 m^n + 2(-1)^n n a_1 m^{n+1}$

$+ (-1)^n n(n+1) a_0 m^{n+2})\frac{1}{m^{n+2}} + ((-1)^n a_1 m^n + (-1)^n a_0 n m^{n+1})$

$\frac{1}{m^{n+1}} + ((-1)^n a_0)|$

$\leq |a_3| + 3n|a_2| + n(n+3)|a_1| + n(n+1)(n+2)|a_0| + |a_2| + 2n|a_1| + n(n+1)|a_0| + |a_1| + n|a_0| + |a_0|$, $(m \geq 1)$

$= \frac{1}{m^n}(|g_2|_{[m,m+\frac{1}{m}]} + (n+1)(n^2+3n+1)|a_0| + (n^2+5n+1)|a_1 + (3n+1)|a_2| + |a_3| = F$ (F)

where $F \in \mathcal{R}_{>0}$ is independent of m. Using the product rule, the condition that $h'''(x) = 0$ in the interval $(m, m+\frac{1}{m})$, is given by;

$n(n-1)(n-2)(x-(m+\frac{1}{m}))^{n-3}(g_2+g_1)(x) + 3n(n-1)(x-(m+\frac{1}{m}))^{n-2}(g_2+g_1)'(x)$

$+3n(x-(m+\frac{1}{m}))^{n-1}(g_2+g_1)''(x) + (x-(m+\frac{1}{m}))^n(g_2+g_1)'''(x) = 0$

which, dividing by $(x-(m+\frac{1}{m}))^{n-3}$, reduces to;

$n(n-1)(n-2)(g_2+g_1)(x) + 3n(n-1)(x-(m+\frac{1}{m}))(g_2+g_1)'(x)+$

$3n(x-(m+\frac{1}{m}))^2(g_2+g_1)''(x) + (x-(m+\frac{1}{m}))^3(g_2+g_1)'''(x) = 0$

and;

$n(n-1)(n-2)g_2(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_2'(x) + 3n(x-(m+\frac{1}{m}))^2 g_2''(x)$

$+(x-(m+\frac{1}{m}))^3 g_2'''(x) = -(n(n-1)(n-2)g_1(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)$

$+3n(x-(m+\frac{1}{m}))^2 g_1''(x) + (x-(m+\frac{1}{m}))^3 g_1'''(x))$ (A)

Without loss of generality, assuming that;

$-(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)+3n(x-(m+\frac{1}{m}))^2 g_1''(x)$

$+(x-(m+\frac{1}{m}))^3 g_1'''(x))|_m = -(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2}$

$-\frac{a_3}{m^3} \geq 0$

we can choose an analytic function $\phi(x)$ on $[m, m+\frac{1}{m}]$ with;

(a). $\phi(x) \leq -(n(n-1)(n-2)g_1(x)+3n(n-1)(x-(m+\frac{1}{m}))g_1'(x)+3n(x-(m+\frac{1}{m}))^2 g_1''(x)$

$+(x-(m+\frac{1}{m}))^3 g_1'''(x))$

(b). $\phi(m) = 0$

The third order differential equation for g_2;

$n(n-1)(n-2)g_2(x)+3n(n-1)(x-(m+\frac{1}{m}))g_2'(x)+3n(x-(m+\frac{1}{m}))^2 g_2''(x)$

$+(x-(m+\frac{1}{m}))^3 g_2'''(x) = \phi(x)$, on $[m, 1+m]$ (B)

with the requirement that $g_2(m) = g_2'(m) = g_2''(m) = 0$, has a solution in $C^3([m, m+\frac{1}{m}))$ by Peano's existence theorem. By the fact (b), we must have that $g_2'''(m) = 0$. Writing the power series for ϕ on $[m, m+\frac{1}{m}]$, as;

$\phi(x) = \sum_{j=0}^{\infty} b_j (x-(m+\frac{1}{m}))^j$

we can use the method of equating coefficients, to obtain a particular solution, with;

$g_{2,part}(x) = \sum_{j=0}^{\infty} a_{j,part}(x-(m+\frac{1}{m}))^j$, with;

$a_{j,part} = \frac{b_j}{n(n-1)(n-2)+3n(n-1)j+3nj(j-1)+j(j-1)(j-2)}$, $(j \geq 3)$

$a_{2,part} = \frac{b_2}{n(n-1)(n-2)+6n(n-1)+3n}$ $a_{1,part} = \frac{b_1}{n(n-1)(n-2)+3n(n-1)}$ $a_{0,part} = \frac{b_0}{n(n-1)(n-2)}$

so that $g_{2,part}$ is analytic as $|a_{j,0}| \leq \frac{|b_j|}{n(n-1)(n-2)}$ for $j \geq 0$.

To solve the homogenous Euler equation;

$n(n-1)(n-2)g_2(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_2'(x) + 3n(x-(m+\frac{1}{m}))^2 g_2''(x)$

$+(x-(m+\frac{1}{m}))^3 g_2'''(x) = 0$ on $[m, m+\frac{1}{m}]$

we can make the substitution $y = m + \frac{1}{m} - x$, to reduce to the equation;

$n(n-1)(n-2)g_{2,m}(y) + 3n(n-1)yg_{2,m}'(y) + 3ny^2 g_{2,m}''(y) + y^3 g_{2,m}'''(y) = 0$ on $[0, \frac{1}{m}]$

with $g_{2,m}(y) = g_2(m + \frac{1}{m} - y)$. Making the further substitution $y = e^u$, and letting $r_{2,m}(u) = g_{2,m}(e^u)$, we have that;

$r_{2,m}'(u) = g_{2,m}'(e^u)e^u$

$r_{2,m}''(u) = g_{2,m}''(e^u)e^{2u} + g_{2,m}'(e^u)e^u$

$r_{2,m}'''(u) = g_{2,m}'''(e^{3u}) + 3g_{2,m}''(e^u)e^{2u} + g_{2,m}'(e^u)e^u$

so that;

$n(n-1)(n-2)g_{2,m}(e^u) + 3n(n-1)e^u g_{2,m}'(e^u) + 3ne^{2u} g_{2,m}''(e^u) + e^{3u} g_{2,m}'''(e^u)$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)e^u(r_{2,m}'(u)e^{-u}) + 3ne^{2u}((r_{2,m}''(u) - g_{2,m}'(e^u)e^u)e^{-2u})$

$+e^{3u}((r_{2,m}'''(u) - 3g_{2,m}''(e^u)e^{2u} - g_{2,m}'(e^u)e^u)e^{-3u})$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r_{2,m}'(u) + 3nr_{2,m}''(u) - 3ng_{2,m}'(e^u)e^u + r_{2,m}'''(u) - 3g_{2,m}''(e^u)e^{2u}$

$-g_{2,m}'(e^u)e^u$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u) - (3n+1)g'_{2,m}(e^u)e^u - 3g''_{2,m}(e^u)e^{2u}$

$= n(n-1)(n-2)r_{2,m}(u) + 3n(n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u) - (3n+1)r'_{2,m}(u)$

$-3e^{2u}((r''_{2,m}(u) - g'_{2,m}(e^u)e^u)e^{-2u})$

$= n(n-1)(n-2)r_{2,m}(u) + (3n^2-6n-1)r'_{2,m}(u) + 3nr''_{2,m}(u) + r'''_{2,m}(u) - 3r''_{2,m}(u) + 3g'_{2,m}(e^u)e^u$

$= n(n-1)(n-2)r_{2,m}(u) + (3n^2-6n-1)r'_{2,m}(u) + 3(n-1)r''_{2,m}(u) + r'''_{2,m}(u) + 3r'_{2,m}(u)$

$= n(n-1)(n-2)r_{2,m}(u) + (3n^2-6n+2)r'_{2,m}(u) + (3n-3)r''_{2,m}(u) + r'''_{2,m}(u) = 0$ (C)

We have that;

$(\lambda^3 + 3(n-1)\lambda^2 + (3n^2-6n+2)\lambda + n(n-1)(n-2))' = 3\lambda^2 + 6(n-1)\lambda + (3n^2-6n+2)$

which has roots when $\lambda = -(n-1) + \frac{1}{\sqrt{3}}$, so that, for large n, the characteristic polynomial of (C) has exactly one real root λ_1 and 2 complex conjugate non-real roots, $\{\lambda_2 + i\lambda_3, \lambda_2 - i\lambda_3\}$. It follows, the general solution of (C) is given by;

$r_{2,m}(u) = A_1 e^{\lambda_1 u} + A_2 e^{\lambda_2 u + i\lambda_3} + A_3 e^{\lambda_2 u - i\lambda_3}$

where $\{A_1, A_2, A_3\} \subset \mathcal{C}$, and, we can obtain a real solution, fitting the corresponding initial conditions, of the form;

$r_{2,m}(u) = B_1 e^{\lambda_1 u} + B_2 e^{\lambda_2 u} cos(\lambda_3 u) + B_3 e^{\lambda_2 u} sin(\lambda_3 u)$

where $\{B_1, B_2, B_3\} \subset \mathcal{R}$. It follows that;

$g_{2,m}(y) = r_{2,m}(ln(y))$

$g_2(x) = g_{2,m}(m + \frac{1}{m} - x) + g_{2,part}(x) = r_{2,m}(ln(m + \frac{1}{m} - x)) + g_{2,part}(x)$

$= B_1 e^{\lambda_1 ln(m+\frac{1}{m}-x)} + B_2 e^{\lambda_2 ln(m+\frac{1}{m}-x)} cos(\lambda_3 ln(m+\frac{1}{m}-x))$

$+B_3 e^{\lambda_2 ln(m+\frac{1}{m}-x)} sin(\lambda_3 ln(m+\frac{1}{m}-x)) + g_{2,part}(x)$ (on $[m, m+\frac{1}{m}]$)

We have that;

$\lambda_1|\lambda_2 + i\lambda_3|^2 = -n(n-1)(n-2)$

$\lambda_1 + \lambda_2 + i\lambda_3 + \lambda_2 - i\lambda_3 = \lambda_1 + 2\lambda_2 = -3(n-1)$

Computing the highest degree in n term of the characteristic polynomial, we obtain that, for $\lambda = \alpha n$;

$\alpha^3 n^3 + 3n(\alpha n)^2 + 3n^2(\alpha n) + n^3 = n^3(\alpha+3)^3 = 0$

so that $\alpha = -3$, $\lambda_1 = -3n + O(1)$ and $2\lambda_2 = -3(n-1) - (-3n + O(1)) = 3 - O(1) = O(1)$

Then, if $B_1 = 0$, we can see that $g_2(x)$ has at most a $\frac{1}{x^{O(1)}}$ singularity at $(m+\frac{1}{m})$, which we can achieve with a 2-parameter family choice for the initial conditions of $\{\phi(m), \phi'(m), \phi''(m)\}$. If;

$-(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2} - \frac{a_3}{m^3} \neq 0$

we can clearly achieve this, while satisfying $(a), (b)$. If;

$-(n(n-1)(n-2)a_0 - \frac{3n(n-1)a_1}{m} + \frac{3na_2}{m^2} - \frac{a_3}{m^3} = 0$

by requiring the the additional property (c);

$\phi'(m) < -(n(n-1)(n-2)g_1(x) + 3n(n-1)(x-(m+\frac{1}{m}))g_1'(x) + 3n(x-(m+\frac{1}{m}))^2 g_1''(x)$

$+(x-(m+\frac{1}{m}))^3 g_1'''(x))'|_m$

we can clearly satisfy $(a), (b)$ as well.

Then, as, for sufficiently large n;

$lim_{x\rightarrow 0}(\frac{B_2 x^n}{x^{O(1)}} sin(\lambda_3 ln(x)) + \frac{B_3 x^n}{x^{O(1)}} cos(\lambda_3 ln(x)))$

$= lim_{x\rightarrow 0}(\frac{B_2 x^n}{x^{O(1)}} sin(\lambda_3 ln(x)) + \frac{B_3 x^n}{x^{O(1)}} cos(\lambda_3 ln(x)))'$

$= lim_{x\rightarrow 0}(\frac{B_2 x^n}{x^{O(1)}} sin(\lambda_3 ln(x)) + \frac{B_3 x^n}{x^{O(1)}} cos(\lambda_3 ln(x)))''$

$= lim_{x\rightarrow 0}(\frac{B_2 x^n}{x^{O(1)}} sin(\lambda_3 ln(x)) + \frac{B_3 x^n}{x^{O(1)}} cos(\lambda_3 ln(x)))''' = 0$

we obtain that $(x-(m+\frac{1}{m}))^n g_2(x)$ extends to a solution in $C^3([m, m+\frac{1}{m}])$, and $(x-(m+\frac{1}{m}))^n(g_2+g_1)(x) \in C^3([m, m+\frac{1}{m}])$. By the fact (a), (A) has no solutions in $(m, m+\frac{1}{m})$, so that $h'''(x) \geq 0$.

We have that;

$|(x-(m+\frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} = |(x-(m+\frac{1}{m})^n(B_2 e^{\lambda_2 ln(m+\frac{1}{m}-x)} cos(\lambda_3 ln(m+\frac{1}{m}-x))$

$+B_3 e^{\lambda_2 ln(m+\frac{1}{m}-x)} sin(\lambda_3 ln(m+\frac{1}{m}-x)) + g_{2,part}(x))|$

$\leq |B_2| m^{\lambda_2 - n} + |B_3| m^{\lambda_2 - n} + m^{-n}|g_{2,part}(x)|$

Noting the right hand side of (a) is bounded by $O(m^n)$ on $[m, m+\frac{1}{m}]$, we can also choose $\phi(x)$ and $g_{2,part}(x)$ to be of $O(m^n)$ on $[m, m+\frac{1}{m}]$, irrespective of the choice of initial conditions $\{\phi(m), \phi'(m), \phi''(m)\}$. We have that $\phi'(m) = O(m^{n+1})$, in the special case, so that choosing $\{B_2, B_3\}$ sufficiently small, noting;

$(x-(m+\frac{1}{m})^n(B_2 e^{\lambda_2 ln(m+\frac{1}{m}-x)} cos(\lambda_3 ln(m+\frac{1}{m}-x))$

$+B_3 e^{\lambda_2 ln(m+\frac{1}{m}-x)} sin(\lambda_3 ln(m+\frac{1}{m}-x)))'|_m = O(max(B_2 m^{n-\lambda_2-1}, B_3 m^{n-\lambda_2-1}))$

we can assume that;

$|(x-(m+\frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} \leq D$

where $D \in \mathcal{R}_{>0}$ is independent of m, so that, using (F);

$|h(x)|_{[m,m+\frac{1}{m}]} \leq |(x-(m+\frac{1}{m})^n g_1(x)|_{[m,m+\frac{1}{m}]} + |(x-(m+\frac{1}{m})^n g_2(x)|_{[m,m+\frac{1}{m}]} \leq F+D$

For the final claim, we have, as $h'''|_{[m,m+\frac{1}{m}]} \geq 0$ or $h'''|_{[m,m+\frac{1}{m}]} \leq 0$, that, using the fundamental theorem of calculus, that;

$\int_m^{m+\frac{1}{m}}|h'''(x)|dx=|\int_m^{m+\frac{1}{m}}h'''(x)dx|$

$=|h''(m+\frac{1}{m})-h''(m)|=|-h''(m)|=|a_2|$

□

Lemma 0.32. *Let f be as in Definition 0.28, then there exists an approximating sequence $\{f_m : m\in\mathcal{N}\}$. Moreover, for sufficiently large m, $|\mathcal{F}(f_m)(k)|\leq\frac{Cm}{|k|^3}$, for $|k|>1$, where $C\in\mathcal{R}_{>0}$, independent of m.*

Proof. Define f_m by setting;

$f_m(x)=f(x)$ for $x\in[-m,m]$

$f_m(x)=h_{1,m}(x)$, for $x\in[-m-\frac{1}{m},-m]$

$f_m(x)=h_{2,m}(x)$, for $x\in[m,m+\frac{1}{m}]$

$f_m(x)=0$, for $x\in(-\infty,-m-\frac{1}{m}]$

$f_m(x)=0$, for $x\in[m,\infty)$

where $\{h_{1,m},h_{2,m}\}$ are the polynomials of degree 5, generated by the data $a_{1,m,0}=f(-m)$, $a_{1,m,1}=f'(-m)$, $a_{1,m,2}=f''(-m)$, $a_{2,m,0}=f(m)$, $a_{2,m,1}=f'(m)$, $a_{2,m,2}=f''(m)$, guaranteed by Lemma 0.30 (or Lemma 0.31. By the construction of Lemma 0.30, we have that (i) in Definition 0.28 holds. By the definition, we have (ii). As f_m is identically zero on $-\infty,-m-\frac{1}{m}]\cup[m,\infty)$, we have that (iii) holds. By the proof of Lemma 0.30, we have that;

$max(|h_{1,m}|_{[m,m+\frac{1}{m}]}|,|h_{2,m}|_{[-m-\frac{1}{m},-m]}|)\leq 16||f||_\infty+7||f'||_\infty+||f''||_\infty$

It follows that;

$\int_{-m-\frac{1}{m}}^{-m}|f_m(x)|dx\leq(16||f||_\infty+7||f'||_\infty+||f''||_\infty)(-m-(-m-\frac{1}{m}))$

$\leq\frac{D}{m}$

$\int_m^{m+\frac{1}{m}}|f_m(x)|dx\leq(16||f||_\infty+7||f'||_\infty+||f''||_\infty)((m+\frac{1}{m})-m)$

$\leq\frac{E}{m}$

where $D = E = (16||f||_\infty + 7||f'||_\infty + ||f''||_\infty)$

proving (iv). For the second claim, we have that;

$\mathcal{F}(f'''_m)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f'''_m(x)e^{-ikx}dx$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}([f''_m(x)e^{-ikx}]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f''_m(x)e^{-ikx}dx$

$= \frac{-ik}{(2\pi)^{\frac{1}{2}}}([f'_m(x)e^{-ikx}]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f'_m(x)e^{-ikx}dx)$

$= \frac{-k^2}{(2\pi)^{\frac{1}{2}}}([f_m(x)e^{-ikx}]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f_m(x)e^{-ikx}dx)$

$= \frac{ik^3}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f_m(x)e^{-ikx}dx)$

so that, for $|k| > 1$;

$|\mathcal{F}(f_m)(k)| = |\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f_m(x)e^{-ikx}dx|$

$= \frac{|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f'''_m(x)e^{-ikx}dx|}{|k|^3}$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}|f'''_m(x)e^{-ikx}|dx$

$= \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}|f'''_m(x)|dx$

$= \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(\int_{-m-\frac{1}{m}}^{-m}|h'''_{1,m}(x)|dx + \int_{-m}^{m}|f'''(x)|dx + \int_{m}^{m+\frac{1}{m}}|h'''_{2,m}(x)|dx)$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(|f''(-m)| + 2m|f'''|_\infty + |f''(m))$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(2||f''||_\infty + 2m||f'''||_\infty)$

$\leq \frac{1}{|k|^3(2\pi)^{\frac{1}{2}}}(2m + 2m||f'''||_\infty)$, $(m > ||f''||_\infty$

$= \frac{Cm}{|k|^3}$

where $C = \frac{1}{(2\pi)^{\frac{1}{2}}}(2 + 2||f'''||_\infty)$

$\square$

Lemma 0.33. *Let $f \in C^3(\mathcal{R})$, with f and $\frac{df}{dx}$ non-oscillatory and of very moderate decrease, with $\{f, f', f'', f'''\}$ bounded, then $\mathcal{F}(f) \in L^1(\mathcal{R})$, and we have that;*

$f(x)=\mathcal{F}^{-1}(\mathcal{F}(f))(x)$

where, for $g\in L^1(\mathcal{R})$*;*

$\mathcal{F}^{-1}(g)(x)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}g(k)e^{ikx}dk$

Proof. By Lemma 0.27, we have that there exists $C\in\mathcal{R}_{>0}$, with $|\mathcal{F}(f)(k)|\leq\frac{C}{|k|^2}$, for sufficiently large k, $(*)$. As f is of very moderate decrease, we have that $|f|^2\leq\frac{D}{|x|^2}$, for $|x|>1$, so that, as $f\in C^0(\mathcal{R})$, we have that $f\in L^2(\mathcal{R})$. It follows that $\mathcal{F}(f)\in L^2(\mathcal{R})$, and $\mathcal{F}(f)|_{[-n,n]}\in L^1(\mathcal{R})$, for any $n\in\mathcal{N}$, $(**)$. Combining $(*),(**)$, we obtain that $\mathcal{F}(f)\in L^1(\mathcal{R})$. Let $\{f_m : m\in\mathcal{N}\}$ be the approximating sequence, given by Lemma 0.32, then, as $f_m\in L^1(\mathcal{R})$, $\mathcal{F}(f_m)$ is continuous and, by Lemma 0.29, converges uniformly to $\mathcal{F}(f)$ on $\mathcal{R}\setminus\{0\}$. It follows that $\mathcal{F}(f)\in C^0(\mathcal{R}\setminus\{0\})$. As $f_m\in C^2(\mathcal{R})$ and $f_m''\in L^1(\mathcal{R})$, we have that there exists constants $D_m\in\mathcal{R}_{>0}$, such that $|\mathcal{F}(f_m)(k)\leq\frac{D_m}{|k|^2}$, for sufficiently large k. Moreover, as $x^nf_m(x)\in L^1(\mathcal{R})$, for $n\in\mathcal{N}$, $\mathcal{F}(f_m)\in C^\infty(\mathcal{R})$. It follows, the Fourier inversion theorem $f_m=\mathcal{F}^{-1}(\mathcal{F}(f_m))$, $(***)$, holds for each f_m, see the proof in [13]. By Lemma 0.29, we have that, for $k\in\mathcal{R}\setminus\{0\}$, $|\mathcal{F}(f)(k)-\mathcal{F}(f_m)(k)|\leq\frac{E}{m}$. Then, for $n\in\mathcal{N}$, $m\in\mathcal{N}$, with $m=n^{\frac{3}{2}}$, using Lemma 0.32, we have, for $x\in\mathcal{R}$, that;

$|\mathcal{F}^{-1}(\mathcal{F}(f))(x)-\mathcal{F}^{-1}(\mathcal{F}(f_m))(x)|=|\mathcal{F}^{-1}(\mathcal{F}(f)(k)-\mathcal{F}(f_m)(k))|$

$=\frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{-n}^{n}(\mathcal{F}(f)(k)-\mathcal{F}(f_m)(k))e^{ikx}dk+\int_{|k|>n}(\mathcal{F}(f)(k)-\mathcal{F}(f_m)(k))e^{ikx}dk|$

$\leq\frac{1}{(2\pi)^{\frac{1}{2}}}(\int_{-n}^{n}|\mathcal{F}(f)(k)-\mathcal{F}(f_m)(k)|dk+\int_{|k|>n}|\mathcal{F}(f)(k)|dk+\int_{|k|>n}|\mathcal{F}(f_m)(k)|dk)$

$\leq\frac{1}{(2\pi)^{\frac{1}{2}}}(\frac{2nE}{m}+\int_{|k|>n}\frac{C}{|k|^2}dk+\int_{|k|>n}\frac{Cm}{|k|^3}dk)$

$\leq\frac{1}{(2\pi)^{\frac{1}{2}}}(\frac{2n}{n^{\frac{3}{2}}}+\frac{2C}{n}+\frac{Cn^{\frac{3}{2}}}{n^2})$

$<\epsilon$

for sufficiently large n, so that, as $\epsilon>0$ was arbitrary, for $x\in\mathcal{R}$;

$lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(x)=\mathcal{F}^{-1}\mathcal{F}(f)(x)$, $(****)$

and, by Definition 0.28, $(***)$, $(****)$;

$$f(x)=lim_{m\to\infty}f_m(x)=lim_{m\to\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(x)=\mathcal{F}^{-1}\mathcal{F}(f)(x)$$

□

Remarks 0.34. *The previous lemma proves an inversion theorem for non-oscillatory functions with very moderate decrease. Such functions belong to* $L^2(\mathcal{R})$ *and an analogous result for Fourier series can be found in* [2], *where convergence is proved almost everywhere rather than everywhere. The corresponding result for transforms is that;*

If $f\in L^p(\mathcal{R})$, $p\in(1,2]$, *then;*

$$f(x)=lim_{R\to\infty}\frac{1}{2\pi}\int_{|k|\leq R}\mathcal{F}(f)(k)e^{ixk}dk$$

for almost every $x\in\mathcal{R}$.

There is also a converse result, which can be found in [7], *but is left as an exercise;*

If $f\in L^1(\mathcal{R})\cap C^0(\mathcal{R})$ *and* $|\mathcal{F}(f)(k)|\leq\frac{A}{|k|}$, *for all* $k\neq 0$, *and* $A\in\mathcal{R}_{\geq 0}$, *then;*

$$f(x)=lim_{R\to\infty}\frac{1}{2\pi}\int_{|k|\leq R}\mathcal{F}(f)(k)e^{ixk}dk$$

for every $x\in\mathcal{R}$.

Lemma 0.35. *Let* $f\in C(\mathcal{R}_{>0})$ *and* $\frac{df}{dx}\in C(\mathcal{R}_{>0})$ *be of very moderate decrease, with* f *and* $\frac{df}{dx}$ *non-oscillatory, and* $lim_{x\to 0}f(x)=0$, $lim_{x\to 0}\frac{df}{dx}=M$, *with* $M\in\mathcal{R}$, *then defining the half Fourier transform* $\mathcal{G}$ *by;*

$$\mathcal{G}(f)(k)=lim_{r\to\infty}\int_0^r f(y)e^{-iky}dy\ (k\neq 0)$$

$$\mathcal{G}(\tfrac{df}{dx})(k)=lim_{r\to\infty}\int_0^r \frac{df}{dx}(y)e^{-iky}dy\ (k\neq 0)$$

we have that $\mathcal{G}(f)$ *and* $\mathcal{G}(\frac{df}{dx})$ *are bounded for* $|k|\geq k_0>0$, *and there exists a constant* $G\in\mathcal{R}_{>0}$, *such that;*

$$|\mathcal{G}(f)(k)|\leq\frac{G}{|k|^2}$$

for sufficiently large k.

Proof. As f is of very moderate decrease, and $lim_{x\rightarrow 0}f(x)=0$, we have that f is bounded and $lim_{x\rightarrow\infty}f(x)=0$. Similarly, $\frac{df}{dx}$ is bounded and $lim_{x\rightarrow\infty}\frac{df}{dx}=0$. As $lim_{x\rightarrow\infty}f(x)=0$, and f is non-oscillatory, we have that, for $k\neq 0$, the indefinite integral;

$$lim_{r\rightarrow\infty}\int_0^r f(y)e^{-iky}dy$$

$$=lim_{r\rightarrow\infty}\int_0^r f(y)cos(ky)dy-ilim_{r\rightarrow\infty}\int_0^r f(y)sin(ky)dy$$

exists. As f is of very moderate decrease and non-oscillatory, we have that $|f(x)|\leq\frac{D}{x}$, for $x>E$, with $E\in\mathcal{R}_{>0}$, and monotone in the interval (E,∞). Using the method of [10], letting $K=max(|f||_{(0,E]})$, we have that;

$$|lim_{r\rightarrow\infty}\int_{-r}^r f(y)cos(ky)dy|\leq KE+(|\int_E^{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}\frac{Dcos(ky)}{y}dy|+|\int_{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}^{\frac{\pi}{2|k|}+\frac{(n_k+1)\pi}{|k|}}\frac{Dcos(ky)}{y}dy|)$$

$$\leq KE+\frac{D}{E}(\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}-E)+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$\leq KE+\frac{D\pi}{E|k|}+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$|lim_{r\rightarrow\infty}\int_{-r}^r f(y)sin(ky)dy|\leq KE+(|\int_E^{\frac{m_k\pi}{|k|}}\frac{Dsin(ky)}{y}dy|+|\int_{\frac{m_k\pi}{|k|}}^{\frac{(m_k+1)\pi}{|k|}}\frac{Dsin(ky)}{y}dy|)$$

$$\leq KE+\frac{D}{E}(\frac{m_k\pi}{|k|}-E)+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$\leq KE+\frac{D\pi}{E|k|}+\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

where $n_k=\mu n(\frac{\pi}{2|k|}+\frac{n\pi}{|k|}\geq E:n\in\mathcal{Z}_{\geq 0})$ and $m_k=\mu n(\frac{n\pi}{|k|}\geq E:n\in\mathcal{Z}_{\geq 0})$

so that;

$$|lim_{r\rightarrow\infty}\int_0^r f(y)e^{-iky}dy|\leq 2KE+\frac{2D\pi}{E|k|}+2\int_E^{E+\frac{\pi}{|k|}}\frac{Dsin(|k|(y-E))}{y}dy$$

$$=2KE+\frac{2D\pi}{E|k|}+2D([\frac{-cos(|k|(y-E))}{|k|y}]_E^{E+\frac{\pi}{|k|}}-\int_E^{E+\frac{\pi}{|k|}}\frac{cos(|k|(y-E))}{y^2}dy)$$

$$=2KE+\frac{2D\pi}{E|k|}+2D(\frac{1}{|k|(E+\frac{\pi}{|k|})}+\frac{1}{E|k|}+\int_E^{E+\frac{\pi}{|k|}}\frac{cos(|k|(y-E))}{y^2}dy)$$

$$\leq 2KE+\frac{2D\pi}{E|k|}+2D(\frac{1}{E|k|+\pi}+\frac{1}{E|k|}+\int_E^{\infty}\frac{1}{y^2}dy)$$

$\leq 2KE + \frac{2D\pi}{E|k|} + 2D(\frac{2}{E|k|} + \frac{1}{E}) = N_k$

Alternatively, letting $F = max(|f||_{(0,\infty)}$, we have that;

$|lim_{r\rightarrow\infty} \int_{-r}^{r} f(y)cos(ky)dy| \leq FE + (|\int_{E}^{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}} Fcos(ky)dy| + |\int_{\frac{\pi}{2|k|}+\frac{n_k\pi}{|k|}}^{\frac{\pi}{2|k|}+\frac{(n_k+1)\pi}{|k|}} Fcos(ky)dy|)$

$\leq FE + F(\frac{\pi}{2|k|} + \frac{n_k\pi}{|k|} - E) + \int_{E}^{E+\frac{\pi}{|k|}} Fsin(|k|(y-E))dy$

$\leq FE + \frac{F\pi}{2|k|} + \int_{E}^{E+\frac{\pi}{|k|}} Fsin(|k|(y-E))dy$

$\leq FE + \frac{F\pi}{2|k|} + \int_{E}^{E+\frac{\pi}{|k|}} Fdy$

$\leq FE + \frac{F\pi}{2|k|} + \frac{F\pi}{|k|}$

$= FE + \frac{3F\pi}{2|k|}$

$|lim_{r\rightarrow\infty} \int_{-r}^{r} f(y)sin(ky)dy| \leq FE + (|\int_{E}^{\frac{m_k\pi}{|k|}} Fsin(ky)dy| + |\int_{\frac{m_k\pi}{|k|}}^{\frac{(m_k+1)\pi}{|k|}} Fsin(ky)dy|)$

$\leq FE + F(\frac{m_k\pi}{|k|} - E) + \int_{E}^{E+\frac{\pi}{|k|}} Fsin(|k|(y-E))dy$

$\leq FE + \frac{F\pi}{2|k|} + \int_{E}^{E+\frac{\pi}{|k|}} Fsin(|k|(y-E))dy$

$\leq FE + \frac{F\pi}{2|k|} + \frac{F\pi}{|k|}$

$= FE + \frac{3F\pi}{2|k|}$

so that;

$|lim_{r\rightarrow\infty} \int_{0}^{r} f(y)e^{-iky}dy| \leq 2FE + \frac{3F\pi}{|k|} = M_k$

In either case, $\mathcal{G}(f)(k)$ and, similarly, $\mathcal{G}(\frac{df}{dx})(k)$ are bounded, for $|k| > k_0 > 0$, ([2]).

We have, using integration by parts, that;

$\mathcal{G}(\frac{df}{dx})(k) = lim_{r\rightarrow\infty} \int_{0}^{r} \frac{df}{dx}(y)e^{-iky}dy$

$= lim_{r\rightarrow\infty}([f(y)e^{-iky}]_{0}^{r} + ik\int_{0}^{r} f(y)e^{-iky}dy)$

[2] $\mathcal{G}(f)(k)$ and $\mathcal{G}(\frac{df}{dx})(k)$ need not be differentiable or even continuous for $k \neq 0$, but see Lemma 0.33 for continuity on $\mathcal{R} \setminus \{0\}$ with stronger assumptions.

$= [f(y)e^{-iky}]_0^\infty + iklim_{r\rightarrow\infty}\int_0^r f(y)e^{-iky}dy$

$= ik\mathcal{G}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|}$, $(\dagger)$

As $\frac{df}{dx}$ is continuous, non-oscillatory and bounded, by the proof of Lemma 0.9 in [10], using underflow, for $r \in \mathcal{R}_{>0}$, we can find $\{F_r, G_r\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > F_r$, we have that;

$|\int_0^r \frac{df}{dx}(y)e^{-iky}dy| < \frac{G_r}{|k|}$, $(**)$

It is easy to see from the proof, that $\{F_r, G_r\}$ can be chosen uniformly in r. Then, from $(**)$, we obtain that, for $|k| > F$;

$|\mathcal{G}(\frac{df}{dx})(k)| < \frac{G}{|k|}$, for $|k| > F$

and, from $(\dagger)$, for $|k| > max(F, 1)$, that;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|} < \frac{G}{|k|^2}$

□

Definition 0.36. *We say that $h : \mathcal{R} \rightarrow \mathcal{C}$ is near analytic if for any $\delta > 0$, $0 < \epsilon < L$, there exists g_1 analytic on (ϵ, L), such that $|h(y) - g_1(y)| < \delta$, and for any $\delta > 0$, $-L < -\epsilon < 0$, there exists g_2 analytic on $(-L, -\epsilon)$, such that $|h(y) - g_2(y)| < \delta$*

Lemma 0.37. *If f satisfies the conditions of Lemma 0.33 or Lemma 0.38, then $\mathcal{F}(f)$ is near analytic. If f satisfies the conditions of Lemma 0.38, then $k\mathcal{F}(f)(k)$ and $\frac{d}{dk}(k\mathcal{F}(f)(k))$ are near analytic.*

Proof. By the proof of Lemma 0.33, we have that $\mathcal{F}(f) \in C^0(\mathcal{R} \setminus \{0\})$. By the Stone-Weierstrass approximation theorem, we can find a polynomial $p_{\epsilon,L,\delta}$ such that $|\mathcal{F}(f)(y) - p_{\epsilon,L,\delta}(y)| < \delta$, for $y \in [\epsilon, L]$. Similarly, we can find a polynomial $p_{-\epsilon,-L,\delta}$ such that $|\mathcal{F}(f)(y) - p_{-\epsilon,-L,\delta}(y)| < \delta$, for $y \in [-L, -\epsilon]$. In particularly, as $p_{\epsilon,L,\delta}$ and $p_{-\epsilon,-L,\delta}$ are analytic on (ϵ, L) and $(-L, -\epsilon)$ respectively, $\mathcal{F}(f)$ is near analytic. The same proof applies if f satisfies the conditions of Lemma 0.38. Similarly, in Lemma 0.38, we have that $k\mathcal{F}(f)(k) \in C^1(\mathcal{R} \setminus \{0\})$, so that $k\mathcal{F}(f)(k) \in C^0(\mathcal{R} \setminus \{0\})$ and $\frac{d}{dk}(k\mathcal{F}(f)(k)) \in C^0(\mathcal{R} \setminus \{0\})$, so we can apply the above proof again.

□

Lemma 0.38. *Let f satisfy the conditions of Lemma 0.33 with the extra assumption that $f \in C^4(\mathcal{R})$, $\{\frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}, \frac{d^4f}{dx^4}\}$ are of moderate decrease, then $k\mathcal{F}(f)(k) \in C^1(\mathcal{R} \setminus \{0\})$, $lim_{k\rightarrow 0}k\mathcal{F}(f)(k) = 0$, for any given $\epsilon > 0$, there exists $\delta > 0$, such that;*

$$max(|\int_0^\delta k\mathcal{F}(f)dk|, |\int_0^\delta \frac{d(k\mathcal{F}(f))}{dk}dk|) < \epsilon$$

Proof. We have that, for $k \neq 0$, as $\frac{df}{dx}$ is of moderate decrease, f is of very moderate decrease and non-oscillatory, and using integration by parts;

$$\mathcal{F}(\frac{df}{dx})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathcal{R}} \frac{df}{dx}(y)e^{-iky}dy$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} lim_{r\rightarrow\infty} \int_{-r}^{r} \frac{df}{dx}(y)e^{-iky}dy$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} lim_{r\rightarrow\infty}([fe^{-iky}]_{-r}^{r} + ik\int_{-r}^{r} f(y)e^{-iky}dy)$$

$$= ik\mathcal{F}(f)(k)$$

so that, for $k \neq 0$, $k\mathcal{F}(f)(k) = -i\mathcal{F}(\frac{df}{dx})(k)$. It follows that, using the MCT, the FTC and the fact that f is of very moderate decrease;

$$lim_{k\rightarrow 0}k\mathcal{F}(f)(k) = -ilim_{k\rightarrow 0}\mathcal{F}(\frac{df}{dx})(k)$$

$$= -\frac{i}{(2\pi)^{\frac{1}{2}}} lim_{k\rightarrow 0} \int_{\mathcal{R}} \frac{df}{dx}(y)e^{-iky}dy$$

$$= -\frac{i}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{R}} \frac{df}{dx}(y)dy$$

$$= -\frac{i}{(2\pi)^{\frac{1}{2}}}[f]_{-\infty}^{\infty}$$

$$= 0$$

As $f \in C^4(\mathcal{R})$, we have that $x\frac{df}{dx} \in C^3(\mathcal{R})$. Moreover, as $\{f, \frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}\}$ are of very moderate decrease, we have that;

$|x\frac{df}{dx}| \leq C_0$

$|\frac{d}{dx}(x\frac{df}{dx})| = |\frac{df}{dx} + x\frac{d^2f}{dx^2}| \leq C_1$

$|\frac{d^2(x\frac{df}{dx})}{dx^2}| = |2\frac{d^2f}{dx^2} + x\frac{d^3f}{dx^3}| \leq C_2$

$|\frac{d^3(xf)}{dx^3}| = |3\frac{d^3f}{dx^3} + x\frac{d^4f}{dx^4}| \leq C_3$

so that $\{x\frac{df}{dx}, \frac{d(x\frac{df}{dx})}{dx}, \frac{d^2(x\frac{df}{dx})}{dx^2}, \frac{d^3(x\frac{df}{dx})}{dx^3}\}$ are bounded. By Lemma 0.32, there exists an approximating sequence g_m, $m \in \mathcal{N}$, for $x\frac{df}{dx}$, with the properties that;

(i). $g_m \in C^2(\mathcal{R})$.

(ii). $g_m|_{[-m,m]} = x\frac{df}{dx}|_{[-m,m]}$

(iii). $\int_{m<|x|<m+\frac{1}{m}} |g_m(x)|dx \leq \frac{D}{m}$

(iv). $g_m|_{|x|>m+\frac{1}{m}} = 0$

Then $f_m = \frac{g_m}{x}$ is an approximating sequence for $\frac{df}{dx}$, with the properties that;

$(i)'$. $f_m \in C^2(\mathcal{R})$.

$(ii)'$. $f_m|_{[-m,m]} = \frac{df}{dx}|_{[-m,m]}$

$(iii)'$. $\int_{m<|x|<m+\frac{1}{m}} |f_m(x)|dx \leq \frac{D}{m^2}$

$(iv)'$. $f_m|_{|x|>m+\frac{1}{m}} = 0$

Following through the proof of Lemma 0.33, we have that $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(\frac{df}{dx})$ on $\mathcal{R} \setminus \{0\}$ and $\mathcal{F}(\frac{df}{dx}) \in C(\mathcal{R} \setminus \{0\})$. As $x^2 f_m = x g_m$ and $x g_m \in L^1(\mathcal{R})$, we have that $\mathcal{F}(f_m$ is twice differentiable, in particularly $\mathcal{F}(f_m) \in C^1(\mathcal{R})$. As f is analytic at infinity, so is $\frac{df}{dx}$. Moreover, as $\frac{df}{dx}$ is of very moderate decrease, $|\frac{df}{dx}|_{\frac{1}{z}}| = |g(z)| \leq C|z|^2$, so that $\frac{g(z)}{z}$ has a removable singularity at 0 and $x\frac{df}{dx}$ is analytic at infinity and non-oscillatory. We have that, for $\{m,n\} \subset \mathcal{N}$, $m \geq n$, differentiating under the integral sign, using the MCT, property (iv) of an approximating sequence, and the fact that $x\frac{df}{dx}$ is of moderate decrease and non-oscillatory, for $k \neq 0$, $\alpha < |k| < \beta$;

$|\frac{d\mathcal{F}(f_m)}{dk} - \frac{d\mathcal{F}(f_n)}{dk}|$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}|\frac{d}{dk}\int_{\mathcal{R}} f_m(y)e^{-iky}dy - \frac{d}{dk}\int_{\mathcal{R}} f_n(y)e^{-iky}dy|$

$= \frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{\mathcal{R}} -iyf_m(y)e^{-iky}dy - \int_{\mathcal{R}} -iyf_n(y)e^{-iky}dy|$

$\leq \frac{1}{(2\pi)^{\frac{1}{2}}}|\int_{\mathcal{R}}(g_m - g_n)(y)e^{-iky}|dy$

$\leq \frac{1}{(2\pi)^{\frac{1}{2}}}(\int_{m<|y|<m+\frac{1}{m}} |g_m|(y)dy + \int_{n<|y|<n+\frac{1}{n}} |g_n|(y)dy + |\int_{n<|y|<m} y\frac{df}{dx}(y)e^{-iky}dy|)$

$\leq \frac{D}{m} + \frac{D}{n} + \frac{C(k)}{n}$ $(*)$

where $C(k)$ is uniformly bounded in the interval $\alpha < |k| < \beta$, so that the sequence $\{\frac{d\mathcal{F}(f_m)}{dk} : m \in \mathcal{N}\}$ is uniformly Cauchy on the interval $\alpha < |k| < \beta$ and converges uniformly on $\mathcal{R} \setminus \{0\}$. As $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(\frac{df}{dx})$ on $\mathcal{R} \setminus \{0\}$, it follows that $\mathcal{F}(\frac{df}{dx}) \in C^1(\mathcal{R} \setminus \{0\})$. As $\mathcal{F}(f) \in L^2(\mathcal{R})$, we have that;

$|\int_0^\delta k\mathcal{F}(f)(k)dk| \leq (\int_0^\delta k^2 dk)^{\frac{1}{2}}||\mathcal{F}(f)||_{L^2(\mathcal{R})}$

$\leq \frac{\delta^{\frac{3}{2}}}{\sqrt{3}}||\mathcal{F}(f)||_{L^2(\mathcal{R})}$

$< \epsilon$

for $\delta < (\frac{\sqrt{3}\epsilon}{||\mathcal{F}(f)||_{L^2(\mathcal{R})}})^{\frac{2}{3}}$

We have that $\frac{d\mathcal{F}(f_m)}{dk} = -i\mathcal{F}(g_m)$ and, as $x\frac{df}{dx} \in L^2(\mathcal{R})$, by a similar calculation to $(*)$, $\{g_m : m \in \mathcal{N}\}$ is a Cauchy sequence with respect to the L^2-norm, so that $\frac{d\mathcal{F}(f_m)}{dk}$ is Cauchy in $L^2(\mathcal{R})$. As $\mathcal{F}$ is an L^2-isometry, using the fact that the limit $\frac{d}{dk}(\mathcal{F}(\frac{df}{dx})(k))$ is of moderate decrease, by Lemma 0.27 and the fact that $x\frac{df}{dx}$ and $\frac{d}{dx}(x\frac{df}{dx})$ are of very moderate decrease and non-oscillatory, we can ignore the tail and use uniform convergence implying L^2-convergence on sets of the form $0 < k_0 < |k| < k_1$. It follows that the pointwise limit $\frac{d}{dk}(\mathcal{F}(\frac{df}{dx})(k)) \in L^2(\mathcal{R})$ and $\frac{d}{dk}(k\mathcal{F}(f)(k)) \in L^2(\mathcal{R})$ as well. We then have that;

$|\int_0^\delta \frac{d}{dk}(k\mathcal{F}(f)(k))dk| \leq \frac{\delta^{\frac{3}{2}}}{\sqrt{3}}||\frac{d}{dk}k\mathcal{F}(f)(k)||_{L^2(\mathcal{R})}$

$< \epsilon$

for $\delta < (\frac{\sqrt{3}\epsilon}{||\frac{d}{dk}(k\mathcal{F}(f)(k))||_{L^2(\mathcal{R})}})^{\frac{2}{3}}$

□

Lemma 0.39. *Let f be light symmetrically asymptotic, then defining $\mathcal{F}(f)$ and $\mathcal{F}(\frac{df}{dx})$ as in Lemma 0.27, we have that, for any $\delta > 0$, there exist constants $\{C_\delta, D_\delta\} \subset \mathcal{R}_{>0}$, such that;*

$|\mathcal{F}(f)(k)| \leq \frac{\delta}{|k|} + \frac{C_\delta}{|k|^2}$, *for* $|k| > D_\delta$

Proof. The proof is a simple generalisation of the proofs of Lemmas 0.26 and 0.27. □

Definition 0.40. *Polars Attempt We say that $g \in C^\infty(\mathcal{R}^3)$ is polar non-oscillatory if, for $0 \leq \theta < \pi$, $-\pi < \phi \leq \pi$, we have that, for $g_{\theta,\phi}$, there exist finitely many point $\{r_{i,\theta,\phi} : 1 \leq i \leq n\} \subset \mathcal{R}_{>0}$, for which $g_{\theta,\phi}|_{(r_{i,\theta,\phi}, r_{i+1,\theta,\phi})}$ is monotone, $2 \leq i \leq n-2$, and $g_{\theta,\phi}|_{(0,r_{1,\theta,\phi})}$ and $g_{\theta,\phi}|_{(r_{n,\theta,\phi},\infty)}$ is monotone. We say that g is polar decaying if, for $0 \leq \theta < \pi$, $-\pi < \phi \leq \pi$, we have that, there exist constants $\{C, D\} \subset \mathcal{R}_{>0}$, such that $|g_{\theta,\phi}(r)| \leq \frac{D}{r^3}$, for $|r| \geq C$.*

Lemma 0.41. *If g is polar non-oscillatory and decaying, we can define;*

$$\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{R\rightarrow\infty} \int_0^R \int_0^\pi \int_{-\pi}^\pi r^2 sin(\theta) g(r,\theta,\phi) e^{-ir(k_1 sin(\theta)cos(\phi) + k_2 sin(\theta) sin(\phi) + k_3 cos(\theta))} dr d\theta d\phi$$

in polar coordinates, $x_1 = rsin(\theta)cos(\phi)$, $x_2 = rsin(\theta)sin(\phi)$, $x_3 = rcos(\theta)$.

where $\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} g(\overline{x}) e^{-i\overline{k}\centerdot\overline{x}} d\overline{x}$

is usually defined for $g \in L^1(\mathcal{R}^3)$.

Proof. We can assume that n is minimal with this property, in which case, the points $\{r_{i,\theta,\phi} : 1 \leq i \leq n\} \subset \mathcal{R}_{>0}$ are local maxima or minima, and, as $g \in C^\infty(\mathcal{R}^3)$, we have that $\frac{\partial g}{\partial r}|_{(r_{i,\theta,\phi},\theta,\phi)} = 0$. By the implicit function theorem, we can find smooth maps $\lambda_i : S^2(1) \rightarrow \mathcal{R}^3$, $1 \leq i \leq n$, such that $Im(\lambda_i) \subset \frac{\partial f}{\partial r} = 0$ and $\{r_{i,\theta,\phi} : 0 \leq \theta < \pi, -\pi < \phi \leq \pi\} = Im(\lambda_i)$. As $S^2(1)$ is compact, we have that $pr_r(Im(\lambda_i))$ defines a closed bounded interval $I_i \subset mathcalR_{>0}$. Moreover, it is straightforward to see, as each $g_{\theta,\phi}$ is a function, that $I_i \cap I_j = \emptyset$, for

$1 \leq i < j \leq n$. Let $m = max(\bigcup_{1\leq i\leq n} I_i)$. Then $g_{\theta,\phi}|_{(m,\infty)}$ is monotone and, if $M = max(|f||_{B(\overline{0},m)})$, as g is decaying, we have that $|g| \leq M$, so that g is bounded. Let $\overline{k} \in \mathcal{R}^3$, with $\overline{k} \neq \overline{0}$, then there exists an open $U_{\overline{k}} \subset (0,\pi) \times (-\pi,\pi)$ with $\overline{k} \cdot (sin(\theta)cos(\phi), sin(\theta)sin(\phi), cos(\theta)) = \nu_{\theta,\phi} \neq 0$, for $(\theta,\phi) \in U_{\overline{k}}$. Let $f_{\theta,\phi} = r^2 sin(\theta) g_{\theta,\phi}$, then $|f_{\theta,\phi}| \leq \frac{D}{r}$, for $r > C$, and the same remarks above, apply to $f_{\theta,\phi}$, as to $g_{\theta,\phi}$.

As $lim_{r\rightarrow\infty} f_{\theta,\phi}(r) = 0$, we have that the indefinite integral;

$$lim_{r\rightarrow\infty} \int_0^r f_{\theta,\phi}(r) e^{-ir(k_1 sin(\theta)cos(\phi)+k_2 sin(\theta)sin(\phi)+k_3 cos(\theta))} dr$$

$$= lim_{r\rightarrow\infty} \int_0^r f_{\theta,\phi}(r) e^{-ir\nu_{\theta,\phi}} dr$$

$$= lim_{r\rightarrow\infty} \int_0^r f_{\theta,\phi}(r) cos(r\nu_{\theta,\phi}) dr - i lim_{r\rightarrow\infty} \int_0^r f_{\theta,\phi}(r) sin(r\nu_{\theta,\phi}) dr$$

exists. As $f_{\theta,\phi}$ is monotone, and $|f_{\theta,\phi}|(r) \leq \frac{D}{r}$, for $r > max(m, C) = E$, using the method of [10], letting $K = max(|f||_{B(\overline{0},E)})$, we have that;

$$|lim_{R\rightarrow\infty} \int_0^R f_{\theta,\phi}(r) cos(r\nu_{\theta,\phi}) dr| \leq KE + K \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{D cos(|\nu_{\theta,\phi}|(r-E))}{r} dr$$

$$|lim_{R\rightarrow\infty} \int_0^R f_{\theta,\phi}(r) sin(r\nu_{\theta,\phi}) dr| \leq KE + K \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{D cos(|\nu_{\theta,\phi}|(r-E))}{r} dr$$

so that;

$$|lim_{R\rightarrow\infty} \int_0^r f_{\theta,\phi}(r) e^{-ir\nu_{\theta,\phi}} dr| \leq 2KE + 2K \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{D cos(|\nu_{\theta,\phi}|(r-E))}{r} dr$$

$$= 2KE + 2KD\left(\left[\frac{-sin(|\nu_{\theta,\phi}|(r-E))}{|\nu_{\theta,\phi}|}\right]_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} - \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{sin(|\nu_{\theta,\phi}|(r-E))}{r^2} dr\right)$$

$$= 2KE + 2KD\left(\frac{1}{|\nu_{\theta,\phi}|(E+\frac{\pi}{2|\nu_{\theta,\phi}|})} - \int_E^{E+\frac{\pi}{2|\nu_{\theta,\phi}|}} \frac{sin(|\nu_{\theta,\phi}|(r-E))}{r^2} dr\right)$$

$$\leq 2KE + 2KD\left(\frac{1}{E|\nu_{\theta,\phi}|+\frac{\pi}{2}} + \int_E^{\infty} \frac{1}{r^2} dr\right)$$

$$\leq 2KE + 2KD\left(\frac{2}{\pi} + \frac{1}{E}\right) = N$$

uniformly, for $(\theta,\phi) \in U_{\overline{k}}$, so that, using the dominated convergence theorem;

$$lim_{R\rightarrow\infty} \int_0^R \int_0^\pi \int_{-\pi}^\pi f(r,\theta,\phi) e^{-ir(k_1 sin(\theta)cos(\phi)+k_2 sin(\theta)sin(\phi)+k_3 cos(\theta))} dr d\theta d\phi$$

exists, and;

$$|lim_{R\rightarrow\infty}\int_0^R\int_0^\pi\int_{-\pi}^\pi f(r,\theta,\phi)e^{-ir(k_1 sin(\theta)cos(\phi)+k_2 sin(\theta)sin(\phi)+k_3 cos(\theta))}drdrd\theta d\phi|$$

$$\leq \int_0^\pi\int_{-\pi}^\pi |(lim_{R\rightarrow\infty}\int_0^R f(r,\theta,\phi)dr|)d\theta d\phi$$

$$\leq 2N\pi^2$$

□

Definition 0.42. *Cartesian We say that $g \in C^\infty(\mathcal{R}^3)$ is Cartesian non-oscillatory if, for $(x,y) \subset \mathcal{R}^2$, there exist finitely many point $\{z_{i,x,y} : 1 \leq i \leq n\} \subset \mathcal{R}$, for which $g_{x,y}|_{(z_{i,x,y},z_{i+1,x,y})}$ is monotone, $2 \leq i \leq n-2$, and $g_{x,y}|_{(-\infty,z_{1,x,y})}$ and $g_{x,y}|_{(z_{n,x,y},\infty)}$ is monotone, and, for fixed $(x,y) \in \mathcal{R}^2$, with $(x,y) \neq (0,0)$, the ordering of $\{g_{rx,ry}(z_{i,rx,ry}) : 1 \leq i \leq n\}$ changes, uniformly in (x,y), at most a finite number of times, with $r \in \mathcal{R}$. We say that $g \in C^\infty(\mathcal{R}^3)$ is slightly decaying if there exists a constant $C \in \mathcal{R}_{>0}$ with $|g(\overline{x})| \leq \frac{C}{|\overline{x}|}$, for $|\overline{x}| > 1$.*

Remarks 0.43. *The components of a causal field $\overline{E}$, obtained using Jefimenko's equations, are slightly decaying (and Cartesian non-oscillatory?) if the charge and current $(\rho,\overline{J})$ have compact support.*

Definition 0.44. *We say that $f : \mathcal{R} \rightarrow \mathcal{R}$ is analytic at infinity, if $f(\frac{1}{x})$ has a convergent power series expansion for $|x| < \epsilon$, $\epsilon > 0$. We say that f is eventually monotone, if there exists $y_0 \in \mathcal{R}_{>0}$ such that $f|_{(-\infty,-y_0)}$ and $f|_{(y_0,\infty)}$ are monotone. We say that $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ is analytic and analytic at infinity, if;*

(i). f is analytic on $\mathcal{R}^3$.

(ii) $f(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})$ has a convergent power series expansion for $|\overline{x}| < \epsilon_{(x_0,y_0,z_0)}$, $\epsilon_{(x_0,y_0,z_0)} > 0$, some $(x_0 : y_0 : z_0) \in P^2(\mathcal{R})$, where $\overline{x} = (x,y,z)$, $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$.

(iii). For every linear transformation T of $\mathcal{R}^3$, $(i),(ii)$ hold for $f\circ T$.

We say that f is analytic at infinity if (ii) holds and (iii) with the restriction that $T \in O(3)$, the orthogonal group.

Lemma 0.45. *$f : \mathcal{R} \rightarrow \mathcal{R}$ is analytic at infinity iff f extends to $\overline{f} : P^1(\mathcal{R}) \rightarrow \mathcal{R}$ analytic on an open neighborhood of the point at infinity. $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ is analytic and analytic at infinity iff f extends* *to* $\overline{f} : P^3(\mathcal{R}) \rightarrow \mathcal{R}$ *such that* $\overline{f}$ *is analytic on* $P^3(\mathcal{R})$ *and constant on* $P^3(\mathcal{R}) \setminus \mathcal{R}^3$. $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ *is analytic at infinity iff* f *extends to* $\overline{f} : P^3(\mathcal{R}) \rightarrow \mathcal{R}$ *such that* $\overline{f}$ *is analytic on an open neighborhood* U *of* $P^3(\mathcal{R}) \setminus \mathcal{R}^3$ *and constant on* $P^3(\mathcal{R}) \setminus \mathcal{R}^3$.

Proof. The first claim follows by observing that if f is analytic at infinity, we can extend f to $P^1(\mathcal{R}$ by defining $\overline{f}(\infty) = g(0$, where g is the analytic power series for $f(\frac{1}{x})$. On the chart $[1 : w]$, we have, for $w \neq 0$, that $\overline{f}([1 : w]) = f(\frac{1}{w}) = g(w)$, and for $w = 0$, $\overline{f}([1 : 0]) = \overline{f}(\infty) = g(0)$, so $\overline{f}$ is analytic in a neighborhood U of the point at infinity. Conversely, if $\overline{f}$ is analytic, then $f(\frac{1}{x}) = \overline{f}([\frac{1}{x} : 1]) = \overline{f}([1 : x])$ is analytic on a neighborhood of 0.

Secondly, observe that if (ii) is satisfied, then for a pair (x_0', y_0', z_0'), (x_0, y_0, z_0), with $x_0' \neq 0$, $y_0' \neq 0$, $z_0' \neq 0$, $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$, then;

$$lim_{w\rightarrow 0} f(\frac{x_0}{w}, \frac{y_0}{w}, \frac{z_0}{w}) = lim_{w\rightarrow 0} g_{x_0x_1x_2}(w, w, w)$$

$$= g_{x_0x_1x_2}(0, 0, 0)$$

$$lim_{w\rightarrow 0} f(\frac{x_0'}{w}, \frac{y_0'}{w}, \frac{z_0'}{w}) = lim_{w\rightarrow 0} f(\frac{x_0'x_0}{wx_0}, \frac{y_0'y_0}{wy_0}, \frac{z_0'z_0}{wz_0})$$

$$= lim_{w\rightarrow 0} f(\frac{x_0}{w\frac{x_0}{x_0'}}, \frac{y_0}{w\frac{y_0}{y_0'}}, \frac{z_0}{w\frac{z_0}{z_0'}})$$

$$= lim_{w\rightarrow 0} g_{x_0x_1x_2}(w\frac{x_0}{x_0'}, w\frac{y_0}{y_0'}, w\frac{z_0}{z_0'})$$

$$= g_{x_0x_1x_2}(0, 0, 0)$$

so that f has a well defined limit at $[x_0 : y_0 : z_0 : 0]$ for any triple $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$.

By (iii), the same is true for $f \circ T$, where T is a linear transformation, so that f has a well defined limit on $P^3(\mathcal{R}) \setminus \mathcal{R}^3$ and we can define an extension $\overline{f} : P^3(\mathcal{R}) \rightarrow P^3(\mathcal{R})$ which is constant on the boundary. We have, by (ii), that, for $w \neq 0$;

$$\overline{f}([1 : y : z : w]) = f(\frac{1}{w}, \frac{y}{w}, \frac{z}{w})$$

$$= f(\frac{1}{w}, \frac{1}{\frac{w}{y}}, \frac{1}{\frac{w}{z}})$$

$= g_{111}(w, \frac{w}{y}, \frac{w}{z})$

and $\overline{f}([1:y:z:0]) = g_{111}(0,0,0)$, so that $\overline{f}([1:y:z:w])$ is analytic for $y \neq 0$ and $z \neq 0$. It follows that $\overline{f}$ is analytic on the set $U \setminus Z$ where $S = (X=0) \cup (Y=0) \cup (Z=0)$, where U is open in $P^3(\mathcal{R})$. Similarly, by (iii), $\overline{f} \circ T = \overline{f \circ T}$ is analytic on $V \setminus S$, where V is open, so that $\overline{f}$ is analytic on $T^{-1}(V) \setminus T^{-1}(S)$. As the sets $U \setminus T(S)$ for U an open neighborhood cover $P^3(\mathcal{R}) \setminus [0:0:0:1]$, it follows that $\overline{f}$ is analytic on $P^3 \setminus [0:0:0:1]$. By (i), f is analytic on $P^3(\mathcal{R})$. Conversely, if $\overline{f}$ is analytic on $P^3(\mathcal{R})$, constant on the boundary $P^3(\mathcal{R}) \setminus \mathcal{R}^3$, let f be its restriction to $\mathcal{R}^3$. Obviously (i) is satisfied. Choose (x_0, y_0, z_0) with $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$, then, for $x \neq 0$, $y \neq 0$, $z \neq 0$;

$f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z}) = \overline{f}([x_0yz : y_0xz : z_0xy : xyz])$

$= \overline{f}([1 : \frac{y_0xz}{x_0yz} : \frac{z_0xy}{x_0yz} : \frac{xyz}{x_0yz}])$

$= \overline{f}([1 : \frac{y_0x}{x_0y} : \frac{z_0x}{x_0z} : \frac{x}{x_0}])$

$= g(\frac{y_0x}{x_0y}, \frac{z_0x}{x_0z}, \frac{x}{x_0})$

where g is analytic. So that $f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z})$ is analytic for $x \neq 0$, $y \neq 0$, $z \neq 0$. We have that, for any linear transformation T of $\mathcal{R}^3$, $\overline{f} \circ T$ is analytic, so that;

$f \circ T(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z}) = f(t_{11}\frac{x_0}{x} + t_{12}\frac{y_0}{y} + t_{13}\frac{z_0}{z}, t_{21}\frac{x_0}{x} + t_{22}\frac{y_0}{y} + t_{23}\frac{z_0}{z},$

$t_{31}\frac{x_0}{x} + t_{32}\frac{y_0}{y} + t_{33}\frac{z_0}{z})$

$= f(\frac{t_{11}x_0yz + t_{12}y_0xz + t_{13}z_0xy}{xyz}, \frac{t_{21}x_0yz + t_{22}y_0xz + t_{23}z_0xy}{xyz}, \frac{t_{31}x_0yz + t_{32}y_0xz + t_{33}z_0xy}{xyz})$

$= \overline{f}([t_{11}x_0yz + t_{12}y_0xz + t_{13}z_0xy : t_{21}x_0yz + t_{22}y_0xz + t_{23}z_0xy :$

$t_{31}x_0yz + t_{32}y_0xz + t_{33}z_0xy : xyz])$

$= \overline{f}([1 : \frac{t_{21}x_0yz + t_{22}y_0xz + t_{23}z_0xy}{t_{11}x_0yz + t_{12}y_0xz + t_{13}z_0xy} : \frac{t_{31}x_0yz + t_{32}y_0xz + t_{33}z_0xy}{t_{11}x_0yz + t_{12}y_0xz + t_{13}z_0xy} : \frac{xyz}{t_{11}x_0yz + t_{12}y_0xz + t_{13}z_0xy}])$

which is analytic for $t_{11}x_0yz + t_{12}y_0xz + t_{13}z_0xy \neq 0$, and considering the other charts, analytic for $t_{21}x_0yz + t_{22}y_0xz + t_{23}z_0xy \neq 0$ and $t_{31}x_0yz + t_{32}y_0xz + t_{33}z_0xy \neq 0$, which as T is invertible occurs if $yz \neq 0$

$xz \neq 0$, $xy \neq 0$ iff $x \neq 0$, $y \neq 0$, $z \neq 0$, so that $f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z})$ is analytic if $T^{-1}(x \neq 0)$, $T^{-1}(y \neq 0)$, $T^{-1}(z \neq 0)$, and as T was arbitrary, $f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z})$ is analytic except at $(0,0,0)$. By complexifying and using Laurent series, using the fact the limit at $(0,0,0)$ exists, $f(\frac{x_0}{x}, \frac{y_0}{y}, \frac{z_0}{z})$ is analytic. So (ii) holds. We can verify (iii) by verifying (ii) for $f \circ T$, which we can do by repeating the argument for (ii) and using the fact $\overline{f} \circ T$ is analytic. The second claim is left to the reader.

□

Lemma 0.46. *If $f : \mathcal{R} \rightarrow \mathcal{R}$, $f \neq 0$ is analytic and analytic at infinity, then it has finitely many zeroes. If $f : \mathcal{R} \rightarrow \mathcal{R}$, $\frac{df}{dx}$ is analytic and analytic at infinity, and $f \neq c$, where $c \in \mathcal{R}$, then f is non-oscillatory. If $f : \mathcal{R} \rightarrow \mathcal{R}$, f is analytic for $|x| > a$, where $a \in \mathcal{R}_{\geq 0}$, analytic at infinity, and $f|_{|x|>a} \neq 0$ then f has finitely many zeroes in the region $|x| > a+1$. If $f : \mathcal{R} \rightarrow \mathcal{R}$, $\frac{df}{dx}$ is analytic for $|x| > a$, analytic at infinity, and $f|_{|x|>a} \neq c$, where $c \in \mathcal{R}$, then f is eventually monotone. If $f : \mathcal{R}^3 \rightarrow \mathcal{R}$, $f \neq 0$ is analytic and analytic at infinity, then, for any line $l \subset \mathcal{R}^3$, $f|_l$ is either zero or has finitely many zeroes, moreover the number of zeros is uniformly bounded, independently of the choice of l. If $f : \mathcal{R}^3 \rightarrow \mathcal{R}$, $\frac{\partial f}{\partial x}$ is analytic and analytic at infinity, and if $\frac{\partial f}{\partial x} \neq 0$, then, for $(y,z) \in \mathcal{R}^2$, $f_{y,z}$ is non-oscillatory. If $f : \mathcal{R}^3 \rightarrow \mathcal{R}$, $\frac{\partial f}{\partial x}$ is analytic for $|\overline{x}| > a$, analytic at infinity, and $\frac{\partial f}{\partial x}|_{|\overline{x}|>a} \neq 0$, then, for $(y,z) \in \mathcal{R}^2$, $f_{y,z}$ is eventually monotone. A similar statement holds for $\{\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\}$, with $f_{x,z}$ and $f_{x,y}$ replacing $f_{y,z}$ respectively.*

Proof. For the first claim, suppose that f has infinitely many zeroes. Then we can find a sequence $\{y_i; i \in \mathcal{N}\}$ with $f(y_i) = 0$. If the sequence is bounded, then by the Bolzano-Weierstrass Theorem, we can find a subsequence $\{y_{i_k}; k \in \mathcal{N}\}$, with $f(y_{i_k}) = 0$, converging to $y \in \mathcal{R}$. By continuity, we have that $f(y) = 0$ and y is a limit point of zeroes. As f is analytic, by the identity theorem, it must be identically zero, contradicting the hypothesis. If the sequence is unbounded, then we can find a subsequence $\{y_{i_k}; k \in \mathcal{N}\}$, with $f(y_{i_k}) = 0$, such that $lim_{k \rightarrow \infty} y_{i_k} = \infty$ or $lim_{k \rightarrow \infty} y_{i_k} = -\infty$. As f is analytic at ∞, we can find $\epsilon > 0$, such that $f(y) = 0$ for $|y| > \frac{1}{\epsilon}$. By the identity theorem again, f is identically zero, contradicting the hypothesis. It follows that f has finitely many zeroes. For the second claim, as $\frac{df}{dx} \neq 0$, by the first part, there exist finitely many points $\{y_1, \ldots, y_n\}$, with $\frac{df}{dx}|_{y_i} = 0$, for $1 \leq i \leq n$, and with $y_i < y_{i+1}$, for $1 \leq i \leq n-1$. In particularly, we have that $f|_{(-\infty, y_1)}$, $f|_{(y_n, \infty)}$ and $f|_{(y_i, y_{i+1})}$ is monotone for

$1 \leq i \leq n-1$, so that f is non-oscillatory. For the third claim, suppose that f has infinitely many zeroes in the region $|x| > a+1$, then we can find a sequence $\{y_i; i \in \mathcal{N}\}$ with $f(y_i) = 0$ and $|y_i > a+1$. As above, if the sequence is bounded, we can find a subsequence $\{y_{i_k}; k \in \mathcal{N}\}$, with $f(y_{i_k}) = 0$, converging to $y \in \mathcal{R}$, with $|y| \geq a+1 > a$. As f is analytic for $|x| > a$, by the identity theorem, it must be identically zero in the region $|x| > a$, contradicting the hypothesis. If the sequence is unbounded, by the same argument as above, f must be identically zero in the region $|x| > a$, contradicting the hypothesis. It follows that f has finitely many zeroes in the region $|x| > a+1$. For the fourth claim, as $\frac{df}{dx}|_{|x|>a} \neq 0$, by the first part, there exist finitely many points $\{y_1, \ldots, y_n\}$, with $\frac{df}{dx}|_{y_i} = 0$, and $|y_i| > a+1$, for $1 \leq i \leq n$. Choose $y_0 > max_{1 \leq i \leq n}(|y_i|)$, then $\frac{df}{dx}|_{|x|>y_0} \neq 0$, so that $f|_{|x|>y_0}$ is monotone. For the fifth claim, we have, by Lemma 0.45 that f extends to $\overline{f}$ analytic on $P^3(\mathcal{R})$, constant on $P^3(\mathcal{R}) \setminus \mathcal{R}^3$. If l is a line in $P^3(\mathcal{R})$ such that, without loss of generality, $l \cap \mathcal{R}^3 \neq \emptyset$, passing through $p = [\epsilon_0 : \epsilon_1 : \epsilon_2 : 1]$, we can choose a chart U centred at p, with the line l corresponding to $x = y = 0$ and such that $\overline{f}$ is analytic on U, defined by a convergent power series $\sum_{i,j,k \geq 0} a_{ijk} x^i y^j z^k$. Substituting $x = y = 0$, we obtain a convergent power series on $l \cap U$, so that $f|_l$ is analytic at infinity. By a similar argument $f|_l$ is analytic, so that by the first claim, $f|_l$ is either zero or has finitely many zeroes. For the uniformity claim, let Z be the zero locus of $\overline{f}$ and consider the relation $R \subset P^3(\mathcal{R}) \times G_{1,3}(\mathcal{R})$ given by $R(x, l)$ iff $x \in Z \cap l$, where $G_{1,3}$ is the Grassmannian of lines in $P^3(\mathcal{R})$. By the previous proof R is a generically finite cover of $G_{1,3}(\mathcal{R})$, with fibres possibly equal to $P^1(\mathcal{R})$. Using the method of Lemma 0.19, we can apply Weierstrass preparation and compactness to get a uniform bound. For the sixth claim, the first part is proved in Lemma 0.19?, $\frac{\partial f_{y_0,z_0}}{\partial x}$ is given by the restriction of $\frac{\partial f}{\partial x}$ to the line $y = y_0, z = z_0$, so by the previous claim, has finitely many zeroes. For the seventh claim, the restriction of $\frac{\partial f}{\partial x}$ to lines l is analytic at infinity, in particularly, by the above, it cannot have an unbounded infinite sequence of zeroes so $f_{y,z}$ is eventually monotone. The last claim follows by symmetry.

□

Definition 0.47. *We say that $g : \mathcal{R}^3 \rightarrow \mathcal{R}$ is of very moderate decrease, if, there exists constants $C \in \mathcal{R}_{>0}$ and $s\imath\mathcal{R}_{>0}$, with;*

$|g(\overline{x})| \leq \frac{C}{|\overline{x}|}$

for $|\overline{x}| > s$, $\overline{x} \in \mathcal{R}^3$

We say that $f : \mathcal{R}^3 \times \mathcal{R} \rightarrow \mathcal{R}$ *is of uniform very moderate decrease, if, for all* $t \in \mathcal{R}$, *there exists a constants* $D \in \mathcal{R}_{>0}$ *and* $s \in \mathcal{R}_0$, *uniform in* t, *with;*

$|f(\overline{x}, t)| \leq \frac{D}{|\overline{x}|}$

for $|\overline{x}| > s$, $\overline{x} \in \mathcal{R}^3$

We say that $f : \mathcal{R}^3 \times \mathcal{R} \rightarrow \mathcal{R}$ *is of very moderate decrease, if, for all* $t \in \mathcal{R}$, *there exists a constants* $D \in \mathcal{R}_{>0}$ *and* $s_t \in \mathcal{R}_0$, *with;*

$|f(\overline{x}, t)| \leq \frac{D}{|\overline{x}|}$

for $|\overline{x}| > s_t$, $\overline{x} \in \mathcal{R}^3$

Lemma 0.48. *The components of the causal fields* $\overline{E}$ *and* $\overline{B}$, *obtained using Jefimenko's equations, are of uniform very moderate decrease and analytic for* $|\overline{x}| > r$, *and analytic at infinity, if, first, the charge and current* $(\rho, \overline{J})$ *are compactly supported and uniformly bounded with* $t \in \mathcal{R}$ *on a volume* $V \subset B(\overline{0}, w)$, *where* $w \in \mathcal{R}_{>0}$, *secondly, the charge* ρ *and the components* j_i *of the current* $\overline{J}$, *for* $1 \leq i \leq 3$, *are smooth and,third, the charge* ρ *and the components* j_i *of the current* $\overline{J}$, *for* $1 \leq i \leq 3$ *are analytic in* t. *If the initial conditions* $\rho_0 \in S(\mathcal{R}^3), \frac{\partial \rho}{\partial t}|_{t=0} \in S(\mathcal{R}^3)$, *with* ρ *defined on* $\mathcal{R}^4$ *by Kirchoff's formula, then* $\rho \in C^\infty(\mathcal{R}^4)$ *and if* ρ_0 *and* $\frac{\partial \rho}{\partial t}|_{t=0}$ *have compact support, then for* $t \in \mathcal{R}$, ρ_t *has compact support, in particularly* $\rho_t \in S(\mathcal{R}^3)$. *If the current* $\overline{J}$ *is defined as in* [14], *with the conditions in the last clause, then, after subtracting a harmonic, time independent, current* $\overline{J}_0(\overline{x})$, *the components* $j_i \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, *and for each* $t \in \mathcal{R}$, $j_{i,t}$ *has compact support and* $j_{i,t} \in S(\mathcal{R}^3)$. *Suppose that the charge* ρ, *obeys the wave equations on* $\mathcal{R}^4$, *with the current* $\overline{J}$ *defined as in* [14], *and with the initial conditions* $\rho_0 \in S(\mathcal{R}^3), \frac{\partial \rho}{\partial t}|_{t=0} \in S(\mathcal{R}^3)$ *and with compact support. Then the fields* $\{\overline{E}, \overline{B}\}$ *are well defined by Jefimenko's equations, as a limit of fields* $\{(\overline{E}_w, \overline{B}_w) : w \in \mathcal{R}_{>0}, w \neq c\}$ *and the components are of uniform very moderate decrease. If the components of the initial conditions* $\rho_0 \in S(\mathcal{R}^3)$ *and* $\frac{\partial \rho}{\partial t}|_{t=0} \in S(\mathcal{R}^3)$ *have compact support in* $B(\overline{0}, w)$, $w \in \mathcal{R}_{>0}$, *and the partial derivatives* $\{\frac{\partial^{i+j+k+l}\rho}{\partial x^i \partial y^j \partial z^k \partial t^l} : (i, j, k, l) \in \mathcal{Z}^4_{\geq 0}\}$ *are analytic in* t, *then the fields* $\{\overline{E}, \overline{B}\}$ *are analytic for* $|\overline{x}| > w$, *uniformly*

in t, and analytic at infinity. If the components of the initial conditions $\rho_0 \in S(\mathcal{R}^3)$ and $\frac{\partial\rho}{\partial t}|_{t=0} \in S(\mathcal{R}^3)$, then the above results hold without the compact support claims, and if $\{\rho_0, (\frac{\partial\rho}{\partial t})_0$ are analytic, then the fields $\{\overline{E}, \overline{B}\}$ are analytic for $|\overline{x}| > w$, uniformly in t, and analytic at infinity.
.

Proof. For the first claim, we have that;

$$|\overline{E}(\overline{r},t)| = \frac{1}{4\pi\epsilon_0}|\int_V \frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau' - \int_V \frac{\dot{\overline{J}}(\overline{r}',t_r)}{c^2|\overline{r}-\overline{r}'|}d\tau'|$$

$$\leq \frac{1}{4\pi\epsilon_0}(\int_V \frac{C_1}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3}{c^2|\overline{r}-\overline{r}'|}d\tau')$$

$$= \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1|\overline{r}|}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2|\overline{r}|}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3|\overline{r}|}{c^2|\overline{r}-\overline{r}'|}d\tau')$$

$$= \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1|\overline{r}-\overline{r}'+\overline{r}'|}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2|\overline{r}-\overline{r}'+\overline{r}'|}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3|\overline{r}-\overline{r}'+\overline{r}'|}{c^2|\overline{r}-\overline{r}'|}d\tau')$$

$$\leq \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1}{|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_1|\overline{r}'|}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_2}{c}d\tau' + \int_V \frac{C_2|\overline{r}'|}{c|\overline{r}-\overline{r}'|}d\tau' + \int_V \frac{C_3}{c^2}d\tau'$$

$$+ \int_V \frac{C_3|\overline{r}'|}{c^2|\overline{r}-\overline{r}'|}d\tau'$$

$$\leq \frac{1}{4\pi\epsilon_0|\overline{r}|}(\int_V \frac{C_1}{w}d\tau' + \int_V \frac{C_1 w}{w^2}d\tau' + \int_V \frac{C_2}{c}d\tau' + \int_V \frac{C_2 w}{cw}d\tau' + \int_V \frac{C_3}{c^2}d\tau'$$

$$+ \int_V \frac{C_3 w}{c^2 w}d\tau')$$

$$\leq \frac{vol(V)}{4\pi\epsilon_0|\overline{r}|}(\frac{C_1}{w} + \frac{C_1}{w} + \frac{C_2}{c} + \frac{C_2}{c} + \frac{C_3}{c^2} + \frac{C_3}{c^2})$$

$$= \frac{D}{\overline{r}}$$

where $\{C_1, C_2, C_3\} \subset \mathcal{R}_{>0}$ are uniform bounds for $\{\rho, \dot{\rho}, |\overline{J}|\}$ on V, $|\overline{r}| > 2w$ and;

$$D = \frac{vol(V)}{4\pi\epsilon_0|}(\frac{2C_1}{w} + \frac{2C_2}{c} + \frac{2C_3}{c^2})$$

We have that, for $1 \leq i \leq 3$, $|e_i| \leq |\overline{E}| \leq \frac{D}{\overline{r}}$, for $|\overline{r}| > 2w$, so the components of $\overline{E}$ are of very moderate decrease.

We have, following the method above, that, for $|\overline{r}| > 2w$;

$$\overline{B}(\overline{r},t) = \frac{\mu_0}{2\pi}|\int_V \frac{\overline{J}(\overline{r}',t_r)\times\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{\dot{\overline{J}}(\overline{r}',t_r)\times\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau|$$

$$\leq \frac{\mu_0}{2\pi}(\int_V \frac{C_3}{|\overline{r}-\overline{r}'|^2}d\tau' + \int_V \frac{C_4}{c|\overline{r}-\overline{r}'|}d\tau')$$

$\leq \frac{E}{|\overline{r}}$

where $C_4 \in \mathcal{R}_{>0}$ is a uniform bound for $|\dot{\overline{J}}|$ on V, $|\overline{r} > 2w$, and;

$E = \frac{\mu_0 vol(V)}{2\pi}(\frac{2C_3}{w} + \frac{2C_4}{c})$

Again, we have that, for $1 \leq i \leq 3$, $|b_i| \leq |\overline{B}| \leq \frac{E}{\overline{r}}$, for $|\overline{r}| > 2w$, so the components of $\overline{B}$ are of very moderate decrease.

For the second claim, expand in coordinates (x, y, z) around a point (x_0, y_0, z_0), with $|\overline{x}_0| > w$, and $|\overline{x} - \overline{x}_0| < \frac{|\overline{x}_0| - w}{4}$. Then, using Newton's expansion;

$(1 + y)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} y^n$, $|y| < 1$

and the fact that if $|\overline{x} - \overline{x}_0| < \frac{|\overline{x}_0| - w}{4} < \frac{|\overline{x}_0 - \overline{r}'|}{4}$, then $|\overline{x} - \overline{x}_0| < \frac{|\overline{x}_0 - \overline{r}'|}{\sqrt{2}}$, so that;

$|\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + \frac{2(\overline{x}-\overline{x}_0)\cdot(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}| \leq \frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + |\frac{2(\overline{x}-\overline{x}_0)\cdot(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}|$

$< \frac{1}{2} + \frac{2|\overline{x}-\overline{x}_0|}{|\overline{x}_0-\overline{r}'|}$

$< \frac{1}{2} + \frac{1}{2} = 1$

it follows;

$\frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2 |\overline{r} - \overline{r}'|} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2[(x-r_1')^2+(y-r_2')^2+(z-r_3')^2]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2[(x-x_0+x_0-r_1')^2+(y-y_0+y_0-r_2')^2+(z-z_0+z_0-r_3')^2]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2[|\overline{x}-\overline{x}_0|^2+|\overline{x_0}-\overline{r}'|^2+2(\overline{x}-\overline{x}_0)\cdot(\overline{x}_0-\overline{r}')]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{c^2|\overline{x}_0-\overline{r}'|[1+\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2}+\frac{2(\overline{x}-\overline{x}_0)\cdot(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2}]^{\frac{1}{2}}} d\tau'$

$= \frac{1}{4\pi\epsilon_0 c^2} \int_V \frac{\dot{j}_1(\overline{r}', t_r)}{|\overline{x}_0-\overline{r}'|} (\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} (\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2} + \frac{2(\overline{x}-\overline{x}_0)\cdot(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2})^n) d\tau'$

We have that;

$\int_V |\frac{\dot{j_1}(\overline{r}',t_r)}{|\overline{x}_0-\overline{r}'|}(\frac{(-1)^n(2n)!}{2^n n!}(\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2}+\frac{2(\overline{x}-\overline{x}_0)\centerdot(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2})^n)|d\tau'$

$\leq \frac{(2n)!}{2^n n!}\int_V \frac{|\dot{j_1}(\overline{r}',t_r)|}{|\overline{x}_0|-w}\sum_{m=0}^n C_m^n(\frac{(\frac{|\overline{x}_0-w}{4})^2}{(|\overline{x}_0|-w)^2})^{n-m}(\frac{2(\frac{|\overline{x}_0|-w}{4})}{\overline{x}_0-w})^m d\tau'$

$= \frac{(2n)!}{2^n n!}\int_V \frac{|\dot{j_1}(\overline{r}',t_r)|}{|\overline{x}_0|-w}\sum_{m=0}^n C_m^n(\frac{1}{16})^{n-m}(\frac{1}{2})^m d\tau'$

$\leq \frac{(2n)!}{2^n n!(|\overline{x}_0|-w)}\int_V |\dot{j_1}(\overline{r}',t_r)|(\frac{9}{16})^n d\tau'$

$\leq \frac{C_1(2n)!(\frac{9}{16})^n}{2^n n!(|\overline{x}_0|-w)}$

and, using Newton's expansion;

$(1-y)^{-\frac{1}{2}}=\sum_{n=0}^\infty \frac{(2n)!}{2^n n!}y^n$, $|y|<1$

we have that;

$\sum_{n=0}^\infty \frac{C_1(2n)!(\frac{9}{16})^n}{2^n n!|\overline{x}_0|-w}=\frac{C_1}{|\overline{x}_0|-w}\frac{1}{(1-\frac{9}{16})^{\frac{1}{2}}}=\frac{4C_1}{\sqrt{7}(|\overline{x}_0|-w)}$

so that, applying the DCT, we have that;

$\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{j_1}(\overline{r}',t_r)}{c^2|\overline{r}-\overline{r}'|}d\tau'$

$=\frac{1}{4\pi\epsilon_0 c^2}\sum_{n=0}^\infty \frac{(-1)^n(2n)!}{2^n n!}\int_V \frac{\dot{j_1}(\overline{r}',t_r)}{|\overline{x}_0-\overline{r}'|}(\frac{|\overline{x}-\overline{x}_0|^2}{|\overline{x}_0-\overline{r}'|^2}+\frac{2(\overline{x}-\overline{x}_0)\centerdot(\overline{x}_0-\overline{r}')}{|\overline{x}_0-\overline{r}'|^2})^n d\tau'$ (†)

and integrating the coefficients of $(x_1-x_{1,0})^i(x_2-x_{2,0})^j(x_3-x_{3,0})^k$ $(i,j,k)\in\mathcal{Z}^3_{\geq 0}$, in the expansion (†), to obtain constants $a_{ijk}\in\mathcal{R}$, $(i,j,k)\in\mathcal{Z}^3_{\geq 0}$, we see that the series;

$\sum_{(i,j,k)\in\mathcal{Z}^3_{\geq 0}} a_{ijk}(x_1-x_{1,0})^i(x_2-x_{2,0})^j(x_3-x_{3,0})^k$

is absolutely convergent for $|\overline{x}-\overline{x}_0|<\frac{|\overline{x}_0|-w}{4}$.

For the third claim, assuming that $\{x,y,z,x_0,y_0,z_0\}\subset\mathcal{R}\setminus\{0\}$, we have that;

$\frac{1}{|\overline{r}-\overline{r}'|}|_{(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})}=\frac{1}{[(\frac{x_0}{x}-r_1')^2+(\frac{y_0}{y}-r_2')^2+(\frac{z_0}{z}-r_3')^2]^{\frac{1}{2}}}$

$=\frac{x}{x_0[(1-\frac{r_1'x}{x_0})^2+(\frac{y_0x}{x_0y}-\frac{r_2'x}{x_0})^2+(\frac{z_0x}{x_0z}-\frac{r_3'x}{x_0})^2]^{\frac{1}{2}}}$

$=\frac{x}{x_0[1+x[-\frac{2r_1'}{x_0}-\frac{2r_2'}{x_0}(\frac{x}{y})(\frac{y_0}{x_0})-\frac{2r_3'}{x_0}(\frac{x}{z})(\frac{z_0}{x_0})]+x^2(\frac{|\overline{r}'|^2}{x_0^2})+(\frac{x}{y})^2(\frac{y_0}{x_0})^2+(\frac{x}{z})^2(\frac{z_0}{x_0})^2]^{\frac{1}{2}}}$ (A)

$$= \frac{y}{y_0[(\frac{x_0 y}{y_0 x}-\frac{r_1' y}{y_0})^2+(1-\frac{r_2' y}{y_0})^2+(\frac{z_0 y}{y_0 z}-\frac{r_3' y}{y_0})^2]^{\frac{1}{2}}}$$

$$= \frac{y}{y_0[1+y(-\frac{2r_2'}{y_0}-\frac{2r_1'}{y_0}(\frac{y}{x})(\frac{x_0}{y_0})-\frac{2r_3'}{y_0}(\frac{y}{z})(\frac{z_0}{y_0}))+y^2(\frac{|\bar{r}'|^2}{y_0^2})+(\frac{y}{x})^2(\frac{x_0}{y_0})^2+(\frac{y}{z})^2(\frac{z_0}{y_0})^2]^{\frac{1}{2}}} \quad (B)$$

$$= \frac{z}{z_0[(\frac{x_0 z}{z_0 x}-\frac{r_1' z}{z_0})^2+(\frac{y_0 z}{z_0 y}-\frac{r_2' z}{z_0})^2+(1-\frac{r_3' z}{z_0})^2]^{\frac{1}{2}}}$$

$$= \frac{z}{z_0[1+z(-\frac{2r_3'}{z_0}-\frac{2r_1'}{z_0}(\frac{z}{x})(\frac{x_0}{z_0})-\frac{2r_2'}{z_0}(\frac{z}{y})(\frac{y_0}{z_0}))+z^2(\frac{|\bar{r}'|^2}{z_0^2})+(\frac{z}{x})^2(\frac{x_0}{z_0})^2+(\frac{z}{y})^2(\frac{y_0}{z_0})^2]^{\frac{1}{2}}} \quad (C)$$

If $0<\epsilon<1$, and $|\frac{x}{y}|<\sqrt{1-\epsilon}|\frac{x_0}{y_0}|$ and $|\frac{x}{z}|<\sqrt{\epsilon}|\frac{x_0}{z_0}|$, then

$\alpha(x,y,z)=(\frac{x}{y})^2(\frac{y_0}{x_0})^2+(\frac{x}{z})^2(\frac{z_0}{x_0})^2<1$

and if;

$|x|<min(\frac{|x_0|\sqrt{1-\alpha}}{\sqrt{2}w},\frac{|x_0|(1-\alpha)}{12w})$

then, in (A);

$x[-\frac{2r_1'}{x_0}-\frac{2r_2'}{x_0}(\frac{x}{y})(\frac{y_0}{x_0})-\frac{2r_3'}{x_0}(\frac{x}{z})(\frac{z_0}{x_0})]+x^2(\frac{|\bar{r}'|^2}{x_0^2})+(\frac{x}{y})^2(\frac{y_0}{x_0})^2+(\frac{x}{z})^2(\frac{z_0}{x_0})^2<1$

(D)

Similarly, if $0<\delta<1$, and $|\frac{y}{x}|<\sqrt{1-\delta}|\frac{y_0}{x_0}|$ and $|\frac{y}{z}|<\sqrt{\delta}|\frac{y_0}{z_0}|$, then

$\beta(x,y,z)=(\frac{y}{x})^2(\frac{x_0}{y_0})^2+(\frac{y}{z})^2(\frac{z_0}{y_0})^2<1$

and if;

$|y|<min(\frac{|y_0|\sqrt{1-\beta}}{\sqrt{2}w},\frac{|y_0|(1-\beta)}{12w})$

then, in (B);

$y(-\frac{2r_2'}{y_0}-\frac{2r_1'}{y_0}(\frac{y}{x})(\frac{x_0}{y_0})-\frac{2r_3'}{y_0}(\frac{y}{z})(\frac{z_0}{y_0}))+y^2(\frac{|\bar{r}'|^2}{y_0^2})+(\frac{y}{x})^2(\frac{x_0}{y_0})^2+(\frac{y}{z})^2(\frac{z_0}{y_0})^2<1$

(E)

and, if $0<\theta<1$, and $|\frac{z}{x}|<\sqrt{1-\theta}|\frac{z_0}{x_0}|$ and $|\frac{z}{y}|<\sqrt{\theta}|\frac{z_0}{y_0}|$, then

$\gamma(x,y,z)=(\frac{x}{y})^2(\frac{y_0}{x_0})^2+(\frac{x}{z})^2(\frac{z_0}{x_0})^2<1$

and if;

$|z| < min(\frac{|z_0|\sqrt{1-\theta}}{\sqrt{2}w}, \frac{|z_0|(1-\theta)}{12w})$

then, in (C);

$z(-\frac{2r'_3}{z_0}-\frac{2r'_1}{z_0}(\frac{z}{x})(\frac{x_0}{z_0})-\frac{2r'_2}{z_0}(\frac{z}{y})(\frac{y_0}{z_0}))+z^2(\frac{|\overline{r}'|^2}{z_0^2})+(\frac{z}{x})^2(\frac{x_0}{z_0})^2+(\frac{z}{y})^2(\frac{y_0}{z_0})^2<1$

(F)

In case (D), we can expand (A) using Newton's theorem, as;

$\frac{x}{x_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}[x[-\frac{2r'_1}{x_0}-\frac{2r'_2}{x_0}(\frac{x}{y})(\frac{y_0}{x_0})-\frac{2r'_3}{x_0}(\frac{x}{z})(\frac{z_0}{x_0})]+x^2(\frac{|\overline{r}'|^2}{x_0^2})+(\frac{x}{y})^2(\frac{y_0}{x_0})^2$

$+(\frac{x}{z})^2(\frac{z_0}{x_0})^2]^n$

$=\sum_{i+j+k\geq 0}a_{ijk}x^i(\frac{x}{y})^j(\frac{x}{z})^k$

with $|\frac{x}{y}|<\sqrt{1-\epsilon}|\frac{x_0}{y_0}|$ and $|\frac{x}{z}|<\sqrt{\epsilon}|\frac{x_0}{z_0}|$, $(*)$. If $|x|>x_1>0$, then if;

$|y|>\frac{|x||\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}$

implies that;

$|y|>\frac{x_1|\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}$

and, for $m\in\mathcal{N}$, we can obtain an expansion of $\frac{1}{y}$ in the region;

$\frac{x_1|\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}<|y|<\frac{mx_1|\frac{y_0}{x_0}|}{\sqrt{1-\epsilon}}$

by noting that, with $c=\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}}$, $|y-c|<c$, so that;

$\frac{1}{y}=\frac{1}{c+y-c}=\frac{1}{c(1+\frac{(y-c)}{c})}$

$=\frac{1}{c}\sum_{n=0}^{\infty}(-1)^n\frac{(y-c)^n}{c^n}$

$=\frac{1}{\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}}}\sum_{n=0}^{\infty}(-1)^n\frac{(y-(\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}}))^n}{(\frac{x_1(1+m)|\frac{y_0}{x_0}|}{2\sqrt{1-\epsilon}})^n}$

....

In case (E), we can expand (B) as;

$\frac{y}{y_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}[y(-\frac{2r_2'}{y_0}-\frac{2r_1'}{y_0}(\frac{y}{x})(\frac{x_0}{y_0})-\frac{2r_3'}{y_0}(\frac{y}{z})(\frac{z_0}{y_0}))+y^2(\frac{|\overline{r}'|^2}{y_0^2})+(\frac{y}{x})^2(\frac{x_0}{y_0})^2$

$+(\frac{y}{z})^2(\frac{z_0}{y_0})^2]^n$

$=\sum_{i+j+k\geq 0}b_{ijk}y^i(\frac{y}{x})^j(\frac{y}{z})^k$

with $|\frac{y}{x}|<\sqrt{1-\delta}|\frac{y_0}{x_0}|$ and $|\frac{y}{z}|<\sqrt{\delta}|\frac{y_0}{z_0}|$

In case (F), we can expand (C) as;

$\frac{z}{z_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}[z(-\frac{2r_3'}{z_0}-\frac{2r_1'}{z_0}(\frac{z}{x})(\frac{x_0}{z_0})-\frac{2r_2'}{z_0}(\frac{z}{y})(\frac{y_0}{z_0}))+z^2(\frac{|\overline{r}'|^2}{z_0^2})+(\frac{z}{x})^2(\frac{x_0}{z_0})^2$

$+(\frac{z}{y})^2(\frac{y_0}{z_0})^2]^n$

$=\sum_{i+j+k\geq 0}c_{ijk}z^i(\frac{z}{x})^j(\frac{z}{y})^k$

with $|\frac{z}{x}|<\sqrt{1-\theta}|\frac{z_0}{x_0}|$ and $|\frac{z}{y}|<\sqrt{\theta}|\frac{z_0}{y_0}|$

....

For the fourth claim, suppose the initial conditions $\rho_0\in S(\mathcal{R}^3),\frac{\partial\rho}{\partial t}|_{t=0}\in S(\mathcal{R}^3)$, have compact support, with ρ defined on $\mathcal{R}^4$ by Kirchoff's formula;

For $t>0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

and, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

then, see [**?**] and the construction in [14], we have that, for $\overline{x}\in\mathcal{R}^3$;

$lim_{t\rightarrow 0+}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}(\overline{x},t)=g(\overline{x})$

$lim_{t\rightarrow 0+}\rho(\overline{x},-t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}(\overline{x},-t)=-g(\overline{x})$

where $g((\overline{x}))=\frac{\partial\rho}{\partial t}|_{t=0}$, so that;

$lim_{t\rightarrow 0-}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0-}\frac{\partial\rho}{\partial t}(\overline{x},t)=lim_{t\rightarrow 0+}-\frac{\partial\rho}{\partial t}(\overline{x},-t)$

$=--g(\overline{x})$

$=g(\overline{x})$

In particular;

$lim_{t\rightarrow 0}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0}\frac{\partial\rho}{\partial t}(\overline{x},t)=g(\overline{x})$

Using the fact that $\rho_0\in S(\mathcal{R}^3)$, $g(\overline{x})\in S(\mathcal{R}^3)$, the transform method, see Lemma 0.4 and uniqueness of the wave equation solution, given the 2 initial conditions, we have for $t>0$;

$\rho(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b(\overline{k})e^{ikct}+d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$\rho(\overline{x},-t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b^-(\overline{k})e^{ikct}+d^-(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$ (X)

where;

$b(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})+\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$d(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})-\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$b^-(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})+\frac{1}{ikc}\mathcal{F}(-g)(\overline{k}))$

$=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})-\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$d^-(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})-\frac{1}{ikc}\mathcal{F}(-g)(\overline{k}))$

$=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})+\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

see also earlier in the paper, so that, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b^-(\overline{k})e^{-ikct}+d^-(\overline{k})e^{ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$ (Y)

Differentiating under the integral sign in (X), we have that, for $t>0$;

$\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})e^{ikct}+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

where $(ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})\in S(\mathcal{R}^3)$ and $(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})\in S(\mathcal{R}^3)$, so that;

$lim_{t\rightarrow 0+}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)$

$=lim_{t\rightarrow 0+}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})e^{ikct}+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k\mathcal{F}(\rho_0)(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$ $(X)'$

Similarly, differentiating under the integral sign in (Y), using the fact that $b^-(\overline{k})+d^-(\overline{k})=\mathcal{F}(\rho_0)(\overline{k})$;

$lim_{t\rightarrow 0-}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$ (Y')

and combining $(X)',(Y)'$, we obtain that;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$

By a similar argument, differentiating under the integral sign, and using the facts that $b(\overline{k})ikc-d(\overline{k}ikc=\mathcal{F}(g)(\overline{k})$ $-ikcb^-(\overline{k})+ikcd^-(\overline{k})=\mathcal{F}(g)(\overline{k})$;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+1}\rho}{\partial x^i\partial y^j\partial z^k\partial t}(\overline{x},t)=\frac{\partial^{i+j+k}g}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$

Similarly, using the fact that $\rho_0\in S(\mathcal{R}^3)$, $\{b(\overline{k}),d(\overline{k})\}\subset L^1(\mathcal{R}^3)$, so we can apply the inversion theorem, we have that;

$lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+2}\rho}{\partial x^i\partial y^j\partial z^k\partial t^2}(\overline{x},t)$

$= lim_{t\rightarrow 0+}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)b(\overline{k})e^{ikct}$

$+(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)(b(\overline{k})+d(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)(\mathcal{F}(\rho_0)(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}c^2(\mathcal{F}(\frac{\partial^{i+j+k}\bigtriangledown^2(\rho_0)}{\partial x^i\partial y^j\partial z^k})(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$= c^2\frac{\partial^{i+j+k}\bigtriangledown^2(\rho_0)}{\partial x^i\partial y^j\partial z^k}(\overline{x})$

and;

$lim_{t\rightarrow 0-}\frac{\partial^{i+j+k+2}\rho}{\partial x^i\partial y^j\partial z^k\partial t^2}(\overline{x},t) = \frac{\partial^{i+j+k}c^2\bigtriangledown^2(\rho_0)}{\partial x^i\partial y^j\partial z^k}(\overline{x})$

As $\rho|_{t>0}$, $\rho|_{t<0}$ obey the wave equation, so do the partial derivatives $\frac{\partial^{i+j+k+l}}{\partial x^i\partial y^j\partial z^k\partial t^l}|_{t>0}$, so that, for $l\geq 1$, l even, $t\neq 0$;

$\frac{\partial^{i+j+k+l}\rho}{\partial x^i\partial y^j\partial z^k\partial t^l}|_{t\neq 0} = c^l(\bigtriangledown^2)^{\frac{l}{2}}(\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k})|_{t\neq 0}$

and, for $l\geq 1$, l odd, $t\neq 0$;

$\frac{\partial^{i+j+k+l}\rho}{\partial x^i\partial y^j\partial z^k\partial t^l}|_{t\neq 0} = c^{l-1}(\bigtriangledown^2)^{\frac{l-1}{2}}(\frac{\partial^{i+j+k+1}\rho}{\partial x^i\partial y^j\partial z^k\partial t})|_{t\neq 0}$

and, using the above, for l even;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+l}\rho(\overline{x},t)}{\partial x^i\partial y^j\partial z^k\partial t^l} = c^l(\bigtriangledown^2)^{\frac{l}{2}}(\frac{\partial^{i+j+k}\rho_0}{\partial x^i\partial y^j\partial z^k})$

and, for l odd;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+l}\rho(\overline{x},t)}{\partial x^i\partial y^j\partial z^k\partial t^l} = c^{l-1}(\bigtriangledown^2)^{\frac{l-1}{2}}(\frac{\partial^{i+j+k}g}{\partial x^i\partial y^j\partial z^k})$

In particularly, as all the partial derivatives of ρ extend continuously to the boundary $t=0$, we have that $\rho\in C^\infty(\mathcal{R}^4)$, and the wave equation is satisfied at $t=0$, $\frac{\partial^2\rho}{\partial t^2} = c^2\bigtriangledown^2(\rho)$, ([3]). By Kirchoff's formula,

[3] It is relatively straightforward calculation to check, using the integral representation of a solution to the wave equation, $\bigtriangledown^2(f)-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=0$ in $\mathcal{R}^3\times[0,\infty)$, generated by the initial data (g,h), that $lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+l}f_t}{\partial x^i\partial x^j\partial z^k\partial t^l} = (c^2\bigtriangledown^2)^{\frac{i}{2}}\frac{\partial^{i+j+k+l}g}{\partial x^i\partial x^j\partial z^k}$ for i even and that $lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+l}f_t}{\partial x^i\partial x^j\partial z^k\partial t^l} = (c^2\bigtriangledown^2)^{\frac{i-1}{2}}\frac{\partial^{i+j+k+l}h}{\partial x^i\partial x^j\partial z^k}$ for i odd. By uniqueness of the wave equation with specified initial conditions (g,h), the same

must be true for Kirchoff's representation. The same result holds for the backward wave equation with initial data $(g, -h)$, so the limit of the partial derivatives is same for $t > 0$ as $t < 0$. We have, if;

$\rho(\overline{x}, t) = \frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} (tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$ $(t > 0)$

$\rho(\overline{x}, t) = \frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, -ct)} (tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$ $(t < 0)$

Then, for $t > 0$, $\rho(\overline{x}, t) = \rho(\overline{x}, -t)$ iff;

$\frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} (tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$

$= \frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} (-tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$

iff $\frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} 2tg(\overline{y}) dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x}, ct)} g(\overline{y}) dS(\overline{y}) = 0$

iff $g(\overline{y}) = 0$

as if $g(\overline{y}_0) \neq 0$, without loss of generality, by continuity, we can choose $t_0 > 0$ sufficiently small with $g|_{\delta B(\overline{y}_0, ct)} > 0$, so that $\int_{\delta B(\overline{y}_0, ct_0)} g(\overline{y}) dS(\overline{y}) > 0$

and, for $t > 0$, $\rho(\overline{x}, t) = -\rho(\overline{x}, -t)$ iff;

$\frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} (tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$

$= \frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} (tg(\overline{y}) - \rho_0(\overline{y}) - D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$

iff $\frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} 2[\rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})] dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x}, ct)} [\rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})] dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x}, ct)} \rho_0(\overline{y}) dS(\overline{y}) + ct \int_{\delta B(\overline{x}, ct)} \bigtriangledown(\rho_0) \cdot d\overline{S} = 0$

iff $\int_{\delta B(\overline{x}, ct)} \rho_0(\overline{y}) dS(\overline{y}) + ct \int_{B(\overline{x}, ct)} div(\bigtriangledown(\rho_0)) dV(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x}, ct)} \rho_0(\overline{y}) dS(\overline{y}) + ct \int_{B(\overline{x}, ct)} \bigtriangledown^2(\rho_0) dV(\overline{y}) = 0$

iff $\rho_0(\overline{y}) = 0$

as if $\rho_0(\overline{y}_0) \neq 0$, by continity, without loss of generality, there exists $\epsilon > 0$, such that, for sufficiently small t_0;

$\int_{\delta B(\overline{y}_0, ct_0)} \rho_0(\overline{y}) dS(\overline{y}) > 4\pi\epsilon c^2 t_0^2$

and, if M is a uniform bound on $\bigtriangledown^2(\rho_0)$

see [4] and [14] and the above, we had for $t>0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

and, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

In particular, for fixed $t_0\in\mathcal{R}$, as ρ_0 and g have compact support, we can see that $\delta B(\overline{x},c|t_0|)\cap Supp(\rho_0,g,D\rho_0)=\emptyset$, for $|\overline{x}_0|>C_{t_0}$, where $C_{t_0}\in\mathcal{R}_{>0}$, so that ρ_{t_0} has compact support as well. As $\rho_{t_0}\in C^\infty(\mathcal{R}^3)$, we then have that $\rho_{t_0}\in S(\mathcal{R}^3)$.

For the fifth claim, with;

$\overline{J}(\overline{x},t)=-c^2\int_{-\infty}^t\bigtriangledown(\rho)ds$

see [14] for the existence of the integral. We have, differentiating under the integral sign, and using the fundamental theorem of calculus, that, for $(i,j,k)\in\mathcal{Z}^3_{\geq 0}$;

$\frac{\partial^{i+j+k}j_1}{\partial x^i\partial y^j\partial z^k}=-c^2\int_{-\infty}^t\frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j\partial z^k}ds\ (Z)$

$\frac{\partial^{i+j+k+1}j_1}{\partial x^i\partial y^j\partial z^k\partial t}=-c^2\frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j\partial z^k}$

and for $l\geq 2$;

$\frac{\partial^{i+j+k+l}j_1}{\partial x^i\partial y^j\partial z^k\partial t^l}=-c^2\frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j\partial z^k\partial t^{l-1}}$

As $(\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k})_0\in S(\mathcal{R}^3)$, and $\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}$ satisfies the wave equation on $\mathcal{R}^4$, by the proof in [14], we have that the integral (Z) is well defined. Then, as $\rho\in C^\infty(\mathcal{R}^4)$, we have that $j_1\in C^\infty(\mathcal{R}^4)$. A similar argument shows that the components $\{j_2,j_3\}\subset C^\infty(\mathcal{R}^4)$. By the fundamental

---

$|ct_0\int_{B(\overline{y}_0,ct_0)}\bigtriangledown^2(\rho_0)dV(\overline{y})|<\frac{4M\pi c^4t_0^4}{3}$

so that, if $4\pi\epsilon c^2t_0^2>\frac{4M\pi c^4t_0^4}{3}$ iff $\frac{3\epsilon}{Mc^2}>t_0^2$, we can choose $0<t_0<\frac{(3\epsilon)^{\frac{1}{2}}}{\sqrt{M}c}$, to obtain;

$\int_{\delta B(\overline{y}_0,ct_0)}\rho_0(\overline{y})dS(\overline{y})+ct_0\int_{B(\overline{y}_0,ct_0)}\bigtriangledown^2(\rho_0)dV(\overline{y})>0$

In either case, we can reflect a solution for $t\geq 0$ to obtain a smooth solution on $\mathcal{R}^4$.

theorem of calculus, we have that;

$$\frac{\partial \overline{J}}{\partial t} = -c^2 \bigtriangledown (\rho)$$

By the previous claim, for $t_0 \in \mathcal{R}$, ρ_{t_0} has compact support, so that $(\bigtriangledown(\rho))_{t_0}$ has compact support and $(\frac{\partial \overline{J}}{\partial t})_{t_0}$ has compact support. It is clear from the above that the compact support V_t of ρ_t and $(\bigtriangledown(\rho))_t$ varies continuously with t, so on the interval $(t_0-\epsilon, t_0+\epsilon)$, $(\frac{\partial \overline{J}}{\partial t})|_{(t_0-\epsilon,t_0+\epsilon)}$ has compact support $W_{t_0,\epsilon}$ in $\mathcal{R}^4$.

$\overline{J}$ satisfies the wave equation on $\mathcal{R}^4$, as, using the fundamental theorem of calculus and the fact that $\bigtriangledown(\rho)$ satisfies the wave equation;

$$\square^2(\overline{J}) = \bigtriangledown^2(\overline{J}) + \frac{1}{c^2}\frac{\partial^2 \overline{J}}{\partial t^2}$$

$$= -c^2(\int_{-\infty}^{t} \bigtriangledown^2(\bigtriangledown(\rho))ds) + \frac{1}{c^2}(-c^2\frac{\partial \bigtriangledown(\rho)}{\partial t})$$

$$= -c^2(\int_{-\infty}^{t} -\frac{1}{c^2}\frac{\partial^2 \bigtriangledown(\rho)}{\partial t^2}ds) - \frac{\partial \bigtriangledown(\rho)}{\partial t}$$

$$= \frac{\partial \bigtriangledown(\rho)}{\partial t}) - \frac{\partial \bigtriangledown(\rho)}{\partial t}$$

$$= \overline{0}$$

By the connecting relation;

$$\bigtriangledown\rho + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$$

we have that $\frac{\partial \overline{J}}{\partial t}$ vanishes outside $Supp(\rho_t)$, and for any $\overline{x} \in \mathcal{R}^3$, there exists two uniformly bounded intervals $[t_{1,\overline{x},-}, t_{2,\overline{x},-}]$, $[t_{1,\overline{x},+}, t_{2,\overline{x},+}]$, for which $\overline{x} \in Supp(\rho_t)$, for $t \in [t_{1,\overline{x},-}, t_{2,\overline{x},-}] \cup [t_{1,\overline{x},+}, t_{2,\overline{x},+}]$. Using the fact that $Supp(\rho_t)$ is moving and $\bigtriangledown(\rho)$ satisfies the wave equation, so uniformly bounded, we can define;

$$\overline{J}_0(\overline{x}) = \int_{t_{1,\overline{x},-}}^{t_{2,\overline{x},-}} \frac{\partial \overline{J}}{\partial t}dt + \int_{t_{1,\overline{x},+}}^{t_{2,\overline{x},+}} \frac{\partial \overline{J}}{\partial t}dt$$

$$= \int_{-\infty}^{\infty} \frac{\partial \overline{J}}{\partial t}dt \text{ (the ultimate value of } \overline{J}(\overline{x},t))$$

with $\overline{J}_0$ bounded. On any ball $B(\overline{0},r)$, we have that $\overline{J} - \overline{J}_0$ eventually vanishes, and, as $div(\overline{J}) - div(\overline{J}_0) = 0$ ultimately on the ball, and $div(\overline{J}) = -\frac{\partial \rho}{\partial t} = 0$, ultimately, otherwise charge would build up, we have that $div(\overline{J}_0)=0$. It follows that $(\rho,\overline{J}-\overline{J}_0)$ satisfies the continuity equation., and the linkage relation;

$$\bigtriangledown\rho+\frac{1}{c^2}\frac{\partial(\overline{J}-\overline{J}_0)}{\partial t}=\overline{0}$$

is still satisfied, as $\overline{J}_0$ is time independent. On any ball $B(\overline{0},r)$, we have that ultimately $\overline{J}-\overline{J}_0=\overline{0}$, so that, as $\square^2(\overline{J})=\overline{0}$ and $\overline{J}_0$ is time independent, ultimately;

$$\bigtriangledown^2(\overline{J}_0)=\square^2(\overline{J}_0)=\square^2(\overline{J})=\overline{0}$$

and $\overline{J}_0$ is harmonic. As the components $\bigtriangledown(\rho)_i$, for $1\leq i\leq 3$, satisfy the wave equation, we have that that there exists constants $C_i\in\mathcal{R}_{>0}$, for which $|\bigtriangledown(\rho)_i(\overline{x},t)|\leq\frac{C_i}{|t|}$ for $1\leq i\leq 3$, so that;

$$|\bigtriangledown(\rho)(\overline{x},t)|\leq\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t|}$$

and;

$$|\overline{J}_0(\overline{x})|=|\int_{t_{1,\overline{x},-}}^{t_{2,\overline{x},-}}-c^2\bigtriangledown(\rho)dt+\int_{t_{1,\overline{x},+}}^{t_{2,\overline{x},+}}-c^2\bigtriangledown(\rho)dt|$$

$$\leq c^2[(t_{2,\overline{x},-}-t_{1,\overline{x},-})+(t_{2,\overline{x},+}-t_{1,\overline{x},+})]|\bigtriangledown(\rho)|_{[t_{1,\overline{x},-},t_{2,\overline{x},-}]\cup[t_{1,\overline{x},-},t_{2,\overline{x},-}]}|$$

$$\leq c^2(t_{2,\overline{x},-}-t_{1,\overline{x},-})\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t_{1,\overline{x},-}|}+c^2(t_{2,\overline{x},+}-t_{1,\overline{x},+})\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t_{1,\overline{x},+}|}$$

$$\leq\frac{C}{|\overline{x}|}$$

as the intervals $[t_{1,\overline{x},-},t_{2,\overline{x},-}]$, $[t_{1,\overline{x},+},t_{2,\overline{x},+}]$ are uniformly bounded, and the hitting times $\{t_{1,\overline{x},-},t_{1,\overline{x},+}\}$ are proportional to the distance $\overline{x}$. It follows, as bounded harmonic functions are constant, that $\overline{J}_0=\overline{0}$, and $\overline{J}$ has compact supports.

The same results hold for $w\neq c$. If $w\neq c$, using Jefimenko's equations, we can prove the existence of fields $(\overline{E}_w,\overline{B}_w)$, for which the components depending on $\overline{J}_w$ have compact support at time t, the support increasing as $w\rightarrow c$ and uniformly bounded. ?$\frac{\partial\overline{J}}{\partial t}$ obeys a wave equation (with speed w)?. Also true that if $(\overline{E},\overline{B})$ are defined from $(\rho,\overline{J})$, using Jefimenko's equations, then $(\frac{\partial\overline{E}}{\partial t},\frac{\partial\overline{B}}{\partial t})$ are defined from $(\frac{\partial\rho}{\partial t},\frac{\partial\overline{J}}{\partial t})$ using Jefimenko's equations, provided the causal solution exists.

For the sixth claim, following the method of [14], and the results in this paper, we can construct charge and current configurations $(\rho_w, \overline{J}_w)$ for $w \in \mathcal{R}_{>0}$, $w \neq c$, such that $\square^2_w(\rho_w) = 0$, $\square^2_w(\overline{J}_w) = \overline{0}$, $\bigtriangledown(\rho) + \frac{1}{w^2}\frac{\partial \overline{J}}{\partial t} = 0$, $\frac{\partial \rho}{\partial t} = -\bigtriangledown \centerdot \overline{J}$, with the same initial conditions (f, g) and support V. All the arguments for charge and current we have used for c, hold in the case $w \neq c$, being careful to replace c with w in the definitions. In this case, the fields $(\overline{E}_w, \overline{B}_w)$ generated by Jefimenko's equations are well defined for $t \in \mathcal{R}$, with respect to charge, as, for given $\overline{x}_0$, the locus of $\{\overline{x} : B(\overline{x}, wt_r) \cap V \neq \emptyset\}$ is bounded, because $wt_r = w(t - \frac{|\overline{x}-\overline{x}_0|}{c})$ contains the factor $\frac{w}{c} \neq 1$, and for current, a similar idea, the proof being the same, as the current obeys the wave equation and has compact support, receding at speed w. Then, we have that $(\rho_w, \overline{J}_w, \overline{E}_w, \overline{B}_w)$ satisfy Maxwell's equations. If we use Kirchoff's formula for $\frac{\partial \rho}{\partial t}$, with initial conditions $(\frac{\partial \rho}{\partial t}|_0, \frac{\partial^2 \rho}{\partial^2 t}|_0) = (\frac{\partial \rho}{\partial t}|_0, -c^2(\bigtriangledown^2 \rho)|_0)$;

$$\frac{\partial \rho}{\partial t}(\overline{x}, t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}, ct)}(t\frac{\partial^2 \rho}{\partial t^2}|_0) + \frac{\partial \rho}{\partial t}{}_0(\overline{y}) +$$

$$D(\frac{\partial \rho}{\partial t}|_0)(\overline{y}) \centerdot (\overline{y} - \overline{x}))dS(\overline{y}) \ (t > 0)$$

$$\frac{\partial \rho}{\partial t}(\overline{x}, t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}, -ct)}(t\frac{\partial^2 \rho}{\partial t^2}|_0) + \frac{\partial \rho}{\partial t}|_0(\overline{y})$$

$$+D(\frac{\partial \rho}{\partial t}|_0)(\overline{y}) \centerdot (\overline{y} - \overline{x}))dS(\overline{y}) \ (t < 0)$$

We then have, using Jefimenko's equations;

$$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho}(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|}d\tau')_1 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\partial \rho}{\partial t}(\overline{r}', t - \frac{|\overline{r}-\overline{r}'|}{c})\frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V [\frac{1}{4\pi c^2 (t - \frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}', c(t - \frac{|\overline{r}-\overline{r}'|}{c}))}(t - \frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0) + \frac{\partial \rho}{\partial t}(\overline{y}, 0)$$

$$+D(\frac{\partial \rho}{\partial t})(\overline{y}, 0) \centerdot (\overline{y} - \overline{r}')]dS(\overline{y})\frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$$

$$+\frac{1}{4\pi\epsilon_0}\int_V [\frac{1}{4\pi c^2 (t - \frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}', -c(t - \frac{|\overline{r}-\overline{r}'|}{c}))}(t - \frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0) + \frac{\partial \rho}{\partial t}(\overline{y}, 0)$$

$$+D(\frac{\partial \rho}{\partial t})(\overline{y}, 0) \centerdot (\overline{y} - \overline{r}')]dS(\overline{y})\frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$$

We can use then use the asymmetry $(r_1 - r_1')$ $r_1 = 0$, $r_1'' = -r_1'$, together with the symmetry, in the integral;

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2(\rho)}{\partial t^2})(\overline{y},0)dS(\overline{y}) = \int_{\delta B(\overline{r}'',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2(\rho)}{\partial t^2})(\overline{y},0)dS(\overline{y})$ $(t=0)$

and vanishing in the integral of $\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))} D(\frac{\partial\rho}{\partial t}|_0)(\overline{y}) \cdot t\overline{z}dS(\overline{y})$ for large $\overline{r}'$, see Lemma 0.49, and the $\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}$ decay in the remaining term, to show that $lim_{w\rightarrow c}(\rho_w,\overline{J}_w,\overline{E}_w,\overline{B}_w)$ exists and define $(\rho_c,\overline{J}_c,\overline{E}_c,\overline{B}_c)$ as $lim_{w\rightarrow c}(\rho_w,\overline{J}_w,\overline{E}_w,\overline{B}_w)$, for the original charge and current combination $(\rho_c,\overline{J}_c)$. It is clear that $(\rho_c,\overline{J}_c,\overline{E}_c,\overline{B}_c)$ satisfies Maxwell's equations, and the configuration $(\overline{E}_c,\overline{B}_c)$ is defined by Jefimenko's equations as an indefinite integral. A detailed exposition of this claim is the the subject of the following.

We are mainly interested in the case $w=c$, but most of the calculations can be adapted to the case $w\neq c$, the important point being to keep the factor c in Jefimenko's equations, ([4]). Unless otherwise stated though, $w=c$. We can assume by the above and the proof in [14], that $\rho\in C^\infty(\mathcal{R}^4)$, for the components j_i, $1\leq i\leq 3$, $j_i\in C^\infty(\mathcal{R}^4)$, for $t\in\mathcal{R}$, ρ_t and $j_{i,t}$ have compact support, and the components j_i satisfy the wave equation $\square^2 j_i=0$, $1\leq i\leq 3$. It follows that the derivatives $\frac{\partial\rho}{\partial t}\in C^\infty(\mathcal{R}^4)$ and $\frac{\partial j_i}{\partial t}\in C^\infty(\mathcal{R}^4)$, $1\leq i\leq 3$, that $\frac{\partial\rho}{\partial t}$ and $\frac{\partial j_i}{\partial t}$, $1\leq i\leq 3$ obey the wave equation and, for $t\in\mathcal{R}$, $\frac{\partial\rho}{\partial t}_t$ and $\frac{\partial j_{i,t}}{\partial t}$, $1\leq i\leq 3$ have compact support. The fields $\{\overline{E},\overline{B}\}$ defined by Jefimenko's equations are given by;

$$\overline{E}(\overline{r},t)=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau'+\int_V\frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau'-\int_V\frac{\dot{\overline{J}}(\overline{r}',t_r)}{c^2|\overline{r}-\overline{r}'|}d\tau'$$

$$\overline{B}(\overline{r},t)=\frac{\mu_0}{2\pi}\int_V\frac{\overline{J}(\overline{r}',t_r)\times\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau'+\int_V\frac{\dot{\overline{J}}(\overline{r}',t_r)\times\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau$$

We have using Kirchoff's formula, that, for $t>0$;

$$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\cdot(\overline{y}-\overline{x}))dS(\overline{y})$$

[4] There may be a point that particles travelling at speed c in the base frame would contradict special relativity, but it is not clear with an extended charge distribution that there are any individual particles. In any case, the associated charge and current configuration $(\rho,\overline{J})$ exists and seems to define fields $(\overline{E},\overline{B})$ satisfying Maxwell's equations with special properties, at least in the case $w>c$. The case when inertial frames travel at speeds $w>c$ is developed in [11].

and, for $t<0$;

$$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$$

so that;

$$(\tfrac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau')_1=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}',t-\frac{|\overline{r}-\overline{r}'|}{c})(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'$$

$$=\frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$$

$$+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})$$

$$+\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$$

$$+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'$$

Let;

$$W_1=\{\overline{r}':\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))\cap B(\overline{0},w)\neq\emptyset\}$$

$$W_2=\{\overline{r}':\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c})\cap B(\overline{0},w)\neq\emptyset\}$$

With the convention $(*)$ below, if $t>0$, we require that $c(t-\frac{|\overline{r}-\overline{r}'|}{c})>0$ iff $|\overline{r}-\overline{r}'|<ct$, so that $W_1\subset B(\overline{0},ct)$, if $t>0$ and $W_1=\emptyset$ if $t\leq 0$. Similarly, we require that $-c(t-\frac{|\overline{r}-\overline{r}'|}{c}>0$ iff $|\overline{r}-\overline{r'}|>ct$, so that, if $t\geq 0$, $W_2\subset\mathcal{R}^3\setminus B(\overline{0},ct)$ and if $t<0$, we obtain no restriction on W_2. In either case, we clearly have, by smoothness of the data, continuity and the fact that $B(\overline{0},ct)$ is bounded for $t>0$, that;

$$|\frac{1}{4\pi\epsilon_0}\int_{W_1}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$$

$$+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'|$$

$$\leq\int_{B(\overline{0},ct)}C_t|\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}|d\tau'$$

$$\leq\frac{C_t}{c}\int_{B(\overline{0},ct)}\frac{1}{|\overline{r}-\overline{r}'|^2}d\tau'$$

$$\leq\frac{C_t}{c}\int_{B(\overline{0},ct)}\frac{1}{|\overline{r}'|^2}d\tau'$$

$$\leq\frac{C_t}{c}\int_0^{\pi}\int_{-\pi}^{\pi}\int_0^{ct}\frac{1}{r^2}r^2|sin(\theta)|drd\theta d\phi$$

$\leq \frac{2\pi^2 C_t}{c}\int_0^{ct} dr$

$\leq 2\pi^2 t C_t$

so that;

$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2} d\tau')_1 = f_1(\overline{r}, t)$

$+\int_{W_2} \frac{1}{4\pi c^2 (t-\frac{|\overline{r}-\overline{r}'|}{c})^2} \int_{\delta B(\overline{r}', -c(t-\frac{|\overline{r}-\overline{r}'|}{c}))} ((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial \rho(\overline{y},0)}{\partial t} + \rho(\overline{y},0)$

$+D\rho(\overline{y},0) \cdot (\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3} d\tau'$

We can assume in the calculation that $\overline{r} \neq \overline{0}$, by changing coordinates with a translation given by $\overline{r}_0$, see below for the corresponding time translation, as we can define a new pair $(\rho^{\overline{r}_0}, \overline{J}^{\overline{r}_0})$ by $\rho^{\overline{r}_0}(\overline{x}, s) = \rho(\overline{x}-\overline{r}_0, s)$ and $\overline{J}^{\overline{r}_0}(\overline{x}, s) = \overline{J}(\overline{x}-\overline{r}_0, s)$, for $(\overline{x}, s) \in \mathcal{R}^4$. The new pair $(\rho^{\overline{r}_0}, \overline{J}^{\overline{r}_0})$ inherits the properties of $(\rho, \overline{J})$, in particular we have that $\rho^{\overline{r}_0} \in C^\infty(\mathcal{R}^4)$, the components of $\overline{J}^{\overline{r}_0}$, $j_i^{\overline{r}_0} \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, $\square^2(\rho^{\overline{r}_0}) = 0$, for $1 \leq i \leq 3$, the continuity equation $\frac{\partial \rho^{\overline{r}_0}}{\partial t} = -\bigtriangledown \cdot \overline{J}^{\overline{r}_0}$ holds, and the connecting relation $\bigtriangledown(\rho^{\overline{r}_0}) + \frac{1}{c^2}\frac{\partial \overline{J}^{\overline{r}_0}}{\partial t} = \overline{0}$. Moreover, we can use Kirchoff's formula with the initial data for $(\rho^{\overline{r}_0}, \overline{J}^{\overline{r}_0})$ given by $(\rho_0^{\overline{r}_0}, (\frac{\partial \rho^{\overline{r}_0}}{\partial t})_0, \overline{J}_0^{\overline{r}_0}, (\frac{\partial \overline{J}^{\overline{r}_0}}{\partial t})_0)$ and we have that, making the substitution $\overline{r}'' = \overline{r}_0 + \overline{r}'$;

$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{0}-\overline{r}'|^2} d\tau')_1 = (\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\overline{r}''-\overline{r}_0, t_r')\hat{\mathfrak{r}}}{|\overline{r}_0-\overline{r}''|^2} d\tau'')_1 =$

$= (\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho^{\overline{r}_0}(\overline{r}'', t_r')\hat{\mathfrak{r}}}{|\overline{r}_0-\overline{r}''|^2} d\tau'')_1$

for the corresponding retarded time $t_r' = t - \frac{|\overline{r}_0-\overline{r}''|}{c}$, and, similarly, for the corresponding terms in Jefimenko's equations.

We have, for $\overline{r}' \neq \overline{0}$, $\overline{r} \neq \overline{0}$, that;

$\delta B(\overline{r}', -c(t-\frac{|\overline{r}-\overline{r}'|}{c}) \cap B(\overline{0}, w) \neq \emptyset$

iff $|\overline{r}' - [-c(t-\frac{|\overline{r}-\overline{r}'|}{c})]\frac{\overline{r}'}{|\overline{r}'|}| \leq w$

iff $|\overline{r}'|\overline{r}'| + (ct - |\overline{r}-\overline{r}'|)\overline{r}'| \leq w|\overline{r}'|$

iff $|\overline{r}'||\overline{r}'|+(ct-|\overline{r}-\overline{r}'|)|\leq w|\overline{r}'|$

iff $||\overline{r}'|+ct-|\overline{r}-\overline{r}'||\leq w$

iff $-w-ct\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w-ct$

so that, if $t>0$;

$W_2=\{\overline{r}':-w-ct\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w-ct\}\cap\mathcal{R}^3\setminus B(\overline{0},ct)$

and, if $t\leq 0$;

$W_2=\{\overline{r}':-w-ct\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w-ct\}$

Letting;

$N=max_{\overline{y}\in B(0,\overline{w})}(|(\frac{\partial\rho}{\partial t})_0|,|\rho_0|,|D\rho_0|,|(D\rho)_0|)$

so that, for $\overline{r}'\in W_2$, using the fact the initial data is supported in $B(\overline{0},w)$;

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}|\frac{\partial\rho(\overline{y},0)}{\partial t}|dS(\overline{y})\leq 4\pi w^2M$

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}|\rho(\overline{y},0)|dS(\overline{y})\leq 4\pi w^2M$

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}|D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))|dS(\overline{y})\leq\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))\cap B(\overline{0},w)}M|(\overline{y}-\overline{r}')|dS(\overline{y})$

$\leq 4\pi w^2M|-c(t-\frac{|\overline{r}-\overline{r}'|}{c})|$

we have that, for s sufficiently large;

$|\int_{W_2\setminus B(\overline{0},s)}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$

$+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'|$

$\leq\int_{W_2\setminus B(\overline{0},s)}\frac{1}{4\pi c^2|t-\frac{|\overline{r}-\overline{r}'|}{c}|^2}(4\pi w^2M|t-\frac{|\overline{r}-\overline{r}'|}{c}|+4\pi w^2M$

$+4\pi w^2M|-c(t-\frac{|\overline{r}-\overline{r}'|}{c})|)\frac{1}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$= w^2M\int_{W_2\setminus B(\bar{0},s)}(\frac{1}{c^3|t-\frac{|\bar{r}-\bar{r}'|}{c}|}+\frac{1}{c^3|t-\frac{|\bar{r}-\bar{r}'|}{c}|^2}+\frac{1}{c^2|t-\frac{|\bar{r}-\bar{r}'|}{c}|})\frac{1}{|\bar{r}-\bar{r}'|^2}d\tau'$

We have that, for s sufficiently large;

$\int_{W_2\setminus B(\bar{0},s)}\frac{1}{|t-\frac{|\bar{r}-\bar{r}'|}{c}||\bar{r}-\bar{r}'|^2}d\tau'=\int_{W_2}\frac{1}{|t-\frac{(|\bar{r}|^2-2\bar{r}\cdot\bar{r}'+|\bar{r}'|^2)^{\frac{1}{2}}}{c}|(|\bar{r}|^2-2\bar{r}\cdot\bar{r}'+|\bar{r}'|^2)}d\tau'$

$=\int_{W_2\setminus B(\bar{0},s)}\frac{1}{|\bar{r}'|^3|\frac{t}{|\bar{r}|'}-\frac{(\frac{|\bar{r}|^2}{|\bar{r}'|^2}-2\frac{\bar{r}\cdot\bar{r}'}{|\bar{r}'|^2}+1)^{\frac{1}{2}}}{c}|(\frac{|\bar{r}|^2}{|\bar{r}'|^2}-2\frac{\bar{r}\cdot\bar{r}'}{|\bar{r}'|^2}+1)}d\tau'$

and, for $|\bar{r}'|>max(t,4|\bar{r}|)$;

$\frac{1}{|\bar{r}'|^3|\frac{t}{|\bar{r}|'}-\frac{(\frac{|\bar{r}|^2}{|\bar{r}'|^2}-2\frac{\bar{r}\cdot\bar{r}'}{|\bar{r}'|^2}+1)^{\frac{1}{2}}}{c}|(\frac{|\bar{r}|^2}{|\bar{r}'|^2}-2\frac{\bar{r}\cdot\bar{r}'}{|\bar{r}'|^2}+1)}\leq\frac{c}{(2c+4)|\bar{r}'|^3}$

so that;

$\int_{W_2\setminus B(\bar{0},max(t,4|\bar{r}|))}\frac{1}{|t-\frac{|\bar{r}-\bar{r}'|}{c}||\bar{r}-\bar{r}'|^2}d\tau'\leq\int_{W_2\setminus B(\bar{0},max(t,4|\bar{r}))}\frac{c}{(2c+4)|\bar{r}'|^3}d\tau'$

Similarly;

$\int_{W_2\setminus B(\bar{0},max(\sqrt{t},4|\bar{r}|))}\frac{1}{|t-\frac{|\bar{r}-\bar{r}'|}{c}|^2|\bar{r}-\bar{r}'|^2}d\tau'\leq\int_{W_2\setminus B(\bar{0},max(\sqrt{t},4|\bar{r}))}\frac{c}{(2c+8)|\bar{r}'|^4}d\tau'$

so that;

$|\int_{W_2\setminus B(\bar{0},max(t,\sqrt{t},4|\bar{r}|))}\frac{1}{4\pi c^2(t-\frac{|\bar{r}-\bar{r}'|}{c})^2}\int_{\delta B(\bar{r}',-c(t-\frac{|\bar{r}-\bar{r}'|}{c}))}((t-\frac{|\bar{r}-\bar{r}'|}{c})\frac{\partial\rho(\bar{y},0)}{\partial t}$

$+\rho(\bar{y},0)+D\rho(\bar{y},0)\cdot(\bar{y}-\bar{r}'))dS(\bar{y})]\frac{(r_1-r_1')}{c|\bar{r}-\bar{r}'|^3}d\tau'|$

$\leq w^2M\int_{W_2\setminus B(\bar{0},max(t,\sqrt{t},4|\bar{r}))}(\frac{(c+1)}{c^2(2c+4)|\bar{r}'|^3}+\frac{1}{c^2(2c+8)|\bar{r}'|^4})d\tau'$

As above, we have that;

$(\frac{1}{4\pi\epsilon_0}\int_{W_2\cap B(\bar{0},max(t,\sqrt{t},4|\bar{r}))}\frac{\rho(\bar{r}',t_r)\hat{\mathfrak{r}}}{|\bar{r}-\bar{r}'|^2}d\tau')_1$ is finite and we claim that

$\int_{W_2\setminus B(\bar{0},max(t,\sqrt{t},4|\bar{r}))}(\frac{(c+1)}{c^2(2c+4)|\bar{r}'|^3}+\frac{1}{c^2(2c+8)|\bar{r}'|^4})d\tau'$

is finite as well. In order to see this, note that up to a bounded region, W_2 is contained in a family of real quadratic surfaces, parametrised by a finite interval $[-\beta,\beta]\supset[-w-ct,w-ct]$ degenerating to the plane $|\bar{r}'|=|\bar{r}-\bar{r}'|$, if $0\in[-w-ct,w-ct]$, ([5]). Compactifying in $P(\mathcal{R}^3)\times$

[5] Noting, that for $d\in\mathcal{R}_{\neq 0}$, $|\bar{r}'|$ sufficiently large, with the interval $(-\beta,\beta)$ symmetric, we have that, denoting by $|\bar{r}'|-|\bar{r}-\bar{r}'|=|d|$, the union of $|\bar{r}'|-|\bar{r}-\bar{r}'|=d$

and $|\overline{r}'|-|\overline{r}-\overline{r}'|=-d$;

$|\overline{r}'|-|\overline{r}-\overline{r}'|=|d|$ or $|\overline{r}'|+|\overline{r}-\overline{r}'|=|d|$

iff $|\overline{r}-\overline{r}'|^2=|\overline{r}'|^2-2|d||\overline{r}'|+|d|^2$

so that, as $|\overline{r}'|+|\overline{r}-\overline{r}'|=|d|$ is bounded in $\mathcal{R}^3$;

$|\overline{r}'|-|\overline{r}-\overline{r}'|=|d|$

iff $|\overline{r}-\overline{r}'|^2=|\overline{r}'|^2-2|d||\overline{r}'|+|d|^2$

iff $R^2-(2r_1r_1'+2r_2r_2'+2r_3r_3')+|\overline{r}'|^2=|\overline{r}'|^2-2|d||\overline{r}'|+d^2$

iff $-(2r_1r_1'+2r_2r_2'+2r_3r_3')-(|d|^2-R^2)=-2|d||\overline{r}'|$

iff $[(2r_1r_1'+2r_2r_2'+2r_3r_3')+(|d|^2-R^2)]^2=4|d|^2(r_1'^2+r_2'^2+r_3'^2)$

iff $4(r_1r_1'+r_2r_2'+r_3r_3')^2+4(r_1r_1'+r_2r_2'+r_3r_3')(|d|^2-R^2)+(|d|^2-R^2)^2$

$=4d^2(r_1'^2+r_2'^2+r_3'^2)$

where $R=|\overline{r}|$. Note that the degenerate case of a single two dimensional plane in $\mathcal{R}^3$ corresponds to the idealised case when the initial charge distribution ρ_0 is supported at a single point.

In coordinates (x,y,z), if we intersect a real generic quadratic surface defined by;

$\alpha x^2+\beta y^2+\gamma z^2+\delta xy+\epsilon xz+\zeta yz+\eta x+\theta y+\iota z+\kappa=0$, $(*)$

where $\{\alpha,\beta,\gamma,\delta,\epsilon,\theta,\eta,\xi,\eta,\iota,\kappa\}\subset\mathcal{R}$, with a real generic plane $\lambda x+\mu y+\nu z=\xi$, we obtain that $x=\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z$, so that substituting in $(*)$;

$\alpha(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)^2+\beta y^2+\gamma z^2+\delta(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)y+\epsilon(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)z+\zeta yz+$

$\eta(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)+\theta y+\iota z+\kappa=0$

which defines a real quadratic curve in the coordinates (y,z). If the curve is generic and unbounded, it cannot be a parabola, a circle or an ellipse, so by the classification of conic sections, must be a hyperbola. By a result in [16], the standard form of a hyperbola is given by;

$\frac{y^2}{a^2}-\frac{z^2}{b^2}=(\frac{y}{a}+\frac{z}{b})(\frac{y}{a}-\frac{z}{b})=1$

so that by a further change of coordinates $\xi=\frac{y}{a}+\frac{z}{b}$, $\eta=\frac{y}{a}-\frac{z}{b}$, we can write this in the standard form $\xi\eta=1$, with asymptotes $\xi=0$, $\eta=0$, defining a curve C' with asymptotes $\{l_1',l_2'\}$. If the original hyperbola C has asymptotes $\{l_1,l_2\}$, and is defined using a set of coefficients $\{c_i:1\leq i\leq 5\}$, with a fixed bound $|c_i|\leq f$, $f\in\mathcal{R}_{>0}$, then there exists a linear transformation $T:\mathcal{R}^2\to\mathcal{R}^2$ and a shift map

$S:\mathcal{R}^2\rightarrow\mathcal{R}^2$ such that $(ST)(C')=C$, $(ST)(l_1')=l_1$, $(ST)(l_2')=l_2$. If $\overline{x}\in C'$ and $\overline{x}'$ is the nearest point to $\overline{x}$ on $l_1'\cup l_2'$, then $|\overline{x}-\overline{x}'|<\frac{\sqrt{2}}{|\overline{x}}$ for $|\overline{x}|>2$. It follows that;

$|(ST)(\overline{x})-(ST)(\overline{x}')|\leq ||T|||\overline{x}-\overline{x}'|$

$<||T||\frac{\sqrt{2}}{|\overline{x}|}$

$=||T||\frac{\sqrt{2}}{|(ST)^{-1}(ST)\overline{x}|}$

so that for $\overline{y}\in C$, we have that, for the nearest point $\overline{y}'\in l_1\cup l_2$;

$|\overline{y}-\overline{y}'|<\frac{\sqrt{2}||T||}{|(ST)^{-1}\overline{y}|}$

$\leq\frac{\sqrt{2}||T||(||T||+1)}{|\overline{y}}$

provided $|(ST)^{-1}\overline{y}|\geq max(|\overline{s}|,2)$, $(*)$, where $\overline{s}$ defines S, as;

$|\overline{y}|=|(ST)(ST)^{-1}(\overline{y})$

$=|T(ST)^{-1}+\overline{s}|$

$\leq(||T||+1)|(ST)^{-1}(\overline{y}|$

provided $|(ST)^{-1}\overline{y}|\geq|\overline{s}|$, in which case;

$|(ST)^{-1}(\overline{y}|\geq\frac{|\overline{y}|}{||T||+1}$

and;

$\frac{1}{|(ST)^{-1}(\overline{y}|)}\leq\frac{(||T||+1)}{|\overline{y}|}$

We can achieve the condition $(*)$ with $|\overline{y}|\geq||T||(|T^{-1}\overline{s}|+max(2,|\overline{s}|))$, as;

$|(ST)^{-1}\overline{y}|\geq max(|\overline{s}|,2)$

iff $|T^{-1}(\overline{y})-T^{-1}\overline{s}|\geq max(|\overline{s}|,2)$

which we can achieve if $|T^{-1}(\overline{y})|\geq|T^{-1}\overline{s}|+max(|\overline{s}|,2)$

but as $|\overline{y}|\leq||T|||T^{-1}(\overline{y})|$

we have that, $|T^{-1}(\overline{y})|\geq\frac{|\overline{y}|}{||T||}$, so if $|\overline{y}|\geq||T||(|T^{-1}\overline{s}|+max(2,|\overline{s}|))$, then $|T^{-1}(\overline{y})|\geq|T^{-1}\overline{s}|+max(|\overline{s}|,2)$

We then obtain that, for $\overline{y}\in C$, for the nearest point $\overline{y}'\in l_1\cup l_2$;

$[-\beta,\beta]$, and using the implicit function theorem, we could choose a finite cover $\{U_1,\ldots,U_n\}$ of $\mathcal{R}^2\setminus B(\overline{0},1)\times[-\beta,\beta]$ and a sequence of maps $f_i: U_i\rightarrow W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r})$ with constants $C_i\in\mathcal{R}_{>0}$ such that $|f_i(\overline{x},t')|\geq C_i|\overline{x}|$, $|det(Jac(f_i))|$ is bounded uniformly in t' by constants $N_i\in\mathcal{R}_{>0}$, and the maps f_i cover $W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r})$. We then have that;

$|\int_{W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r}))}(\frac{(c+1)}{c^2(2c+4)|\overline{r}'|^3}+\frac{1}{c^2(2c+8)|\overline{r}'|^4})d\tau'|$

$\leq\sum_{i=1}^n|\int_{U_i}f_i^*(\frac{(c+1)}{c^2(2c+4)|\overline{r}'|^3}+\frac{1}{c^2(2c+8)|\overline{r}'|^4})|det(Jac(f_i))|dxdxydt'$

$\leq\sum_{i=1}^n\int_{U_i}N_i(\frac{(c+1)}{C_i^3c^2(2c+3)|(x,y)|^3}+\frac{1}{C_i^3c^2(2c+8)|(x,y)|^4})dxdydt'$

$\leq\sum_{i=1}^n\frac{2N_i\beta}{C_i^3}\int_{\mathcal{R}^2\setminus B(\overline{0},1)}(\frac{(c+1)}{c^2(2c+3)|(x,y)|^3}+\frac{1}{c^2(2c+8)|(x,y)|^4})dxdy$

$\leq\sum_{i=1}^n\frac{4\pi N_i\beta}{C_i^3}\int_{r>1}(\frac{(c+1)r}{c^2(2c+3)r^3}+\frac{r}{c^2(2c+8)r^4})dr$

$=\sum_{i=1}^n\frac{4\pi N_i\beta}{C_i^3}\int_{r>1}(\frac{(c+1)}{c^2(2c+3)r^2}+\frac{1}{c^2(2c+8)r^3})dr$

$=\sum_{i=1}^n\frac{4\pi N_i\beta}{C_i^3}(\frac{(c+1)}{c^2(2c+3)}+\frac{1}{2c^2(2c+8)})$

This proves that $(\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau')_1$ is finite and well defined.

We then have that, using Kirchoff's formula for $\frac{\partial\rho}{\partial t}$, with initial conditions $(\frac{\partial\rho}{\partial t}|_0,\frac{\partial^2\rho}{\partial^2 t}|_0)=(\frac{\partial\rho}{\partial t}|_0,-c^2(\bigtriangledown^2\rho)|_0)$;

$\frac{\partial\rho}{\partial t}(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(t\frac{\partial^2\rho}{\partial t^2}|_0)+\frac{\partial\rho}{\partial t}_0(\overline{y})+$

$D(\frac{\partial\rho}{\partial t}|_0)(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})\ (t>0)$

$\frac{\partial\rho}{\partial t}(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(t\frac{\partial^2\rho}{\partial t^2}|_0)+\frac{\partial\rho}{\partial t}|_0(\overline{y})$

$+D(\frac{\partial\rho}{\partial t}|_0)(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})\ (t<0)$

that, using Jefimenko's equations;

$|\overline{y}-\overline{y}'|\leq\frac{E}{|\overline{y}}$

for $|\overline{y}|\geq D$, where $D=||T||(|T^{-1}\overline{s}|+max(2,|\overline{s}|))$, $E=\sqrt{2}||T||(||T||+1)$.

$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|}d\tau')_1 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\partial\rho}{\partial t}(\overline{r}',t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$= \frac{1}{4\pi\epsilon_0}\int_V [\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)+\frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$+\frac{1}{4\pi\epsilon_0}\int_V [\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)+\frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$ (QQ)

with the convention that $\delta B(\overline{x}_0,r_0)=\emptyset$, when $r_0\leq 0$, $(*)$, using the fact that, for fixed $t\in\mathcal{R}_{<0}$, $t-\frac{|\overline{r}-\overline{r}'|}{c}<0$, and for $t\in\mathcal{R}_{\geq 0}$, $t-\frac{|\overline{r}-\overline{r}'|}{c}=0$ iff $\overline{r}'\in\delta B(\overline{r},ct)$, with $d\tau'(\delta B(\overline{r},ct))=0$. Without loss of generality, we have that $\{(\frac{\partial^2\rho}{\partial^2 t})_0,(\frac{\partial\rho}{\partial t})_0,\rho_0,(D\rho)_0(D\frac{\partial\rho}{\partial t})_0\}$ are supported on $B(\overline{0},\overline{w})$, for some $w\in\mathcal{R}_{>0}$, and, using continuity, we let;

$M=max_{\overline{y}\in B(0,\overline{w})}(|(\frac{\partial^2\rho}{\partial t^2})_0|,|(\frac{\partial\rho}{\partial t})|_0|,|D(\frac{\partial\rho}{\partial t})|_0|)$

We can change the time coordinate, as we can define a new pair $(\rho^t,\overline{J}^t)$ by $\rho^t(\overline{x},s)=\rho(\overline{x},s+t)$ and $\overline{J}^t(\overline{x},s)=\overline{J}(\overline{x},s+t)$, for $(\overline{x},s)\in\mathcal{R}^4$. The new pair $(\rho^t,\overline{J}^t)$ inherits the properties of $(\rho,\overline{J})$, in particular we have that $\rho^t\in C^\infty(\mathcal{R}^4)$, the components of $\overline{J}^t$, $j_i^t\in C^\infty(\mathcal{R}^4)$, $1\leq i\leq 3$, $\square^2(\rho^t)=0$, $\square^2 j_i^t=0$, for $1\leq i\leq 3$, the continuity equation $\frac{\partial\rho^t}{\partial t}=-\bigtriangledown\centerdot\overline{J}^t$ holds, and the connecting relation $\bigtriangledown(\rho^t)+\frac{1}{c^2}\frac{\partial\overline{J}^t}{\partial t}=\overline{0}$. Moreover, we can use Kirchoff's formula with the initial data for $(\rho^t,\overline{J}^t)$ given by $(\rho_0^t,(\frac{\partial\rho^t}{\partial t})_0,\overline{J}_0^t,(\frac{\partial\overline{J}^t}{\partial t})_0)=(\rho_t,(\frac{\partial\rho}{\partial t})_t,\overline{J}_t,(\frac{\partial\overline{J}}{\partial t})_t)$ and we have that;

$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau')_1=(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho^t}(\overline{r}',t_r')\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau')_1$

for the corresponding retarded time $t_r'=-\frac{|\overline{r}-\overline{r}'|}{c}$, and, similarly, for the corresponding terms in Jefimenko's equations.

We can assume in this calculation, that $\overline{r}$ is disjoint from the a ball $B(\overline{0},s)$ containing the support of $\{(\frac{\partial^2\rho}{\partial t^2})_0,(\frac{\partial\rho}{\partial t})|_0,D(\frac{\partial\rho}{\partial t})|_0\}$. This is because, if t is fixed, then we have for a sufficiently large $t'>t$, that $\overline{r}$ is disjoint from a ball $B(\overline{x}_0,s)$ containing the support of $\{(\frac{\partial^2\rho}{\partial t^2})_{t'},(\frac{\partial\rho}{\partial t})|_{t'},D(\frac{\partial\rho}{\partial t})|_{t'}\}$. Then, using the uniqueness property, we have that $\rho(\overline{x,t})$ is determined by the shifted initial conditions $\{(\frac{\partial^2\rho}{\partial t^2})_{t'},(\frac{\partial\rho}{\partial t})|_{t'},D(\frac{\partial\rho}{\partial t})|_{t'}\}$. By a change of coordinates, $\overline{x}'=\overline{x}+\overline{x}_0$, and considering $\rho^{\overline{x}_0}$, we can assume that $\overline{x}_0=\overline{0}$, $\overline{r}$ is disjoint from $B(\overline{0},s)$, with the support of $\{(\frac{\partial^2\rho}{\partial t^2})_{t'},(\frac{\partial\rho}{\partial t})|_{t'},D(\frac{\partial\rho}{\partial t})|_{t'}\}$ contained in $B(\overline{0},s)$. By a further change of coordinates, $t''=t+t'$, and considering $\rho^{t'}$, we can assume that $t'=0$, with the original t moving to $t-t'$, so that we can assume $t<0$, but we can't assume that $t=0$.

It follows, as $t<0$, that in (QQ), we can ignore the term;

$(\frac{1}{4\pi\epsilon_0}\int_V\frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|}d\tau')_1=\frac{1}{4\pi\epsilon_0}\int_V\frac{\partial\rho}{\partial t}(\overline{r}',t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$=\frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)+\frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

and, we are left, simplifying the radius, from (QQ) with;

$+\frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)+\frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$ (QQQ)

If $\overline{d}\in B(\overline{0},s)$, we let;

$V_{\overline{d},t}=\{\overline{r}'\in\mathcal{R}^3:\overline{d}\in\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\}$

$=\{\overline{r}'\in\mathcal{R}^3:|\overline{d}-\overline{r}'|=-ct+|\overline{r}-\overline{r}'|\}$

so that, in (QQQ), we have that $V=\bigcup_{\overline{d}\in B(\overline{0},s)}V_{\overline{d},t}$

As $B(\overline{0},s)$ is open, we can choose $\delta_{\overline{d}}>0$ such that $B(\overline{d},\delta_{\overline{d}})\subset B(\overline{0},s)$. By the calculation above, we can assume that the real unbounded hypersurface $V_{\overline{d},t}$ is a real quadratic surface and, by the calculation below, that the asymptotic cone $Z_{\overline{d},t}$ is a union of lines parametrised over a finite interval. For a line l appearing in the asymptotic cone, fixing $0<\epsilon<\delta_{\overline{d}}$, and $r(\epsilon)$ sufficiently large, we can assume that for $\overline{r}'\in l\cap(\mathcal{R}^3\setminus B(\overline{0},r(\epsilon)))$, there exists $\overline{r}''\in V_{\overline{d},t}$ with $\overline{r}'-\overline{r}''|<\epsilon$, see footnote 5, so that;

$\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)=(\delta B(\overline{r}'',-ct+|\overline{r}-\overline{r}'|)+(\overline{r}'-\overline{r''}))\cap B(\overline{0},s)$

and, as $\overline{d}' = \overline{d} + (\overline{r}' - \overline{r''}) \in B(\overline{d}, \delta_{\overline{d}}) \subset B(\overline{0}, s)$, that $\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|) \cap B(\overline{0}, s) \neq \emptyset$ and passes through $\overline{d}' \in B(\overline{0}, s)$ with $|\overline{d}' - \overline{d}| < \epsilon$. Let $P_{\overline{d}}$ be the plane passing through $\overline{d}$, with $P_{\overline{d}}$ perpendicular to l and intersecting l at $\overline{p}_{\overline{d}}$. Let $T_{\overline{d}'}$ be the tangent plane to $\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|)$ at $\overline{d}'$, intersecting l at $\overline{p}_{\overline{d}'}$, so that we can assume, for sufficiently large $r(\epsilon)$, that $|\overline{p}_{\overline{d}} - \overline{p}_{\overline{d}'}| < \epsilon$. Let $\overline{r}'_{opp} = \overline{p}_{\overline{d}} - (\overline{r}' - \overline{p}_{\overline{d}} = 2\overline{p}_{\overline{d}} - \overline{r}'$. Then, for sufficiently large $r(\epsilon)$, we have that $\overline{r}'_{opp} \in l \cap (\mathcal{R}^3 \setminus B(\overline{0}, r(\epsilon)))$, $\delta B(\overline{r}'_{opp}, -ct + |\overline{r} - \overline{r}'_{opp}|) \cap B(\overline{0}, s) \neq \emptyset$ and passes through $\overline{d}'_{opp} \in B(\overline{0}, s)$ with $|\overline{d}'_{opp} - \overline{d}'| < \epsilon$. We have that;

(i). Using the facts that $|\frac{\partial \rho}{\partial t}|_0| \leq M$ on $B(\overline{0}, s)$, the surface measure of $\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|) \cap B(\overline{0}, s)$ is at most $2\pi s^2$, $\overline{r}'_{opp} = 2\overline{p}_{\overline{d}} - \overline{r}'$, we have, for sufficiently large $r(\epsilon)$, that;

$$|\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2} \int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (t - \tfrac{|\overline{r}-\overline{r}'|}{c})(\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y}) \tfrac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+ \tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2} \int_{\delta B(\overline{r}'_{opp}, -ct+|\overline{r}-\overline{r}'_{opp}|)} (t - \tfrac{|\overline{r}-\overline{r}'_{opp}|}{c})(\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y}) \tfrac{(r_1 - r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$$

$$= |\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})} \tfrac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2} \int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y})$$

$$+ \tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})} \tfrac{(r_1 - r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2} \int_{\delta B(\overline{r}'_{opp}, -ct+|\overline{r}-\overline{r}'_{opp}|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y})|$$

$$= |[\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})} \tfrac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2} + \tfrac{1}{4\pi\epsilon_0} \tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})} \tfrac{(r_1 - r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}] \int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y})$$

$$+ [\tfrac{1}{4\pi\epsilon_0} \tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})} \tfrac{(r_1 - r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}](\int_{\delta B(\overline{r}'_{opp}, -ct+|\overline{r}-\overline{r}'_{opp}|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0) dS(\overline{y})$$

$$- \int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0) dS(\overline{y}))|$$

$$= |\tfrac{1}{16\pi^2\epsilon_0 c^3}[\tfrac{(r_1 - r_1')((t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2 - (t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2 (t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2} + \tfrac{(r_1 - r_1') + (r_1 - r'_{1,opp})}{(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$\int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y}) + [\tfrac{1}{4\pi\epsilon_0} \tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})} \tfrac{(r_1 - r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp}, -ct+|\overline{r}-\overline{r}'_{opp}|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0) dS(\overline{y}) - \int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0) dS(\overline{y}))|$$

$$= |\tfrac{1}{16\pi^2\epsilon_0 c^3}[\tfrac{(r_1 - r_1')((t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2 - (t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2 (t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2} + \tfrac{2r_1 - 2p_{\overline{d},1}}{(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}]$$

$$\int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)} (\tfrac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y}) + [\tfrac{1}{4\pi\epsilon_0} \tfrac{1}{4\pi c^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})} \tfrac{(r_1 + r_1' - 2p_{\overline{d},1})}{c|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$\leq \frac{Ms^2}{8\pi\epsilon_0 c^3}|\frac{(r_1-r'_1)((t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}|+\frac{Ms^2}{8\pi\epsilon_0 c^3}|\frac{2r_1-2p_{\overline{d},1}}{(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}|$$

$$+|\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})}\frac{(r_1+r'_1-2p_{\overline{d},1})}{c|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}|$$

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

$$\leq \frac{Ms^2}{\pi\epsilon_0 c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0 c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0 c^3}\frac{1}{|(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})||\overline{r}+\overline{r}'-2p_{\overline{d}}|}$$

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

(P)

(following the method in (ii), noting the $O(|\overline{r}'|^3)$ term cancels in the first long term to obtain $\frac{O(|\overline{r}'|)O(|\overline{r}'|^2)}{O(|\overline{r}'|^6)}=\frac{1}{O(|\overline{r}'|^3)}$)

Change coordinates, so that the azimuth angle θ of the sphere $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, is centred on the line passing through $\{\overline{r}',\overline{d}'\}$, giving coordinates;

$$\overline{r}'+sin(\theta)cos(\phi)\overline{x}+sin(\theta)sin(\phi)\overline{y}+cos(\theta)(\overline{d}'-\overline{r}')$$

$$(0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi)$$

for a choice of orthogonal vectors $\{\overline{x},\overline{y},\overline{d}'-\overline{r}'\}$ with modulus $-ct+|\overline{r}-\overline{r}'|$. Similarly, choose the azimuth angle θ_{opp} of the sphere $\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)$, is centred on the line passing through $\{\overline{r}'_{opp},\overline{d}'_{opp}\}$, giving coordinates;

$$\overline{r}'+sin(\theta_{opp})cos(\phi_{opp})\overline{x}_{opp}+sin(\theta_{opp})sin(\phi_{opp})\overline{y}_{opp}+cos(\theta_{opp})(\overline{d}'_{opp}-\overline{r}'_{opp})$$

$$(0\leq\theta_{opp}\leq\pi,-\pi\leq\phi_{opp}\leq\pi)$$

for a choice of orthogonal vectors $\{\overline{x}_{opp},\overline{y}_{opp},\overline{d}'_{opp}-\overline{r}'_{opp}\}$ with modulus $-ct+|\overline{r}-\overline{r}'_{opp}|$. We have, for points $\{\overline{q}',\overline{q}'_{opp}\}$ of intersection between $B(\overline{0},s)$ and $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, $B(\overline{0},s)$ and $\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)$ that;

$\theta(\overline{q}')\simeq sin(\theta(\overline{q}'))\leq\frac{2s}{-ct+|\overline{r}-\overline{r}'|}$

$\theta_{opp}(\overline{q}'_{opp})\simeq sin(\theta_{opp}(\overline{q}'_{opp}))\leq\frac{2s}{-ct+|\overline{r}-\overline{r}'_{o}pp|}$ (TT)

and, for sufficiently large $r(\epsilon)$, choosing $\{\overline{x},\overline{y},\overline{x}_{opp},\overline{y}_{opp}\}$ compatibly, we may assume that;

$|\overline{q}'-\overline{q}'_{opp}|\leq 2\epsilon$

for $\{\overline{q}',\overline{q}'_{opp}\}$ defined by coordinates $\theta=\theta_{opp}$, $\phi=\phi_{opp}$ with $0\leq\theta\leq max(\theta_{max},\theta_{max,opp})$, where;

$\theta_{max}=max_{0\leq\phi\leq 2\pi}\theta(\overline{q}')$

for $\overline{q}'$ in $B(\overline{0},s)\cap\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, with coordinates $\{\theta,\phi\}$, and;

$\theta_{max,opp}=max_{0\leq\phi\leq 2\pi}\theta_{opp}(\overline{q}'_{opp})$

for $\overline{q}'_{opp}$ in $B(\overline{0},s)\cap\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)$, with coordinates $\{\theta_{opp},\phi_{opp}\}$

It follows that, for sufficiently large $r(\epsilon)$, using the surface measure $dS=r^2 sin(\theta)$, the fact (TT) and $r^2(1-cos(\frac{1}{r}))=O(1)$, and footnote 5, for sufficiently large r;

$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$

$\leq 2\epsilon|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(-ct+|\overline{r}-\overline{r}'_{opp}|)^2\int_0^{max(\theta_{max},\theta_{max,opp})}sin(\theta)d\theta$

$=2\epsilon|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(-ct+|\overline{r}-\overline{r}'_{opp}|)^2(1-cos(max(\theta_{max},\theta_{max,opp})))$

$\leq C\epsilon$

$\leq\frac{D}{|\overline{r}'+1|}$

where $\{C,D\}\subset\mathcal{R}_{>0}$.

It follows from (P), for sufficiently large $r(\epsilon)$, following the method of (ii), that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{Ms^2}{\pi\epsilon_0 c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0 c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0 c^3}\frac{D}{|\overline{r}'+1|}\frac{1}{|(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})||\overline{r}+\overline{r}'-2p_{\overline{d}}|}$

$\leq \frac{E_1}{|\overline{r}'|^3}$

where $E_1\in\mathcal{R}_{>0}$.

(ii). Using the facts that $|\frac{\partial\rho}{\partial t}|_0|\leq M$ on $B(\overline{0},s)$, the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ is at most $2\pi s^2$, $\overline{r}'_{opp}=2\overline{p}_{\overline{d}}-\overline{r}'$, we have, for sufficiently large $r(\epsilon)$, that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{1}{4\pi\epsilon_0 c}\frac{2\pi Ms^2}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}+\frac{1}{4\pi\epsilon_0 c}\frac{2\pi Ms^2}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}$

$=\frac{Ms^2}{8\pi c\epsilon_0(ct-|\overline{r}-\overline{r}'|)^2|\overline{r}-\overline{r}'|}+\frac{Ms^2}{8\pi c\epsilon_0(ct-|\overline{r}_1+\overline{r}'|)^2|\overline{r}_1+\overline{r}'|}$

$=\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}-\overline{r}'|^3|\frac{ct}{|\overline{r}-\overline{r}'|}+1|^2}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3|(\frac{ct}{|\overline{r}_1+\overline{r}'|}-1)|^2}$

$\leq \frac{Ms^2}{4\pi c\epsilon_0|\overline{r}-\overline{r}'|^3}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3}$

$\leq \frac{3Ms^2}{8\pi c\epsilon_0|\overline{r}'|^3}$

$=\frac{E_2}{|\overline{r}'|^3}$

where $\overline{r}_1=\overline{r}-2\overline{p}_{\overline{d}}$, $E_2\in\mathcal{R}_{>0}$.

(iii). We have that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$=|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}(-ct+|\overline{r}-\overline{r}'|)\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{z}(\overline{y}))]dS(\overline{y})\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}(-ct+|\overline{r}-\overline{r}'_{opp}|)\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot$
$(\overline{z}_{opp}(\overline{y})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'|)}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}_{opp}(\overline{y})dS(\overline{y})|$
(NN)

Letting $\overline{z}_0=\frac{(\overline{d}'-\overline{r}')}{-ct+|\overline{r}-\overline{r}'|}$, so that $|\overline{z}_0|=1$, R the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$, using Lemma 0.49, following the method of (i), we have that, for sufficiently large $r(\epsilon)$;

$|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|$

$=|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})+\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot$

$\overline{z_0}dS(\overline{y})|$

$\leq|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})|+|[\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)]\centerdot$

$\overline{z_0}dS(\overline{y})|$

$\leq Rmax_{\overline{y}\in B(\overline{0},s)}|D(\frac{\partial\rho}{\partial t})(\overline{y},0)||\overline{z}(\overline{y})-\overline{z_0}|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})\centerdot$
$\overline{z_0}|$

$\leq RMmax_{\overline{y}\in B(\overline{0},s)}|\overline{z}(\overline{y})-\overline{z_0}|+|\overline{z_0}||\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RM|(1-cos(\theta_{max},sin(\theta_{max})|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})$

$-\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|+|\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$=\sqrt{2}RM(1-cos(\theta_{max}))^{\frac{1}{2}}+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})-\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RMF\theta_{max}+G\epsilon$

$\leq\frac{2sH}{-ct+|\overline{r}-\overline{r}'|}+\frac{W}{|1+\overline{r}'|}$

$=\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|}$

where $\{F,G,W,H,A_1,B_1\}\subset\mathcal{R}_{>0}$. Similarly, there exist $\{A_2,B_2\}\subset\mathcal{R}_{>0}$, such that

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|\leq\frac{A_2}{-ct+|\overline{r}-\overline{r}'_{op}|}+\frac{B_2}{|1+\overline{r}'_{op}|}$$

$$=\frac{A_2}{-ct+|\overline{r}+\overline{r}'-2\overline{p}_{\overline{d}}|}+\frac{B_2}{|1+2\overline{p}_{\overline{d}}-\overline{r}'|}$$

so that, from (NN), following the method of (ii)

$$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$$

$$\leq\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}')}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$$

$$+\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{-ct+|\overline{r}+\overline{r}'-2\overline{p}_{\overline{d}}|}+\frac{B_2}{|1+2\overline{p}_{\overline{d}}-\overline{r}'|})$$

$$=\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|(t-\frac{|\overline{r}-\overline{r}'|}{c})||\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$$

$$+\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c}||\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{-ct+|\overline{r}+\overline{r}'-2\overline{p}_{\overline{d}}|}+\frac{B_2}{|1+2\overline{p}_{\overline{d}}-\overline{r}'|})$$

$$\leq\frac{E_3}{|\overline{r}'|^3}$$

where $E_3\in\mathcal{R}_{>0}$ $((i),(ii),(iii))$

By the calculation below, we can assume that the asymptotic cone $Z_{\overline{d},t}$ of the real unbounded hypersurface $V_{\overline{d},t}$ is a union of lines parametrised over a finite interval $[\alpha,\beta]$. It follows that we can define maps $\theta_1:\mathcal{R}\times[\alpha,\beta)\rightarrow Z_{\overline{d},t}$, $\theta_2:\mathcal{R}\times[\alpha,\beta)\rightarrow Z_{\overline{d},t}$, such that for fixed $\gamma\in[\alpha,\beta)$, $\theta_1(r,\gamma)\in l_{\gamma,\overline{d},1}$, $\theta_2(r,\gamma)\in l_{\gamma,\overline{d},2}$, $r\in\mathcal{R}$, where the intersection curve $C_{\gamma,\overline{d}}$ has the two real asymptotes $\{l_{\gamma,\overline{d},1},l_{\gamma,\overline{d},2}\}$, and, such that, for $i\in\{1,2\}$;

(i). $\theta_i(0,\gamma)=p_{\overline{d},\gamma,i}$, (using the notation above)

(ii). $\theta_i(r,\gamma)_{opp}=\theta_1(-r,\gamma)$

(iii). There exist $R_i\subset\mathcal{R}_{>0}$ with θ_i diffeomorphisms outside $[-R_i,R_i]\times[\alpha,\beta)$, with the partial derivatives uniformly bounded.

(iv). $Im(\theta_1|_{\mathcal{R}\setminus[-R_1,R_1]\times[\alpha,\beta)}\cap Im(\theta_2|_{\mathcal{R}\setminus[-R_2,R_2]\times[\alpha,\beta)})=\emptyset$

(v). For $r_2 > r_1 > R_i$, $|\theta_i(r_2,\gamma) - \theta_i(r_1,\gamma)| = r_2 - r_1$

It follows from $(iii),(v)$ that the pullback;

$\theta_1|^*_{\mathcal{R}\setminus[-R_1,R_1]\times[\alpha,\beta)}(dLeb|_{Z_{\overline{d},t}}) = |\frac{\partial\theta_1}{\partial r}\times\frac{\partial\theta_1}{\partial\gamma}|drd\gamma = f(r,\gamma)drd\gamma$

has the property that $f(r,\gamma)$ has order $O(r)$, uniformly in γ and $f(r,\gamma) = f(-r,\gamma)$, for $r\in\mathcal{R}_{>0}$. For $R\in\mathcal{R}_{>0}$, with $R > R_i$, can define the regions $S_{R,i}\subset\mathcal{R}\times[\alpha,\beta)$, by;

$S_{R,i} = \{(r',\gamma) : R_i \leq |r'| \leq R, \gamma\in[\alpha,\beta)\}$

with corresponding regions $\theta_i(S_{R,i})\subset Z_{\overline{d},t}$

Then, by the calculation above, letting;

$H(\overline{r}') = \frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

we have that, for $r > R_i$;

$|\theta_1^*H(r,\gamma) + \theta_1^*H(-r,\gamma)| \leq \frac{C}{r^3}$

$|f(r,\gamma)| \leq Dr$

$|(\theta_1^*H(r,\gamma) + \theta_1^*H(-r,\gamma))f(r,\gamma)| \leq \frac{CD}{r^2}$

and;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})} H(\overline{r}')d\overline{r}' = lim_{R\rightarrow\infty,R>R_i}\int_{S_{R,i}}(\theta_1^*H)(r,\gamma)f(r,\gamma)drd\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}[\int_{R_i}^R\theta_1^*H(r,\gamma)f(r,\gamma)dr + \int_{-R}^{-R_i}\theta_1^*H(r,\gamma)f(r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}[\int_{R_i}^R\theta_1^*H(r,\gamma)f(r,\gamma)dr + \int_{-R}^{-R_i}\theta_1^*H(-r,\gamma)f(-r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}\int_{R_i}^R(\theta_1^*H(r,\gamma) + \theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma$

$= \int_{[\alpha,\beta)}\int_{R_i}^\infty(\theta_1^*H(r,\gamma) + \theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma$

where, letting $G(\gamma) = \int_{R_i}^\infty(\theta_1^*H(r,\gamma) + \theta_1^*H(-r,\gamma))f(r,\gamma)dr$;

$|G(\gamma)| \leq \int_{R_i}^\infty\frac{CD}{r^2}dr = [\frac{-CD}{r}]_{R_i}^\infty$

$= \frac{CD}{R_i}$

so that;

$lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}\int_{R_i}^{\infty}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma = \int_{[\alpha,\beta)}G(\gamma)d\gamma$

exists and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}\int_{R_i}^{\infty}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma| \leq \frac{CD(\beta-\alpha)}{R_i}$

It follows;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}H(\overline{r}')d\overline{r}'$

exisrs, and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}H(\overline{r}')d\overline{r}'| \leq \frac{CD(\beta-\alpha)}{R_i}$

as well. (UU)

..... Let the lines appearing in the asymptotic cone $Z_{\overline{d},t}$, parametrised by $[\alpha,\beta)$, correspond to the system of hyperplanes H_γ, $\gamma\in[\alpha,\beta)$ with fixed locus $Q_{\overline{d},t}$. Then, for $\overline{x}\in V_{\overline{d},t}\setminus Q_{\overline{d},t}$, $|\overline{x}|$ sufficiently large, let $\overline{x}_{near}$ be the nearest point on the asymptotic line $l_{\overline{d},t,\gamma}$, and $\overline{x}_{opp}$ be the nearest point on $V_{\overline{d},t}\cap H_\alpha$ to $(\overline{x}_{near})_{opp}$. By a simple adaptation of the above argument $((i),(ii),(iii))$, we have that, for $|\overline{r}'|$ sufficiently large, there exists $C\in\mathcal{R}_{>0}$, with;

$|h(\overline{r}')+h(\overline{r}'_{opp})| \leq \frac{C}{|\overline{r}'|^3}$

where $h(\overline{r}') = (\frac{1}{4\pi\epsilon_0}\frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|})_1$

(Follow argument of (UU), using the facts, that for sufficiently large $|\overline{r}'|$, $dV_{\overline{d},t}\simeq dZ_{\overline{d},t}$, $dV_{\overline{d},t} = g(\overline{r}')dZ_{\overline{d},t}$, with $g(\overline{r}')\simeq g(\overline{r}'_{opp})$, for the surface measures on $V_{\overline{d},t}$ and $Z_{\overline{d},t}$ respectfully, as;

$\frac{\partial\theta_1}{\partial r}(\overline{r}',\gamma) \simeq \frac{\partial\theta_1}{\partial r}(\overline{r}'_{opp},\gamma)$

for the appropriate parametrisation θ_1, so that;

$|h(\overline{r}')g(\overline{r}')dZ_{\overline{d},t}+h(\overline{r}'_{opp})g(\overline{r}'_{opp})dZ_{\overline{d},t}|$

$\leq |(h(\overline{r}')+h(\overline{r}'_{opp}))g(\overline{r}')dZ_{\overline{d},t}|+|h(\overline{r}'_{opp})(g(\overline{r}'_{opp})-g(\overline{r}'))dZ_{\overline{d},t}|$

$=O(\frac{1}{R^3})O(R)drd\gamma+O(\frac{1}{R^2})O(\frac{1}{R})drd\gamma=O(\frac{1}{R^2})drd\gamma.$

Same idea for asymptotic cones defined below in Lemma 0.51, reflecting the branch at infinity.)

.... If $t=0$, then $W_2=\{\overline{r}':-w\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w\}$, and, by the calculation in footnote 5, we can, for sufficiently large $\overline{r}'$, characterise W_2 as a family of quadratic surfaces, parametrised by $[0,w]$, degenerating to the plane $\overline{r}'=|\overline{r}-\overline{r}'|$. We denote by W_2^s, for $0\leq s\leq w$ the locus;

$\{\overline{r}':|\overline{r}'|-|\overline{r}-\overline{r}'|=s\}\cup\{\overline{r}':|\overline{r}'|-|\overline{r}-\overline{r}'|=-s\}$

characterised, for $s\neq 0$, by the quadratic real surface V_s in footnote 5, with W_0 being the plane $\{\overline{r}':|\overline{r}'|=|\overline{r}-\overline{r}'|\}$. Fixing $s_0\neq 0$, for a real generic hyperplane H_{s_0}, using footnote 5, the intersection $V_{s_0}\cap H_{s_0}$ is a real unbounded generic quadratic curve $C_{s_0}\subset H_{s_0}$. In particularly, by the classification of real quadratic curves as conic sections, C_{s_0} is generic hyperbolic and has two real asymptotes $\{l_{s_0,1},l_{s_0,2}\}$. If we take a generic real 1-dimensional pencil of hyperplanes $\{H_{s_0,r}:r\in\mathcal{R}\}$, such that $\bigcup_{r\in\mathcal{R}}H_{s_0,r}=\mathcal{R}^3$, with base locus l_{s_0}, then clearly;

$\bigcup_{r\in\mathcal{R}}(V_{s_0}\cap H_{s_0,r})=V_{s_0}$

and, using O-minimality, there exists finitely many open bounded intervals $\{I_j:1\leq j\leq n\}$ for which $V_{s_0}\cap H_{s_0,r}$ is finite, $r\in\bigcup_{1\leq j\leq n}I_j$. Let $P_{s_0}=\mathcal{R}\setminus\bigcup_{1\leq j\leq n}I_j$, and we still have that;

$\bigcup_{r\in P_{s_0}}(V_{s_0}\cap H_{s_0,r})=V_{s_0}$

We define the two dimensional asymptotic cone Z_{s_0} of V_{s_0} to be $\bigcup_{r\in P_{s_0}}l_{s_0,r,1}\cup l_{s_0,r,2}$ where the intersection curve $C_{s_0,r}$ has the two real asymptotes $\{l_{s_0,r,1},l_{s_0,r,2}\}$. By choosing the base locus l_{s_0} to intersect V_{s_0} in a finite number of points and noting that for a sufficiently generic family, $\overline{V_{s_0}}\cap H_{s_0,r}\cap W=0$, in coordinates $[X,Y,Z,W]$, where $\overline{V_{s_0}}$ is the projective closure of V_{s_0} in $P(\mathcal{R}^3)$, is mobile, and compact, so can be paramertised analytically by a finite interval. we can assume that

P_{s_0} is a finite interval I_{s_0} when parametrising Z_{s_0} and V_{s_0}, so that;

$\bigcup_{r\in I_{s_0}}(V_{s_0}\cap H_{s_0,r})=V_{s_0}$

Let $d\tau'_{s_0}$ be the surface measure on Z_{s_0} obtained from the pullback of Lebesgue measure with the inclusion of Z_{s_0} in $\mathcal{R}^3$ and, similarly, let $d\tau'_{s_0,r,1}$ and $d\tau'_{s_0,r,2}$ be the line measures on $l_{s_0,r,1}$ and $l_{s_0,r,2}$, obtained from the pullback of Lebesgue measure, and let $d\tau'_{s_0,r,1,2}$ be the union of the measures on $l_{s_0,r,1}\cup l_{s_0,r,2}$. .... If $t_1<t_2$, with $\{t_1,t_2\}\subset\mathcal{R}$, and $\{V_{t_1},V_{t_2}\}$ denote the compact supports of $\{\rho_{t_1},\rho_{t_2}\}$, then as the supports vary continuously, and $\overline{J}_t$ and ρ_t are compactly supported for each $t\in[t_1,t_2]$, $\overline{J}_t$ and ρ_t are uniformly compacted supported for $t\in[t_1,t_2]$ in a ball $B(\overline{0},p)$, for some $p\in\mathcal{R}_{>0}$. In particularly;

$\int_{V_{t_1}}\rho_{t_1}dV=\int_{B(\overline{0},p)}\rho_{t_1}dV$

$\int_{V_{t_2}}\rho_{t_2}dV=\int_{B(\overline{0},p)}\rho_{t_2}dV$

For $t\in[t_1,t_2]$, using the continuity equation, the divergence theorem and the fact $\overline{J}_t$ is uniformly compacted supported for $t\in[t_1,t_2]$ in $B(\overline{0},p)$, we have that;

$\frac{d}{dt}(\int_{B(\overline{0},p)}\rho_t dV)=\int_{B(\overline{0},p)}\frac{\partial\rho}{\partial t}dV$

$=\int_{B(\overline{0},p)}div(\overline{J})_t dV$

$=\int_{\delta B(\overline{0},p)}\overline{J}_t\cdot d\overline{S}dV$

$=0$

so that;

$\int_{B(\overline{0},p)}\rho_{t_1}dV=\int_{B(\overline{0},p)}\rho_{t_2}dV$

$\int_{V_{t_1}}\rho_{t_1}dV=\int_{V_{t_2}}\rho_{t_2}dV$

In particularly, $\frac{d}{dt}(\int_{V_t}\rho_t dV)=0$, ([6]). The same argument applies for $\frac{\partial\rho}{\partial t}$, with associated current $\overline{J}_1=-c^2\bigtriangledown(\rho)$ and compact supports W_t, $t\in\mathcal{R}$, obeying the wave equation $\square^2(\overline{J}_1)=\overline{0}$. It follows from the Reynold's transport theorem, ([7]), the divergence theorem and the fact that $\overline{J}_1$ vanishes outside W_t and V_t, that;

6

In fact, the result is true for $(\rho,\overline{J})$ satisfying the continuity equation, when $\overline{J}$ fails to have compact support, and the components j_i, for $1\leq i\leq 3$, are uniformly of rapid decay, in the sense, that for any finite interval $[t_1,t_2]$, there exists constants $C_{1,2,i,n}\in\mathcal{R}_{>0}$ such that $|j_i(\overline{x},t)|\leq\frac{C_{1,i,2,n}}{|\overline{x}|^n}$ for $t\in[t_1,t_2]$ and $|\overline{x}|>1$. In order to see this, suppose that on a finite interval (t_1,t_2), ρ is supported uniformly on $B(\overline{0},p)$. and $\frac{d}{dt}\int_{V_t}\rho dV\neq 0$, for some $t\in[t_1,t_2]$. Then there exists an interval $(t_0-\epsilon,t_0+\epsilon)\subset(t_1,t_2)$, such that, without loss of generality, $\frac{d}{dt}\int_{V_t}\rho dV|_{(t_0-\epsilon,t_0+\epsilon)}>0$, and, by the intermediate value theorem, we can assume that $\int_{V_t}\rho dV|_{(t_0-\epsilon,t_0+\epsilon)}$ is strictly increasing, with $\int_{V_{t_0+\epsilon}}\rho_{t_0+\epsilon}dV-\int_{V_{t_0}}\rho_{t_0}dV>\delta>0$, $(*)$. Using the hypotheses on $\overline{J}$, we can choose $r>p$ sufficienly large such that for $t\in(t_0-\epsilon,t_0+\epsilon)$, $|\int_{\delta B(\overline{0},r)}\overline{J}_t\centerdot d\overline{S}|<\delta_1$, and by the continuity equation, for $t\in(t_0-\epsilon,t_0+\epsilon)$;

$|\frac{d}{dt}\int_{B(\overline{0},r)}pdV|=|\int_{B(\overline{0},r)}\frac{\partial\rho}{\partial t}dV|$

$=|-\int_{B(\overline{0},r)}div(\overline{J})dV|$

$=|\int_{\delta B(\overline{0},r)}\overline{J}\centerdot d\overline{S}|$

$<\delta_1$

and the intermediate value theorem;

$|\int_{B(\overline{0},r)}p_{t_0+\epsilon}dV-\int_{B(\overline{0},r)}p_{t_0}dV|<\delta_1\epsilon$

so choosing $\delta_1=\frac{\delta}{2\epsilon}$, we obtain that;

$|\int_{B(\overline{0},r)}p_{t_0+\epsilon}dV-\int_{B(\overline{0},r)}p_{t_0}dV|=|\int_{V_{t_0+\epsilon}}p_{t_0+\epsilon}dV-\int_{V_{t_0}}p_{t_0}dV|$

$<\frac{\delta}{2}$

which contradicts $(*)$.

[7] The Reynolds transport theorem is true in this case, but is not the usual form, as, due to the failure of analyticity, there can be jumps in the support. There is also an issue with using the formula $\rho\overline{v}=\overline{J}$, when substituting for the velocity of the area element. This could be resolved in [15].

$\int_{V_t} \bigtriangledown^2(\rho) dV = \frac{1}{c^2} \int_{V_t} \frac{\partial^2 \rho}{\partial t^2} dV$

$= \frac{1}{c^2}(\frac{d}{dt}(\int_{V_t} \frac{\partial \rho}{\partial t} dV) - \int_{V_t} div(\overline{J}_1))$

$= -\frac{1}{c^2} \int_{V_t} div(\overline{J}_1) dV$

$= -\frac{1}{c^2} \int_{\delta V_t} \overline{J}_1 \cdot d\overline{S}$

$= 0$

In particular, at $t = 0$, we can assume that;

$\int_{V_0} \bigtriangledown^2(\rho_0) dV = \int_{V_0} (\sum_{i=1}^{3} (\frac{\partial^2 \rho}{\partial x_i^2})_0) dV = 0$ (O), ([8]).

We can define antiderivatives, by letting;

$p^a(\overline{x}, t) = \int_{-\infty}^{t} p(\overline{x}, s) ds$

$\overline{J}^a(\overline{x}, t) = \int_{-\infty}^{t} \overline{J}(\overline{x}, s) ds$ (if the integral exists)

As is easily checked, if $p \in C^\infty(\mathcal{R}^4)$ and the components $j_i \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, then $\rho^a \in C^\infty(\mathcal{R}^4)$ and the components $j_i^a \in C^\infty(\mathcal{R}^4)$, for $1 \leq i \leq 3$. The wave equation holds for ρ^a and $\overline{J}^a$, as, using the fundamental theorem of calculus, differentiating under the integral sign, the result about he left hand limit in [14], and using the fact that ρ satisfies the wave equation;

$\square^2(\rho^a) = \int_{-\infty}^{t} \bigtriangledown^2(\rho) ds - \frac{1}{c^2} \frac{\partial \rho}{\partial t}$

$= \int_{-\infty}^{t} \frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} ds - \frac{1}{c^2} \frac{\partial \rho}{\partial t}$

---

[8] Note that you can also deduce this, using the divergence theorem, and the fact that $\bigtriangledown(\rho_0)$ vanishes on δV_0;

$\int_{V_0} \bigtriangledown^2(\rho_0) dV = \int_{\delta V_0} \bigtriangledown \cdot (\bigtriangledown(\rho_0)) dV$

$= \int_{\delta V_0} \bigtriangledown(\rho_0) \cdot d\overline{S}$

$= 0$

$= \frac{1}{c^2}\frac{\partial\rho}{\partial t} - \frac{1}{c^2}\frac{\partial\rho}{\partial t}$

$= 0$

and;

$\square^2(\overline{J}^a) = \int_{-\infty}^t \bigtriangledown^2(\overline{J})ds - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \int_{-\infty}^t \frac{1}{c^2}\frac{\partial^2\overline{J}}{\partial t^2}ds - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t} - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \overline{0}$

Differentiating under the integral sign and using the fundamental theorem of calculus, the fact that the continuity equation holds for $(\rho, \overline{J})$, the continuity equation holds as;

$\frac{\partial\rho^a}{\partial t} + \bigtriangledown \cdot \overline{J}^a$

$= \rho + \int_{-\infty}^t \bigtriangledown \cdot \overline{J}ds$

$= \rho + \int_{-\infty}^t + \int_{-\infty}^t -\frac{\partial\rho}{\partial s}ds$

$= \rho - \rho = 0$

and, differentiating under the integral sign, using the fundamental calculus of calculus and the connecting relation for $(\rho, \overline{J})$, the connecting relation holds;

$\bigtriangledown(\rho^a) + \frac{1}{c^2}\frac{\partial\overline{J}^a}{\partial t}$

$= \int_{-\infty}^t \bigtriangledown(\rho)ds + \frac{1}{c^2}\overline{J}$

$= \int_{-\infty}^t -\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}ds + \frac{1}{c^2}\overline{J}$

$= -\frac{1}{c^2}\overline{J} + \frac{1}{c^2}\overline{J}$

$= \overline{0}$,([9])

.... Then the fields $\{\overline{E},\overline{B}\}$ are well defined by Jefimenko's equations and the components are of uniform very moderate decrease.

□

Lemma 0.49. *Cancellation Lemma*

Let $g \in C^{\infty}(\mathcal{R}^3)$ with compact support $V \subset \mathcal{R}^3$, then for a hyperplane $H \subset \mathcal{R}^3$, we have that;

$\int_{V\cap H} \triangledown(g)d\mu = \overline{0}$

where μ is Lebesgue measure on $V \cap H$.

Proof. With out loss of generality, we can assume that $V = B(\overline{0}, r)$, for some $r \in \mathcal{R}_{>0}$ and H is a hyperplane passing through $\overline{0}$, with the equation $\alpha x+\beta y+\gamma z = 0$. Assume first that $\{\alpha, \beta, \gamma\} \subset \mathcal{R}$ are distinct and non zero. Let $pr_{12}, pr_{13}, pr_{23}$ be the projections onto the coordinates $(x, y), (x, z), (y, z)$. Let;

---

[9] We don't necessarily have that $(\rho^a, \overline{J}^a)$ has compact supports. On a finite interval $[t_1, t_2]$, for sufficiently large $\overline{x}$, we have $\frac{\partial \rho^a}{\partial t} = \rho = 0$, and;

$\triangledown^2(\rho_a) = \frac{1}{c^2}\frac{\partial^2 \rho^a}{\partial^2 t}$

$= 0$

Let $h(\overline{x})$ define ρ^a for sufficiently large $\overline{x}$, then, as $\mathcal{R}^3 = \bigcup_{t\in\mathcal{R}} Supp(\rho_t)^c$;

$\triangledown^2(h(\overline{x})) = \Box^2(h(\overline{x})) = 0$

everywhere. We can repeat the argument for the antiderivative $\overline{J}^a$ to obtain $\overline{c}(\overline{x})$ defining $\overline{J}^a$ for sufficiently large $\overline{x}$. so, as $\mathcal{R}^3 = \bigcup_{t\in\mathcal{R}} Supp(\overline{J}_t)^c$, we have that $\triangledown^2(\overline{c}(\overline{x})) = \Box^2(\overline{c}(\overline{x})) = \overline{0}$, and, clearly, for the pair $(h(\overline{x}), \overline{c}(\overline{x}))$, we have that;

$div(\overline{c}(\overline{x})) = -\frac{\partial h}{\partial t} = 0$

$\triangledown(h)(\overline{x}) = -c^2\frac{\partial \overline{c}(\overline{x})}{\partial t}$

$= \overline{0}$

and $(\rho^a - h(\overline{x}), \overline{J}^a - \overline{c}(\overline{x}))$ has compact supports and inherits all the properties above for $(\rho^a, \overline{J}^a)$.

$g_{12}(x,y)=g(x,y,z(x,y))=g(x,y,-\frac{\alpha x}{\gamma}-\frac{\beta y}{\gamma})$

$g_{13}(x,z)=g(x,y(x,z),z)=g(x,-\frac{\alpha x}{\beta}-\frac{\gamma z}{\beta},z)$

$g_{23}(y,z)=g(x(y,z),y,z)=g(-\frac{\beta y}{\alpha}-\frac{\gamma z}{\alpha},y,z)$

Then, by the chain rule;

$\frac{\partial g_{12}}{\partial x}|_{(x,y)}=(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}-\frac{\alpha}{\gamma}\frac{\partial g}{\partial z})_{(x,y,z(x,y))}$

$\frac{\partial g_{12}}{\partial y}|_{(x,y)}=(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}-\frac{\beta}{\gamma}\frac{\partial g}{\partial z})_{(x,y,z(x,y))}$

so that;

$\frac{\beta}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)}-\frac{\alpha}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)}=\frac{(\beta-\alpha)}{\gamma}(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y})|_{(x,y,z(x,y))}$

and;

$(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y})|_{(x,y,z(x,y))}=\frac{\gamma}{(\beta-\alpha)}(\frac{\beta}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)}-\frac{\alpha}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)})$

and, a similar calculation holds for $\{g_{13},g_{23}\}$. It follows that, using Fubini's theorem, the fundamental theorem of calculus and the fact that g_{12} vanishes on $\delta(pr_{12}(V\cap H))$;

$\int_{V\cap H}(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y})d\mu=\int_{pr_{12}(V\cap H)}(\frac{\beta}{\beta-\alpha}\frac{\partial g_{12}}{\partial x}-\frac{\alpha}{\beta-\alpha}\frac{\partial g_{12}}{\partial x})|_{(x,y)}c_{12}(\alpha,\beta,\gamma)dxdy$

$=0$

where $c_{12}(\alpha,\beta,\gamma)\in\mathcal{R}$ is non-zero. Similarly, using $\{pr_{13},pr_{23}\}$;

$\int_{V\cap H}(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial z})d\mu=\int_{V\cap H}(\frac{\partial g}{\partial y}+\frac{\partial g}{\partial z})d\mu$

so that;

$\int_{V\cap H}\frac{\partial g}{\partial x}d\mu=-\int_{V\cap H}\frac{\partial g}{\partial y}d\mu=\int_{V\cap H}\frac{\partial g}{\partial z}=-\int_{V\cap H}\frac{\partial g}{\partial x}d\mu$

and;

$\int_{V\cap H}\frac{\partial g}{\partial x}d\mu=0$

Similarly;

$\int_{V\cap H}\frac{\partial g}{\partial y}d\mu=\int_{V\cap H}\frac{\partial g}{\partial z}d\mu=0$

and;

$\int_{V\cap H}\bigtriangledown(g)d\mu=\overline{0}$

By continuity, the result holds for any hyperplane H as the initial assumption was generic.

□

Lemma 0.50. *Uniqueness of Representation of Arcs*

Suppose that $\overline{x}\in\mathcal{R}^3\setminus B(\overline{0},s)$ *such that* $\delta B(\overline{x},r)\cap B(\overline{0},s)\neq\emptyset$*, then there exists a unique* $0\leq w\leq s$ *such that* $B(\overline{0},w)$ *intersects* $B(\overline{x},r)$ *at a single point* $\overline{p}_{\overline{x},r}$*, with the property that the spheres* $\delta B(\overline{x},r)$ *and* $\delta B(\overline{0},w)$ *share a tangent plane at* $\overline{p}_{\overline{x},r}$*.*

Proof. Suppose that $\overline{0}\notin B(\overline{x},r)$. Let l be the line connecting the points $\{\overline{0},\overline{x}\}$, intersecting the sphere $\delta B(\overline{0},s)$ at $\overline{q}$. Then $\overline{q}\in B(\overline{x},r)$, otherwise $\delta B(\overline{x},r)\cap B(\overline{0},s)=\emptyset$. We have that $\delta B(\overline{x},r)\cap B(\overline{0},s)$ partitions $B(\overline{0},s)$ into 2 disjoint, connected regions, and the regions containing $\overline{0}$ and $\overline{q}$ are distinct. It follows that the line l between $\overline{0}$ and $\overline{q}$ intersects $B(\overline{x},r)$ at the point $\overline{p}_{\overline{x},r}\in B(\overline{0},s)$. Choose $0\leq w\leq s$ such that $\delta B(\overline{0},w)$ passes through $\overline{p}_{\overline{x},r}$. Then, as the tangent planes to the spheres $\delta B(\overline{0},w)$ and $\delta B(\overline{x},r)$ at $\overline{p}_{\overline{x},r}$ are both perpendicular to l and pass through $\overline{p}_{\overline{x},r}$, they must coincide. Suppose that the spheres $\delta B(\overline{0},w)$ and $\delta B(\overline{x},r)$ share a further intersection point $\overline{p}'$ with the properties that the tangent planes at $\overline{p}'$ coincide, then the lines l and l', where l' connects the points $\{\overline{0},\overline{p}'\}$, both pass through $\overline{0}$ and $\overline{x}$, so must coincide and $\overline{p}'\in l$. Then, as $\overline{p}_{\overline{x},r}$ and $\overline{p}'$ are distinct, it follows that $\overline{p}'\notin\delta B(\overline{x},r)$.

□

Lemma 0.51. *Fix* $0<w\leq s$ *and with* $\overline{r}\notin B(\overline{0},s)$*,* $t<0$*, let* $V_w(\overline{x})$ *be the locus defined by;*

$B(\overline{0},w)$ *intersects* $B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ *at a single point* $\overline{p}_{\overline{x}}$*, with the property that the spheres* $\delta B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ *and* $\delta B(\overline{0},w)$ *share a tangent plane at* $\overline{p}_{\overline{x},r}$*.*

Then, $V_w(\overline{x}) \subset V_w^1(\overline{x}$*, where;*

$$V_w^1(\overline{x}) \equiv \exists\lambda\exists\overline{y}[(|\overline{y}| = w) \wedge ((|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}|)$$

$$\vee(|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|) \vee (|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct)) \wedge \overline{x} = \lambda\overline{y}]$$

and $V_w^1(\overline{x})$ *is generically a double cover of* $\delta B(\overline{0}, w)$*, and there exists parallel planes* $\{P_1, P_2\} \subset \mathcal{R}^3$*, such that, either;*

$V_w^1(\overline{x})$ *is bounded*

when $(P_1 \cap \delta B(\overline{0}, w)) = (P_2 \cap \delta B(\overline{0}, w)) = \emptyset$*, or;*

$V_w^1(\overline{x})$ *blows up at an exceptional locus* $Z_a \subset \delta B(\overline{0}, w)$

where $Z_a = (P_1 \cap \delta B(\overline{0}, w)) \cup (P_2 \cap \delta B(\overline{0}, w))$ *is the union of* 2 *circles on* $\delta B(\overline{0}, w)$*. For specific, non-generic* w*, these circles can coincide, but, in the generic case, when* Z_a *has two components,* $V_w(\overline{x})$ *basically has two asymptotic cones among* $Cone_1(\overline{0}, P_1 \cap \delta B(\overline{0}, w))$ *and* $Cone_2(\overline{0}, P_2 \cap \delta B(\overline{0}, w))$ *corresponding to distinct* $\{P_1, P_2\}$*, with a single pair of infinite opposite branches along asymptotes, which are bounded translations of the lines of the cones. The cover splits into a bounded and unbounded component centred along the asymptotes. In a special case of this generic behaviour, again corresponding to specific* w*,* $V_w^1(\overline{x})$ *can blow up along one component of* Z_a *and remain bounded over the other component. There is another special case, due to a specific link between* t *and* $\overline{r}$*, which can occur for non generic* w*, but it exhibits similar behaviour to the generic case.*

Proof. By the proof of Lemma 0.50, we have that;

$$V_w(\overline{x}) \equiv \exists\lambda_{\neq 0}\exists\overline{y}[(|\overline{y}| = w) \wedge (|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}|) \wedge \overline{x} = \lambda\overline{y}]$$

Making the substitutions $\overline{x} = \lambda\overline{y}$ and $|\overline{y}| = w$, we have that;

$$|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}| \iff |\lambda\overline{y} - \overline{y}| = -ct + |\lambda\overline{y} - \overline{r}|$$

$$\iff |\lambda - 1||\overline{y}| = -ct + |\lambda\overline{y} - \overline{r}|$$

$$\iff w|\lambda - 1| = -ct + |\lambda\overline{y} - \overline{r}|$$

$\implies w^2(\lambda-1)^2 = c^2t^2 + (\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2$

$-2ct|\lambda\overline{y} - \overline{r}|$

$\implies 4c^2t^2[(\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2]$

$= [w^2(\lambda-1)^2 - c^2t^2 - (\lambda y_1 - r_1)^2 - (\lambda y_2 - r_2)^2 - (\lambda y_3 - r_3)^2]^2$

$\iff 4c^2t^2[\lambda^2 w^2 - 2\lambda\overline{y}\centerdot\overline{r} + |\overline{r}|^2] = [-2\lambda w^2 + w^2 - c^2t^2 + 2\lambda\overline{y}\centerdot\overline{r} - |\overline{r}|^2]^2$

$\iff \lambda^2(4c^2t^2w^2 - (2\overline{y}\centerdot\overline{r} - 2w^2)^2) + \lambda(-8c^2t^2\overline{y}\centerdot\overline{r} - 2(2\overline{y}\centerdot\overline{r} - 2w^2)$

$(w^2 - c^2t^2 - |\overline{r}|^2)) + (4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2) = 0 \ (AA)$

If we reverse the two $\implies$ steps, we obtain the alternatives;

$w^2(\lambda-1)^2 = c^2t^2 + (\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2$

$+2ct|\lambda\overline{y} - \overline{r}|$

and $w|\lambda - 1| = ct + |\lambda\overline{y} - \overline{r}|$ or $w|\lambda - 1| = -ct - |\lambda\overline{y} - \overline{r}|$

which gives;

$|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|$ or $|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct$

so that the condition (AA) defines the admissible λ in the formula;

$V_w^1(\overline{x}) \equiv \exists\lambda_{\neq 0}\exists\overline{y}[(|\overline{y}| = w) \wedge ((|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}|)$

$\vee(|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|) \vee (|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct)) \wedge \overline{x} = \lambda\overline{y}]$

with $V_w(\overline{x}) \subset V_w^1(\overline{x})$. By the quadratic formula, we have that, if $a \neq 0$;

$\lambda = \frac{-b+/-\sqrt{b^2-4ac}}{2a} = \frac{\gamma_1}{a}$ or $\frac{\gamma_2}{a}$

where;

$a = 4c^2t^2w^2 - (2\overline{y}\centerdot\overline{r} - 2w^2)^2$

$b = -8c^2t^2\overline{y} \centerdot \overline{r} - 2(2\overline{y} \centerdot \overline{r} - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)$

$c = 4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2$ (QQ), ([10]).

Let;

$a(z) = 4c^2t^2w^2 - (2z - 2w^2)^2$

$b(z) = -8c^2t^2z - 2(2z - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)$

$c(z) = 4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2$

Then $a(z) \in \mathcal{R}[z]$ is a polynomial of degree 2, $b(z) \in \mathcal{R}[z]$ is a polynomial of degree 1 iff;

$-8c^2t^2 - 4(w^2 - c^2t^2 - |\overline{r}|^2) \neq 0$

iff $4|\overline{r}|^2 - 4c^2t^2 - 4w^2 \neq 0$

iff $|\overline{r}| \neq \sqrt{\frac{4w^2+4c^2t^2}{4}}$

iff $|\overline{r}| \neq \sqrt{w^2 + c^2t^2}$

and $c(z)$ is a constant. We have that $c(z) = 0$

iff $4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2 = 0$

---

[10]Generically the two roots corresponding to λ must provide one of the three alternatives;

(i). $|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}|$

(ii). $|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|$

(iii). $|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct$

for the corresponding $\overline{x} = \lambda y$. Clearly the points on $V_w^1(\overline{x})$ corresponding to case (iii) are bounded, so if we obtain any infinite points, they must correspond to cases (i) or (ii). By Lemma 0.55, the infinite points on opposite sides of the asymptotic line which we find below, must correspond to both cases (i) and (ii). To obtain cancellation, we therefore need to include the opposite time $-t$ in the calculation, which we can do by considering $\dot{\rho} + \dot{\rho}^{-2t}$, where $\dot{\rho}^s(\overline{x}, t) = \dot{\rho}(\overline{x}, t - s)$.

iff $|\overline{r}|^2 = \frac{2w^2+2c^2t^2+/-\sqrt{(2c^2t^2+2w^2)^2-4(w^2-c^2t^2)^2}}{2}$

iff $|\overline{r}|^2 = w^2 + c^2t^2 + / - \sqrt{(c^2t^2 + w^2)^2 - (w^2 - c^2t^2)^2}$

$|\overline{r}|^2 = w^2 + c^2t^2 + / - \sqrt{4w^2c^2t^2}$

iff $|\overline{r}|^2 = w^2 + c^2t^2 + 2wct = (w + ct)^2$ or $|\overline{r}|^2 = w^2 + c^2t^2 - 2wct = (w - ct)^2$

iff $|\overline{r}| = |w + ct|$ or $|\overline{r}| = |w - ct| = w - ct$

which can happen, with roots at 0 and $-\frac{b}{a}$, the finite point, calculated in $(*)$ below being 0. However, we consider the generic case when $c(z) \neq 0$, leaving further consideration of the other case to the reader.

Let;

$Z_a = \{\overline{x} \in \delta B(\overline{0}, w) : a(\overline{y} \centerdot \overline{r}) = 0\}$

$Z_b = \{\overline{x} \in \delta B(\overline{0}, w) : b(\overline{y} \centerdot \overline{r}) = 0\}$

As $a(z)$ has degree 2, we have, by the quadratic formula, that;

$$a = 4c^2t^2w^2 - (2\overline{y} \centerdot \overline{r} - 2w^2)^2 = 0 \text{ iff } \overline{y} \centerdot \overline{r} = \frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2} \quad (PP)$$

which has at most 2 real solutions, corresponding to at most 2 (possibly empty) parallel intersection circles of the sphere $\delta B(\overline{0}, w)$ with parallel planes $\{P_{1,a}, P_{2,a}\}$. We will consider the generic case with two nonempty parallel circles, $\{C_{1,a}, C_{2,a}\}$, which are not points, leaving the other cases to the reader, so that $Z_a = C_{1,a} \cup C_{2,a}$, ([11]). We have that $b(z)$ has degree at most 1, with at most 1 real solution, corresponding to at most 1 (possibly empty) intersection circle C_b of the sphere with a plane P_b, parallel to $P_{1,a}$ and $P_{2,a}$. Again, we will consider the generic

[11] The case when a has repeated roots, by the formula (PP) occurs when $1 - \frac{4(1-c^2t^2)}{w^2} = 0$, iff $w^2 = 4(1 - c^2t^2)$, we can exclude this case by assuming $t^2 > \frac{1}{c}$ by moving the initial conditions sufficiently far enough in advance of t and changing coordinates. Alternatively, we can obtain at most 2 possible solutions for w, which will account for a set of measure zero in the final integration, see footnote refcoincides. Observe that when a has two real roots, they cannot be maxima or minima, so a will change sign on opposite sides of the intersection circles $C_{1,a}$ and $C_{2,a}$.

case when C_b is nonempty and not a point, leaving the other cases to the reader. We have that C_b coincides with one of the circles $C_{1,a}$ or $C_{2,a}$ iff $a=b=0$;

iff $(w^2-c^2t^2-r^2)(4w^2-4\overline{y}\centerdot\overline{r})=8c^2t^2\overline{y}\centerdot\overline{r}$ and (PP) holds

iff $(w^2-c^2t^2-r^2)(4w^2-4(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}))=8c^2t^2(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})$

which can happen, in which case $V_w(\overline{x}$ does not blow up along C_b. Again, we leave this case to the interested reader.

For $\overline{y}\in\delta B(\overline{0},w)\setminus Z_a$, we have that $p(\lambda,\overline{y},\overline{r})=0$, where $p(z,\overline{y},\overline{r})\in\mathcal{R}[z]$ is a polynomial of degree 2, with coefficients in $\{\overline{y},\overline{r}\}$, having at most 2 real roots.

Using the fact that;

$|\overline{y}\centerdot\overline{r}|\leq|\overline{y}||\overline{r}|=wr$

$|a|\leq(4c^2t^2w^2+(2wr+2w^2)^2)=C_1$

$|b|\leq 8c^2t^2wr+2(2wr+2w^2)(w^2+c^2t^2+r^2)=C_2$

$|c|\leq 4c^2t^2r^2+(w^2+c^2t^2+r^2)^2=C_3$

where $\{C_1,C_2,C_3\}\subset\mathcal{R}_{>0}$. Denoting the possible real roots of $p(\lambda,\overline{y},\overline{r})$ by $\{\frac{\gamma_1}{a},\frac{\gamma_2}{a}\}$, we have;

$max(|\gamma_1|,|\gamma_2|)\leq\frac{|b|+|\sqrt{b^2-4ac}|}{2}\leq\frac{C_2+\sqrt{C_2^2+4C_1C_3}}{2}=C_4$

where $C_4\subset\mathcal{R}_{>0}$. Then, if;

$|a|=|4c^2t^2w^2-(2\overline{y}\centerdot\overline{r}-2w^2)^2|>\epsilon>0$

it follows;

$max(|\frac{\gamma_1}{a}|,|\frac{\gamma_2}{a}|)\leq\frac{C_4}{\epsilon}$

In particular $V_w(\overline{x})$ can only blow up along the exceptional locus Z_a, ([12]).

In the generic case, with $C_{1,a} \neq \emptyset$, $C_{2,a} \neq \emptyset$, $C_{1,a} \neq C_{2,a}$, not points, we define the 2 asymptotic cones of $V_w(\overline{x})$ by;

$Cone(C_{1,a}) = \bigcup_{\overline{y} \in C_{1,a}} l_{\overline{0},\overline{y}}$

$Cone(C_{2,a}) = \bigcup_{\overline{y} \in C_{2,a}} l_{\overline{0},\overline{y}}$

where $l_{\overline{0},\overline{y}}$ is the line joining $\overline{0}$ and $\overline{y} \in C_{i,a}$, for $i \in \{1,2\}$.

We have that $Cone(C_{1,a}) \cap Cone(C_{2,a}) = \emptyset$ unless $pr^*(C_{1,a}) = C_{2,a}$, where pr^* is the orthogonal projection defined by the perpendicular line l passing through $\overline{0}$, perpendicular to the parallel planes $P_{1,a}$ and $P_{2,a}$, onto $P_{2,a}$, in which case $Cone(C_{1,a}) = Cone(C_{2,a})$. Again, we consider this generic case, leaving the case $Cone(C_{1,a}) = Cone(C_{2,a})$ to the reader.

We obtain no real roots, iff $b^2 - 4ac < 0$

iff $[-8c^2t^2\overline{y} \centerdot \overline{r} - 2(2\overline{y} \centerdot \overline{r} - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)]^2$

$-4[4c^2t^2w^2 - (2\overline{y} \centerdot \overline{r} - 2w^2)^2][4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2] < 0$

iff $q(\overline{y} \centerdot \overline{r}) < 0$, where $q \in \mathcal{R}[x]$ is a polynomial of degree at most 2, which by continuity determines an open set $Y_w \subset \mathcal{R}^3$, so that $X_w = Y_w \cap \delta B(\overline{0}, w)$ is open. We can exclude X_w from our calculations as the fibre is empty, and assume $b^2 - 4ac \geq 0$.

We obtain a repeated real root at $\frac{-b}{2a}$ iff;

$b^2 - 4ac = 0$

iff $[-8c^2t^2\overline{y} \centerdot \overline{r} - 2(2\overline{y} \centerdot \overline{r} - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)]^2$

$-4[4c^2t^2w^2 - (2\overline{y} \centerdot \overline{r} - 2w^2)^2][4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2] = 0$

---

[12] We can also note that if $C_{1,a} = C_{2,a} = \emptyset$ then $|a| > \epsilon_0$ on $\delta B(\overline{0}, w)$, and $max(|\frac{\gamma_1}{a}|, |\frac{\gamma_2}{a}|) \leq \frac{C_4}{\epsilon_0} = C_5$, where $C_5 \in \mathcal{R}_{>0}$, $V_w(\overline{x}) \subset B(\overline{0}, C_5 w)$ and $V_w(\overline{x})$ is bounded.

which again determines 2 intersection circles $Z_{rep} \subset B(\overline{0}, w)$, parallel to the circles $Z_a \cup Z_b$. Again, we consider the generic case that Z_{rep} is distinct from $Z_a \cup Z_b$, leaving the other cases to the reader, ([13]).

---

[13] If Z_a and Z_b are distinct, with $Z_{rep} = Z_a$, then $b^2 - 4ac = 0$ and $a = 0$, so $b = 0$, so that Z_a and Z_b have an intersection, which is a contradiction. Similarily, if Z_a and Z_b are distinct, with $Z_{rep} = Z_b$, then $b^2 - 4ac = 0$ and $b = 0$, so $ac = 0$, and $c = 0$, the blow up behaviour along Z_a being similar to the generic case. If $Z_b \subset Z_a$ with $Z_b \subset Z_{rep}$, then, we must have that $a = b = 0$, and;

$$\frac{-4w^2(w^2-c^2t^2-|\overline{r}|^2)}{-8c^2t^2-4(w^2-c^2t^2-|\overline{r}|^2)} = \frac{w^2+/w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$$

which, for fixed $\{t, |\overline{r}|\}$ has at most 8 solutions for w, $(*)$. Suppose that the spheres $\delta B(\overline{x}, -ct + |\overline{x} - \overline{r}|)$ and $\delta B(\overline{0}, w)$ share a tangent plane at $\overline{p}_{\overline{x},r}$, for some $0 < w < s$, so that the line $l_{\overline{0},\overline{p}_{\overline{x},r}}$ passes through $\overline{x}$. Without loss of generality, suppose that $|\overline{p}_{\overline{x},r}| < |\overline{x}|$ $\overline{p}_{\overline{x},r} = \frac{|\overline{p}_{\overline{x},r}|}{|\overline{x}|}\overline{x}$. Assume $\overline{x} \neq \overline{r}$ and consider the function $f_{\overline{x}}$ defined, for small λ by;

$$f_{\overline{x}}(\lambda) = -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - |\overline{x} + \lambda\overline{x} - \overline{p}_{\overline{x},r}|$$

$$= -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - |(1+\lambda)\overline{x} - \frac{|\overline{p}_{\overline{x},r}|}{|\overline{x}|}\overline{x}|$$

$$= -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - (1+\lambda - \frac{|\overline{p}_{\overline{x},r}|}{|\overline{x}|})|\overline{x}|$$

$$= -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - (1+\lambda)|\overline{x}| + |\overline{p}_{\overline{x},r}|$$

$$= -ct + [((1+\lambda)x_1 - r_1)^2 + ((1+\lambda)x_2 - r_2)^2 + ((1+\lambda)x_3 - r_3)^2]^{\frac{1}{2}} - (1+\lambda)|\overline{x}| + |\overline{p}_{\overline{x},r}|$$

$$= -ct + g_{\overline{x}}(\lambda) - (1+\lambda)|\overline{x}| + |\overline{p}_{\overline{x},r}|$$

in coordinates $\overline{x} = (x_1, x_2, x_3)$, $\overline{r} = (r_1, r_2, r_3)$, with $f_{\overline{x}}(0) = -ct + |\overline{x} - \overline{r}| - |\overline{x} - \overline{p}_{\overline{x},r}| = 0$, $g_{\overline{x}}(0) = |\overline{x} - \overline{r}|$. Then;

$$\frac{df}{d\lambda} = \frac{1}{2g_{\overline{x}}(\lambda)}(2((1+\lambda)x_1 - r_1)x_1 + 2((1+\lambda)x_2 - r_2)x_2 + 2((1+\lambda)x_3 - r_3)x_3) - |\overline{x}|$$

$$= \frac{1}{g_{\overline{x}}(\lambda)} < (1+\lambda)\overline{x} - \overline{r}, \overline{x} > -|\overline{x}|$$

$$= \frac{1}{g_{\overline{x}}(\lambda)}[(1+\lambda)|\overline{x}|^2 - < \overline{r}, \overline{x} >] - |\overline{x}|$$

so that $\frac{df}{d\lambda}(0) = 0$

iff

$$|\overline{x}|^2 - < \overline{r}, \overline{x} > = < \overline{x}, \overline{x} - \overline{r} > = |\overline{x}||\overline{x} - \overline{r}|$$

which implies that $\overline{r} \in l_{\overline{0},\overline{x}}$. Excluding this solution, as $f_{\overline{x}}$ is analytic, by O-minimality, for $\epsilon > 0$, we can assume that $f_{\overline{x}} = 0 \cap [-\epsilon, \epsilon]$ is a finite union of points and intervals. No interval can contain 0, as then $\frac{df}{d\lambda}(0)=0$, so that $f_{\overline{x}}\neq 0$ on some set of the form $(-\epsilon,\epsilon)\setminus\{0\}$. In particularly, this implies that we can obtain tangency of $\delta B(\overline{x}_1,-ct+|\overline{x}_1-\overline{r}|)$ with $\delta B(\overline{0},\overline{p}_{\overline{x},r,1})$ for mobile points $\overline{x}_1$ and $\overline{p}_{\overline{x},r,1}$ along the line $l_{\overline{0},\overline{p}_{\overline{x},r}}$. If $\overline{x}=\overline{r}$ or $\overline{r}\in l_{\overline{0},\overline{x}}$, we either have $|\overline{x}|<|\overline{r}|$, in which case, it is clear we can move $\overline{x}$ along $l_{\overline{0},\overline{x}}$ and obtain mobile points, or $|\overline{x}|>|\overline{r}|$, in which case we can move $\overline{x}$ through $\overline{r}$ towards $\overline{0}$, and eventually obtain mobile points, $(**)$. From $(**)$, the possible $0<w<s$ can represent arcs with the property that;

$B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ intersects $B(\overline{0},s)$

and such that the spheres $\delta B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ and $\delta B(\overline{0},w)$ share a tangent plane at $\overline{p}_{\overline{x},r}$, see Lemma 0.51, is not discrete. It follows that the case $(*)$ accounts for a set of measure zero in the final parametrisation and doesn't effect the finiteness of the integral. When $w-ct=|\overline{r}|$, $\overline{y}=w$, $\overline{r}=(w-ct)\frac{\overline{y}}{|\overline{y}}=(1-\frac{ct}{w})\overline{y}$, $(***)$, we obtain, as above, that there exist solutions to $V_w(\overline{x}$ for $|\overline{x}|\geq|\overline{r}|$, $\overline{x}\in l_{\overline{0},\overline{y}}$. This corresponds to the case $a(\overline{y}\centerdot\overline{r})=b(\overline{y}\centerdot\overline{r})=c(\overline{y}\centerdot\overline{r})=0$, where;

$a(z)=4c^2t^2w^2-(2z-2w^2)^2$

$b(z)=-8c^2t^2z-2(2z-2w^2)(w^2-c^2t^2-|\overline{r}|^2)$

$c(z)=4c^2t^2|\overline{r}|^2-(w^2-c^2t^2-|\overline{r}|^2)^2$

We have from $(***)$ that;

$\overline{y}\centerdot\overline{r}=\overline{y}\centerdot(1-\frac{ct}{w})\overline{y}$

$=(1-\frac{ct}{w})|\overline{y}|^2$

$=(1-\frac{ct}{w})w^2$

$=w(w-ct)$

so that;

$a(\overline{y}\centerdot\overline{r})=a(w(w-ct))$

$=4c^2t^2w^2-(2w(w-ct)-2w^2)^2$

$=0$

$b(\overline{y}\centerdot\overline{r})=b(w(w-ct))$

$=-8c^2t^2w(w-ct)-2(2w(w-ct)-2w^2)(w^2-c^2t^2-|\overline{r}|^2)$

Assuming $b^2-4ac\geq 0$, we obtain that $\gamma_1=0$ or $\gamma_2=0$ iff;

$\sqrt{b^2-4ac}=b$ or $\sqrt{b^2-4ac}=-b$ iff $(b^2-4ac)=b^2$

iff $4ac=0$

iff $a=0$ or $c=0$

iff $4c^2t^2w^2-(2\overline{y}\centerdot\overline{r}-2w^2)^2=0$ or $4c^2t^2|\overline{r}|^2-(w^2-c^2t^2-|\overline{r}|^2)^2=0$

iff $\overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$ or $4c^2t^2r^2-(w^2-c^2t^2-r^2)^2=0$

iff $\overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$ or $r=|w+ct|$ or $r=w-ct$

iff Case 1. $\overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$

or Case 2. $r=|w+ct|$ or $r=w-ct$

In Case 2, for $a\neq 0$, we obtain exactly 2 real roots $\frac{-b}{a}$ and 0, uniformly in $\overline{y}$.

In Case 1, with $b\neq 0$, we have, using Newton's expansion of $(1+y)^{\frac{1}{2}}$, for $|y|<1$, that;

$lim_{a\rightarrow 0}\frac{-b+\sqrt{b^2-4ac}}{2a}=lim_{a\rightarrow 0}\frac{-b+b(1-\frac{4ac}{b^2})^{\frac{1}{2}}}{2a}$

$=lim_{a\rightarrow 0}\frac{-b+b(1+\frac{y}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}y^n)}{2a}|_{y=-\frac{4ac}{b^2}}$

$=-8c^2t^2w(w-ct)-2(2w(w-ct)-2w^2)(w^2-c^2t^2-(w-ct)^2)$

$=0$

$c(\overline{y}\centerdot\overline{r})=c(w(w-ct))$

$=4c^2t^2(w-ct)^2-(w^2-c^2t^2-(w-ct)^2)^2$

$=0$

$= lim_{a\rightarrow 0}\frac{\frac{b(-\frac{4ac}{b^2})}{2}+b\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}(-\frac{4ac}{b^2})^n}{2a}$

$= lim_{a\rightarrow 0}\frac{1}{2}[-\frac{2c}{b}+\sum_{n=2}^{\infty}\frac{(-1)^{2n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}\frac{(4c)^n a^{n-1}}{b^{2n-1}}]$

$= \frac{-c}{b}$ $(*)$

and, with $b\neq 0$;

$lim_{a\rightarrow 0}\frac{-b-\sqrt{b^2-4ac}}{2a} = lim_{a\rightarrow 0}\frac{-b-b(1-\frac{4ac}{b^2})^{\frac{1}{2}}}{2a}$

$= lim_{a\rightarrow 0}\frac{-b-b(1+\frac{y}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}y^n)}{2a}|_{y=-\frac{4ac}{b^2}}$

$= lim_{a\rightarrow 0}\frac{-2b-\frac{b(-\frac{4ac}{b^2})}{2}-b\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}(-\frac{4ac}{b^2})^n}{2a}$

$= lim_{a\rightarrow 0}[-\frac{b}{a}+\frac{c}{b}-\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{2n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}\frac{(4c)^n a^{n-1}}{b^{2n-1}}]$

$= lim_{a\rightarrow 0}(-\frac{b}{a}+\frac{c}{b})$ $(**)$, ([14])

Letting $\overline{p}_{i,a}$ denote the centres of the blow up circle $S_{i,a}$, $1\leq i\leq 2$, and $\overline{q}_{i,a} = l_{\overline{0},\overline{p}_{i,a}}\cap\delta B(\overline{0},w)$, if $\overline{y}\in S_{i,a}$, we let $S_{i,\overline{y},a}$ denote the great circle passing through $\overline{y}$ and $\overline{q}_{i,a}$. Then, without loss of generality, we have that the region;

[14] This is a first order approximation for $V_w(\overline{x})$. We introduce the angle θ below and consider the leading term $-\frac{b}{a}$ which blows up as $a\rightarrow 0$. Strictly speaking, letting $d=-\frac{1}{2}[\sum_{n=2}^{\infty}\frac{(-1)^{2n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}\frac{(4c)^n a^{n-1}}{b^{2n-1}}]$, we have that;

$-\frac{b}{a}+\frac{c}{b}+d = -\frac{b}{a}(1-\frac{ca}{b^2}-\frac{da}{b})$

If we define $b_{new}(\theta)=b(\theta)(1-\frac{ca}{b^2}-\frac{da}{b})(\theta)$, with $a(0)=0$, so that;

$b_{new}(0)=b(0)$

$b'_{new}(\theta)=b'(\theta)(1-\frac{ca}{b^2}-\frac{da}{b})(\theta)+b(\theta)(\frac{2ca}{b^3}-\frac{c'a}{b^2}-\frac{ca'}{b^2}+\frac{da}{b^2}-\frac{d'a}{b}-\frac{da'}{b})$

$b'_{new}(0)=b'(0)+b(0)(-\frac{c(0)a'(0)}{b^2(0)}-\frac{d(0)a'(0)}{b(0)})$

$=b'(0)-\frac{c(0)a'(0)}{b(0)}-d(0)a'(0)$

the proof goes through replacing the instances of $\{b(0),b'(0)\}$ with $\{b_{new}(0),b'_{new}(0)\}$, which are all finite.

$a > 0 \cap B(\overline{0}, w) = \bigcup_{\overline{y} \in S_{1,a}} (S_{1,\overline{y},a} \cap a > 0)$

$a < 0 \cap B(\overline{0}, w) = \bigcup_{\overline{y} \in S_{1,a}} (S_{1,\overline{y},a} \cap a < 0)$

with $a < 0$ situated between the intersections $S_{1,a}$ and $S_{2,a}$, $a > 0$ situated above and below the intersections $S_{1,a}$ and $S_{2,a}$ on $\delta B(\overline{0}, w)$, and blow ups of opposite signs, see footnote 13, along $S_{1,\overline{y},a}$ at $\overline{y}$ and the corresponding opposite point $\overline{y}' \in S_{1,\overline{y},a} \cap S_{1,a}$ and points

$\overline{y}'', \overline{y}''' \subset S_{1,\overline{y},a} \cap S_{2,a}$, ([15]).

---

[15] In this case, $Cone_{1,a}$ and $Cone_{2,a}$ have the following asymptotic property. Fix $\overline{y} \in S_{1,a}$, and form the plane $Q_{1,\overline{y},a}$ determined by $l_{\overline{0},\overline{y}}$ and the tangent to the great circle $S_{1,\overline{y},a}$ at $\overline{y}$, so that $S_{1,\overline{y},a} \subset Q_{1,\overline{y},a}$. For a fixed $\overline{y}' \in S_{1,\overline{y},a}, \backslash \overline{y}$, let θ denote the angle between $l_{\overline{0},\overline{y}}$ and $l_{\overline{0},\overline{y}'}$ in the plane $Q_{1,\overline{y},a}$ and let $a(\theta) = a(\overline{y}')$, $b(\theta) = b(\overline{y}')$. Considering the first order approximation $-\frac{b}{a}$ for $V_w(\overline{x})$ along $S_{1,\overline{y},a}$, defining $V_{w,1}(\overline{x})$, we have that $|\overline{x}| = -\frac{b(\theta)w}{a(\theta)}$. Let pr^* be the orthogonal projection from $Q_{1,\overline{y},a}$ onto $l_{\overline{0},\overline{y}}$, and let $pr^*(\overline{x}) \in l_{\overline{0},\overline{y}}$ be the corresponding point, so that $pr^*(\overline{x})$ is the nearest point to $\overline{x}$ on $l_{\overline{0},\overline{y}}$, with $x = |\overline{x} - pr^*(\overline{x})|$ and $R = |pr^*(\overline{x})|$. By elementary trigonometry, assuming $\theta > 0$, we have that;

$x = |-\frac{b(\theta)wsin(\theta)}{a(\theta)}|$, $R = |-\frac{b(\theta)wcos(\theta)}{a(\theta)}|$, $\frac{x}{R} = tan(\theta)$ $(*)$

As the circles $S_{1,a}$ and $S_{2,a}$ are distinct and non-empty, we can factor a as $(\overline{y} \centerdot \overline{r} - \alpha)(\overline{y} \centerdot \overline{r} - \beta)$, where $S_{1,a}$ is defined by $(\overline{y} \centerdot \overline{r} = \alpha) \cap \delta B(\overline{0}, w)$, $S_{2,a}$ is defined by $(\overline{y} \centerdot \overline{r} = \beta) \cap \delta B(\overline{0}, w)$. Rotating coordinates so that $\overline{y}$ is situated at $(w, 0, 0)$, $\overline{y}'$ at $(wcos(\theta), wsin(\theta), 0)$, we have that;

$wr_1 - \alpha = 0$

where $\overline{r} = (r_1, r_2, r_3)$, and, without loss of generality, we can assume that $r_2 \neq 0$,. This follows as if we rotate $\overline{y}$ to $(w, 0, 0)$, $\overline{y}'$ to $(wcos(\theta), wsin(\theta), 0)$, with $r_2 = 0$, rotate $\overline{y}$ to $(0, w, 0)$ and $\overline{y}'$ to $(0, wcos(\theta), wsin(\theta))$ with $r_3 = 0$, and rotate $\overline{y}$ to $(0, 0, w)$ and $\overline{y}'$ to $(wsin(\theta), 0, wcos(\theta))$ with $r_1 = 0$, then $\overline{r} \in l_{\overline{0},\overline{y}}$, which we can exclude, as it accounts for a set of measure zero in the final integration. It follows that;

$a(\theta) = (wcos(\theta)r_1 + wsin(\theta)r_2 - \alpha)\gamma(\theta) = (\frac{w\alpha cos(\theta)}{w} + wsin(\theta)r_2 - \alpha)\gamma(\theta)$

$= (\alpha(cos(\theta) - 1) + wsin(\theta)r_2)\gamma(\theta)$ $(**)$

with $\gamma(0) \neq 0$, so that, from $(*)$,$(**)$;

$cos(\theta) = |-\frac{a(\theta)R}{b(\theta)w}|$

$= |-\frac{R}{b(\theta)w}||[(\alpha(cos(\theta) - 1) + wsin(\theta)r_2)\gamma(\theta)]|$ (L)

so that, using the power series expansions $cos(\theta) = 1 + O(\theta^2)$, $sin(\theta) = \theta + O(\theta^3)$;

$1 + O(\theta^2) = -\frac{R}{b(\theta)w}(\alpha O(\theta^2) + wr_2\theta + O(\theta^3)\gamma(\theta)$

and, rearranging;

$\theta = |\frac{-b(0)}{Rr_2}|(\frac{1}{|\gamma(0)|} + O(\theta))$ (D)

so $\theta = O(\frac{1}{R})$ $(***)$

---

so that, from $(*),(***)$;

$tan(\theta)=O(\frac{1}{R})$

$x=Rtan(\theta)=O(1)$

and, as $|\overline{x}|cos(\theta)=R=|pr^{*}(\overline{x})|$, we have that;

$|\overline{x}|-|pr^{*}(\overline{x})|=\frac{R}{cos(\theta)}-R$

$=R(1+\frac{\theta^{2}}{2}+O(\theta^{4}))-R$

$=O(\theta^{2})$

$=O(\frac{1}{R^{2}})$

From (D);

$\theta=|\frac{-b(0)}{Rr_{2}}|(\frac{1}{|\gamma(0)|}+O(\theta))\ (D)$

$=|\frac{-b(0)}{\gamma(0)Rr_{2}}|(1+O(\theta))$

so that;

$\theta(1+O(\theta))^{-1}=\theta(1+O(\theta))=|\frac{-b(0)}{\gamma(0)Rr_{2}}|$

$||\frac{-b(0)}{\gamma(0)Rr_{2}}|-\theta|\leq|\theta(1+O(\theta))-\theta|=O(\theta^{2})=O(\frac{1}{R^{2}})$

so that;

$||-\frac{b(0)}{\theta\gamma(0)r_{2}}|-R|\leq O(\frac{1}{R^{2}})O(\frac{R}{\theta})=O(\frac{1}{R^{2}})O(R^{2})=O(1)$

We have that, using (L);

$x=Rtan(\theta)=|-\frac{wb(\theta)sin(\theta)}{[(\alpha(cos(\theta)-1)+wsin(\theta)r_{2})\gamma(\theta)]}|$

and using L'Hopital's rule;

$lim_{\theta\rightarrow 0}x=|-\frac{lim_{\theta\rightarrow 0}(wb(\theta)sin(\theta))'}{lim_{\theta\rightarrow 0}[(\alpha(cos(\theta)-1)+wsin(\theta)r_{2})\gamma(\theta)]'}|$

$=|-\frac{lim_{\theta\rightarrow 0}(wb'(\theta)sin(\theta)+wb(\theta)cos(\theta))}{lim_{\theta\rightarrow 0}[(-\alpha sin(\theta)+wcos(\theta)r_{2})\gamma(\theta)+((\alpha(cos(\theta)-1)+wsin(\theta)r_{2}))\gamma'(\theta)]}|$

$=|-\frac{wb(0)}{wr_{2}\gamma(0)}|$

$=|-\frac{b(0)}{r_{2}\gamma(0)}|$

so that the line formed by the translation of $l_{\overline{0},\overline{y}}$ by a perpendicular distance of $|-\frac{b(0)}{r_{2}\gamma(0)}|$ in the plane $Q_{1,\overline{y},a}$ is actually an asymptote. Moreover, as x is analytic

If $\overline{y} \in S_{1,a}$ is fixed, with corresponding $\{S_{1,\overline{y},a}, Q_{1,\overline{y},a}\}$, then as $b \neq 0$ along $C_{1,a}$, we can assume that for small enough $|\theta| < \delta$, see footnote 15, $|b(\theta)| > \epsilon$, uniformly in $\overline{y} \in S_{1,a}$, so that $|\frac{-c}{b(\theta)}| \leq \frac{|c|}{\epsilon}$, and the root found in $(*)$ has a maximum value M, varying $|\theta| < \delta$ and $\overline{y} \in S_{1,a}$. For the root $\overline{x}(\theta) = -\frac{b(\theta)}{a(\theta)} + \frac{c}{b(\theta)}\overline{w}$, defined by $(**)$, we can assume that

---

in θ, $x - |\frac{b(0)}{r_2\gamma(0)}| = O(\theta) = O(\frac{1}{R})$.

We also have, using (L), and L'Hopital's rule twice, that;

$$|lim_{\theta\rightarrow 0}(-\frac{b(0)}{\theta r_2\gamma(0)} - R)|$$

$$= |lim_{\theta\rightarrow 0}|(-\frac{b(0)}{r_2\theta\gamma(0)}| + |\frac{b(\theta)wcos(\theta)}{[\alpha(cos(\theta)-1)+wsin(\theta)r_2]\gamma(\theta)}|)|$$

$$= lim_{\theta\rightarrow 0}|\frac{-b(0)\gamma(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2]+b(\theta)wr_2cos(\theta)\theta\gamma(0)}{r_2\gamma(0)\gamma(\theta)\theta[\alpha(cos(\theta)-1)+wsin(\theta)r_2]}|$$

$$= lim_{\theta\rightarrow 0}|\frac{-b(0)\gamma'(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2]-b(0)\gamma(\theta)[-\alpha sin(\theta)+wcos(\theta)r_2]+b'(\theta)wr_2\gamma(0)cos(\theta)\theta+b(\theta)wr_2\gamma(0)[cos(\theta)-\theta sin(\theta)]}{\gamma(0)\gamma'(\theta)\theta[\alpha(cos(\theta)-1)+wsin(\theta)r_2]+\gamma(0)\gamma(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2-\alpha\theta sin(\theta)+wcos(\theta)\theta r_2]}|$$

$$= lim_{\theta\rightarrow 0}|\frac{E(\theta)}{F(\theta)}|$$

where;

$$E(\theta) = -b(0)\gamma''(\theta)[\alpha(cos(\theta)-1) + wsin(\theta)r_2] - 2b(0)\gamma'(\theta)[-\alpha sin(\theta)$$

$$+wcos(\theta)r_2] - b(0)\gamma(\theta)[-\alpha cos(\theta) - wsin(\theta)r_2] + b''(\theta)wr_2\gamma(0)\theta cos(\theta)$$

$$+2b'(\theta)wr_2\gamma(0)[cos(\theta) - \theta sin(\theta)] + b(\theta)wr_2\gamma(0)[-2sin(\theta) - \theta cos(\theta)]$$

$$F(\theta) = \gamma(0)\gamma''(\theta)\theta[\alpha(cos(\theta)-1) + wsin(\theta)r_2] + \gamma(0)\gamma'(\theta)[\alpha(cos(\theta)-1)$$

$$+wsin(\theta)r_2] + \gamma(0)\gamma'(\theta)\theta[-\alpha sin(\theta) + wcos(\theta)r_2] + \gamma(0)\gamma'(\theta)[\alpha(cos(\theta)-1)$$

$$+wsin(\theta)r_2 - \alpha\theta sin(\theta) + wcos(\theta)\theta r_2] + \gamma(0)\gamma(\theta)[-2\alpha sin(\theta) + 2wcos(\theta)r_2$$

$$-\alpha\theta cos(\theta) - wsin(\theta)\theta r_2]$$

so that;

$$|lim_{\theta\rightarrow 0}(-\frac{b(0)}{r_2\theta\gamma(0)} - R)| = |\frac{-2b(0)\gamma'(0)wr_2+b(0)\gamma(0)\alpha+2b'(0)wr_2\gamma(0)}{2\gamma(0)^2wr_2}|$$

It follows, as $\frac{-b(0)}{r_2\theta\gamma(0)} - R$ is analytic in θ, that;

$$||\frac{-b(0)}{r_2\theta\gamma(0)}| - R| - |\frac{-2b(0)\gamma'(0)wr_2+b(0)\gamma(0)\alpha+2b'(0)wr_2\gamma(0)}{2\gamma(0)^2wr_2}| = O(\theta) = O(\frac{1}{R})$$

for sufficiently small δ, $|\overline{x}|>Mw$, and we can unambiguously define $\overline{x}(\theta)_{opp}=\overline{x}(-\theta)$, ([16]).

Returning to the case notation above, we have that, Case 3, $a=b=0$ iff $(1-c^2t^2-r^2)=\frac{8c^2t^2\overline{y}.\overline{r}}{4w^2-4\overline{y}.\overline{r}}$

iff $(1-c^2t^2-r^2)=\frac{8c^2t^2(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})}{4w^2-4(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})}$

corresponding to specific values of w, a situation considered in the footnote above. We have that $V_w(\overline{x})$ is bounded over one of the components of Z_a and exhibits a blow up behaviour over the other component.

In Case 1, not Case 3, as we have seen in the above footnotes, we obtain two components, with one component having a pair of infinite opposite branches parallel to the lines in the asymptotic cones, and a bounded component corresponding to $\frac{-c}{b}$ over the singular locus Z_a.

In Case 2, we again, by a similar calculation to $(*)$, obtain two components, with one component having a pair of infinite opposite branches parallel to the lines in the asymptotic cones, and a bounded component corresponding to the root 0.

□

Lemma 0.52. *Cancellation along asymptotes*

We have that, along the line $l_{\overline{0},\overline{y},sh}$, the integrals;

$$(i)\ \frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}$$

[16] By the calculation in footnote 15, we have that $\{\overline{x},\overline{x}_{opp}\}$ vary as $O(\frac{1}{\theta})$ with the angle θ. Moreover, by Lemma 0.55, for sufficiently small θ, if $\overline{x}$ corresponds to $-ct$, then $\overline{x}_{opp}$ corresponds to ct. By the definition of $V_w^1(\overline{x})$, $B(\overline{x},-ct+|\overline{x}-\overline{r})$ and $B(\overline{x}_{opp},ct+|\overline{x}_{opp}-\overline{r})$ pass through $\{\overline{y}',\overline{y}''\}\subset B(\overline{0},w)$, touching $\delta B(\overline{0},w)$, with $|\overline{y}-\overline{y}'|=2w|\theta|=O(\frac{1}{R})$ and centred on "oppposite" sides of $B(\overline{0},w)$. As the boundaries $\delta B(\overline{x},-ct+|\overline{x}-\overline{r})$ and $\delta B(\overline{x}_{opp},-ct+|\overline{x}_{opp}-\overline{r})$ limit to the tangent planes of $\overline{y}'$ and $\overline{y}''$ for sufficiently large $\{\overline{x},\overline{x}_{opp}\}$ and the points $\overline{y}'$ and $\overline{y}''$ approach each other as we increase R, this will be enough to obtain cancellation in the indefinite integral, following the method above. Moreover, by the calculation in footnote 15, we can assume that $\overline{x}$ and $\overline{x}_{opp}$ in the limit as $\theta\rightarrow 0$ approach the same line consisting of a bounded translate of the line $l_{\overline{0},\overline{y}}$ in the plane $Q_{1,\overline{y},a}$

$(ii)\ \frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}$

$(iii)\ \frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}$

are $O(\frac{1}{R^3})$*, with* $R=|\overline{r}'|$

Proof. Using the notation in Lemma 0.51, we consider the restriction of $V_w(\overline{x})$ to a cover of $S_{1,\overline{y},a}$, for $\overline{y}\in S_{1,a}$. For $\overline{r}''(\theta)\in V_w(\overline{x})|_{S_{1,\overline{y},a}}$, let $\overline{r}'$ be the nearest point to $\overline{r}''$ on the asymptote $l_{\overline{0},\overline{y},sh}$, where $l_{\overline{0},\overline{y},sh}$ is a shift of $l_{\overline{0},\overline{y}}$ by the perpendicular distance $c_{\overline{y}}=|\frac{-b(0)}{r_2\gamma(0)}|$ in the plane $S_{1,\overline{y},a}$. Then, by the result of Lemma 0.51, we have, for any $0<\epsilon<1$, that;

$$|\overline{r}''(\theta)-\overline{r}'(\theta)|<\epsilon=O(\tfrac{1}{R})$$

for sufficiently small θ, with $|\overline{r}'(\theta)-\overline{v}_{\overline{y}}|=R$ and $|\overline{v}_{\overline{y}}|=|c_{\overline{y}}|$ and $|\overline{r}''(\theta)|-|pr^*(\overline{r}''(\theta))|<\frac{E}{R^2}$, where $|pr^*(\overline{r}''(\theta))|=|\overline{r}'(\theta)-\overline{v}_{\overline{y}}|=R$, so that, for sufficiently small $\theta(R)$ or large $R(\theta)$;

$$R-1<R-\tfrac{E}{R^2}<|\overline{r}''(\theta)|<R+\tfrac{E}{R^2}<R+1$$

$$R-2<R-1-\epsilon<|\overline{r}'(\theta)|<R+1+\epsilon<R+2$$

We also have that, by the result of Lemma 0.51, that, for sufficiently small $\theta(R)$, $0<\epsilon'<1$;

$$|\overline{r}''(\theta)|-|(\tfrac{-b(0)}{\theta r_2\gamma(0)})|=|\overline{r}''(\theta)|-|pr^*(\overline{r}''(\theta))|+|pr^*(\overline{r}''(\theta))|-|(\tfrac{-b(0)}{\theta r_2\gamma(0)})|$$

$$<\tfrac{E}{R^2}+\epsilon'+\epsilon_{\overline{y}}$$

$$=\tfrac{E}{R^2}+O(\tfrac{1}{R})+\epsilon_{\overline{y}}$$

$$<2\epsilon'+\epsilon_{\overline{y}}$$

where $\epsilon_{\overline{y}}=|\frac{-2b(0)\gamma'(0)wr_2+b(0)\gamma(0)\alpha+2b'(0)wr_2\gamma(0)}{2\gamma(0)^2wr_2}|$

and similarly;

$|\overline{r}''(\theta)_{opp}| - |(\frac{-b(0)}{-\theta r_2\gamma(0)})|$

$= |\overline{r}''(\theta)| - |pr^*(\overline{r}''(\theta)_{opp})| + |pr^*(\overline{r}''(\theta)_{opp})| - |(\frac{b(0)}{\theta r_2\gamma(0)})|$

$< \frac{E}{R^2} + \epsilon' + \epsilon_{\overline{y}}$

$< 2\epsilon' + \epsilon_{\overline{y}}$

so that;

$|\overline{r}''(\theta)| - |\overline{r}''(\theta)_{opp}| = |\overline{r}''(\theta)| - |(\frac{b(0)}{\theta r_2\gamma(0)})| + |(\frac{b(0)}{\theta r_2\gamma(0)})| - |\overline{r}''(\theta)_{opp}|$

$\leq 4\epsilon' + 2\epsilon_{\overline{y}}$

$= O(\frac{1}{R}) + 2\epsilon_{\overline{y}}$

and;

$|\overline{r}'(\theta)| - |\overline{r}'(\theta)_{opp}| = (|\overline{r}'(\theta)| - |\overline{r}''(\theta)|) + (|\overline{r}''(\theta)| - |\overline{r}''(\theta)_{opp}|) + (|\overline{r}''(\theta)_{opp}| -$

$|\overline{r}'(\theta)_{opp}|)$

$\leq (|\overline{r}'(\theta) - \overline{r}''(\theta)|) + (|\overline{r}''(\theta)| - |\overline{r}''(\theta)_{opp}|) + (|\overline{r}''(\theta)_{opp} - \overline{r}'(\theta)_{opp}|)$

$\leq 4\epsilon' + 2\epsilon_{\overline{y}} + 2\epsilon$

$= O(\frac{1}{R}) + 2\epsilon_{\overline{y}}$

In particularly, as, by Pythagoras' Theorem;

$|\overline{r}'(\theta)|^2 + |c_{\overline{y}}|^2 = |pr^*(\overline{r}'(\theta))|^2$

$|\overline{r}'(\theta)_{opp}|^2 + |c_{\overline{y}}|^2 = |pr^*(\overline{r}'(\theta)_{opp})|^2$

we have;

$= (|pr^*(\overline{r}'(\theta))|^2 - |c_{\overline{y}}|^2)^{\frac{1}{2}} - (|pr^*(\overline{r}'(\theta)_{opp})|^2 - |c_{\overline{y}}|^2)^{\frac{1}{2}}$

$= |\overline{r}'(\theta)| - |\overline{r}'(\theta)_{opp}|$

$\leq 4\epsilon' + 2\epsilon + 2\epsilon_{\overline{y}}$

so that, using Newton's expansion;

$|pr^*(\overline{r}'(\theta))|(1 - \frac{|c_{\overline{y}}|^2}{|pr^*(\overline{r}'(\theta))|^2})^{\frac{1}{2}} - |pr^*(\overline{r}'(\theta)_{opp})|(1 - \frac{|c_{\overline{y}}|^2}{|pr^*(\overline{r}'(\theta)_{opp})|^2})^{\frac{1}{2}}$

$\leq 4\epsilon' + 2\epsilon + 2\epsilon_{\overline{y}}$

$|pr^*(\overline{r}'(\theta))| - |pr^*(\overline{r}'(\theta)_{opp})| \leq 4\epsilon' + 2\epsilon + O(\frac{1}{R^2}) + 2\epsilon_{\overline{y}}$

and we can assume that for sufficiently small θ;

$pr^*(\overline{r}'(\theta)) = -pr^*(\overline{r}'(\theta)_{opp}) + \overline{\epsilon} + \overline{w}_{\overline{y}}$

with $|\epsilon| < 4\epsilon' + 3\epsilon$, $|\overline{w}_{\overline{y}}| = 2\epsilon_{\overline{y}}$, and;

$-\overline{r}'(\theta) = -(\overline{v}_{\overline{y}} + pr^*(\overline{r}'(\theta)))$

$= -\overline{v}_{\overline{y}} - pr^*(\overline{r}'(\theta))$

$= -\overline{v}_{\overline{y}} + pr^*(\overline{r}'(\theta)_{opp}) - \overline{\epsilon} - \overline{w}_{\overline{y}}$

$= -\overline{v}_{\overline{y}} + (\overline{r}'(\theta)_{opp} - \overline{v}_{\overline{y}}) - \overline{\epsilon} - \overline{w}_{\overline{y}}$

$= \overline{r}'(\theta)_{opp} - 2\overline{v}_{\overline{y}} - \overline{\epsilon} - \overline{w}_{\overline{y}}$

$= \overline{r}'(\theta)_{opp} - (2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}}) + O(\frac{1}{R})$

$\simeq \overline{r}'(\theta)_{opp} - (2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}})$

For the asymptote $l_{\overline{0},\overline{y},sh}$, with $\overline{r}' \in l_{\overline{0},\overline{y},sh}$, $|\overline{r}'| = R$, sufficiently large, there exists a unique $\overline{r}'' \in V_w(\overline{x})$, with $pr^*(\overline{r}'') = \overline{r}'$, where pr^1 is the orthogonal projection onto $l_{\overline{0},\overline{y},sh}$ in the plane $Q_{1,\overline{y},a}$. If $|pr^*(\overline{r}'')| = S$, then $|\overline{r}'' - \overline{r}'| = O(\frac{1}{S})$, $|\overline{r}'' - pr^*(\overline{r}'')| = O(\frac{1}{S^2})$, so that;

$S - \frac{1}{S} + O(\frac{1}{S^2}) \leq R \leq S + \frac{1}{S} + O(\frac{1}{S^2})$

so that $|\overline{r}'' - \overline{r}'| = O(\frac{1}{R}) = O(\frac{1}{S})$. We have that;

$\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s) = (\delta B(\overline{r}'', -ct+|\overline{r}-\overline{r}'|)+(\overline{r}'-\overline{r''}))\cap B(\overline{0},s)$

$\simeq (\delta B(\overline{r}'', -ct+|\overline{r}-\overline{r}''|)+(\overline{r}'-\overline{r''}))\cap B(\overline{0},s)$

with a radial adjustment of at most $|\overline{r}'-\overline{r}''|$, and $\delta B(\overline{r}'', -ct+|\overline{r}-\overline{r}''|)$ passes through $\overline{y}''$ with $|\overline{y}''-\overline{y}| = w\theta = O(\frac{1}{S}) = O(\frac{1}{R})$. It follows that $\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)$ passes through $\overline{y}'$ with $|\overline{y}''-\overline{y}'| = O(\frac{1}{R})$, $|\overline{y}-\overline{y}'| = O(\frac{1}{R})$. Similarly, we have that for the pair $\{\overline{r}'_{opp}, \overline{r}''_{opp}\}$;

$\overline{r}'_{opp} = (2\overline{v}_{\overline{y}}+\overline{w}_{\overline{y}})-\overline{r}'+O(\frac{1}{S})$

$= (2\overline{v}_{\overline{y}}+\overline{w}_{\overline{y}})-\overline{r}'+O(\frac{1}{R})$

$= 2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})$

where $\overline{z}_{\overline{y}} = \frac{1}{2}(2\overline{v}_{\overline{y}}+\overline{w}_{\overline{y}})$, so that $|\overline{r}''_{opp}-\overline{r}'_{opp}| = O(\frac{1}{R}) = O(\frac{1}{S})$. Moreover;

$\delta B(\overline{r}'_{opp}, ct+|\overline{r}-\overline{r}'_{opp}|)\cap B(\overline{0},s) = (\delta B(\overline{r}''_{opp}, ct+|\overline{r}-\overline{r}'_{opp}|)+(\overline{r}'_{opp}-\overline{r''}_{opp}))\cap B(\overline{0},s)$

$\simeq (\delta B(\overline{r}''_{opp}, ct+|\overline{r}-\overline{r}''_{opp}|)+(\overline{r}'_{opp}-\overline{r''}_{opp}))\cap B(\overline{0},s)$

with a radial adjustment of at most $|\overline{r}'_{opp}-\overline{r}''_{opp}|$, and $\delta B(\overline{r}''_{opp}, ct+|\overline{r}-\overline{r}''_{opp}|)$ passes through $\overline{y}''_{opp}$ with $|\overline{y}''_{opp}-\overline{y}| = w\theta = O(\frac{1}{S}) = O(\frac{1}{R})$. It follows that $\delta B(\overline{r}'_{opp}, ct+|\overline{r}-\overline{r}'_{opp}|)$ passes through $\overline{y}'_{opp}$ with $|\overline{y}''_{opp}-\overline{y}'_{opp}| = O(\frac{1}{R})$, $|\overline{y}-\overline{y}'_{opp}| = O(\frac{1}{R})$.

We have that;

(i). Using the facts that $|\frac{\partial\rho}{\partial t}|_0| \leq M$ on $B(\overline{0},s)$, the surface measure of $\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ is at most $2\pi s^2$, $\overline{r}'_{opp} = 2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})$, we have, for sufficiently large $R = |\overline{r}'|$, that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp}, ct+|\overline{r}-\overline{r}'_{opp}|)}(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$= |\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})}\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})|$$

$$=|[\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})}\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}+\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})$$

$$+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}](\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})$$

$$-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$=|\frac{1}{16\pi^2\epsilon_0c^3}[\frac{(r_1-r'_1)((-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}+\frac{(r_1-r'_1)+(r_1-r'_{1,opp})}{(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$=|\frac{1}{16\pi^2\epsilon_0c^3}[\frac{(r_1-r'_1)((-t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}+\frac{2r_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R})}{(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}$$

$$\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})}\frac{(r_1+r'_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R}))}{c|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$\leq\frac{Ms^2}{8\pi\epsilon_0c^3}|\frac{(r_1-r'_1)((-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}|$$

$$+\frac{Ms^2}{8\pi\epsilon_0c^3}|\frac{2r_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R})}{(t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}|+|\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})}\frac{(r_1+r'_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R}))}{c|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}|$$

$$|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

$$\leq\frac{Ms^2}{\pi\epsilon_0c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0c^3}\frac{1}{|(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})||\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}$$

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

(P)

where, we follow the method in (ii) below, noting the $O(|\overline{r}'|^3)$ term cancels in the first long term to obtain $\frac{O(|\overline{r}'|)O(|\overline{r}'|^2)}{O(|\overline{r}'|^6)}=\frac{1}{O(|\overline{r}'|^3)}$.

Change coordinates, so that the azimuth angle θ of the sphere $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$ is centred on the line passing through $\{\overline{r}',\overline{y}'\}$, giving coordinates;

$\overline{r}'+sin(\theta)cos(\phi)\overline{x}+sin(\theta)sin(\phi)\overline{y}+cos(\theta)(\overline{y}'-\overline{r}')$

$(0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi)$

for a choice of orthogonal vectors $\{\overline{x},\overline{y},\overline{y}'-\overline{r}'\}$ with modulus $-ct+|\overline{r}-\overline{r}'|$. Similarly, choose the azimuth angle θ_{opp} of the sphere $\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)$ is centred on the line passing through $\{\overline{r}'_{opp},\overline{y}'_{opp}\}$, giving coordinates;

$\overline{r}'+sin(\theta_{opp})cos(\phi_{opp})\overline{x}_{opp}+sin(\theta_{opp})sin(\phi_{opp})\overline{y}_{opp}+cos(\theta_{opp})(\overline{y}'_{opp}-\overline{r}'_{opp})$

$(0\leq\theta_{opp}\leq\pi,-\pi\leq\phi_{opp}\leq\pi)$

for a choice of orthogonal vectors $\{\overline{x}_{opp},\overline{y}_{opp},\overline{y}'_{opp}-\overline{r}'_{opp}\}$ with modulus $ct+|\overline{r}-\overline{r}'_{opp}|$. We have, for points $\{\overline{q}',\overline{q}'_{opp}\}$ of intersection between $B(\overline{0},s)$ and $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, $B(\overline{0},s)$ and $\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)$ that;

$\theta(\overline{q}')\simeq sin(\theta(\overline{q}'))\leq\frac{2s}{-ct+|\overline{r}-\overline{r}'|}$

$\theta_{opp}(\overline{q}'_{opp})\simeq sin(\theta_{opp}(\overline{q}'_{opp}))\leq\frac{2s}{ct+|\overline{r}-\overline{r}'_{opp}|}$ (TT)

Let $\{m,m',m'_{opp}\}$ be perpendicular lines to the asymptotic line l containing $\{\overline{r}',\overline{r}'_{opp}\}$ with centre $\overline{z}_{\overline{y}}+O(\frac{1}{R})$, passing through the points $\{\overline{y},\overline{y}',\overline{y}'_{opp}\}$, with $\overline{p}=m\cap l$, $\overline{p}'=m'\cap l$, $\overline{p}'_{opp}=m'_{opp}\cap l$. Let $\{P,P',P'_{opp}\}$ be planes passing through $\{\overline{y},\overline{y}',\overline{y}'_{opp}\}$, perpendicular to the lines formed by translating l by the vectors $\{\overline{y}'-\overline{p}',\overline{y}'_{opp}-\overline{p}'_{opp}\}$ respectively. Let $v=|\overline{y}-\overline{p}|$, $v'=|\overline{y}'-\overline{p}'|$, $v'_{opp}=|\overline{y}'_{opp}-\overline{p}'_{opp}|$, $k'=|\overline{r}'-\overline{p}'|$, $k'_{opp}=|\overline{r}'_{opp}-\overline{p}'_{opp}$, then by elementary trigonometry, the angles $\{\alpha',\alpha'_{opp}\}$ between the lines $\{l,l_{\overline{y}',\overline{r}'}\}$ and $\{l,l_{\overline{y}'_{opp},\overline{r}'_{opp}}\}$ are given by;

$\alpha'\simeq tan(\alpha')=\frac{v'}{k'}=\frac{|\overline{y}'-\overline{p}'|}{|\overline{r}'-\overline{p}'|}$

$\alpha'_{opp}\simeq tan(\alpha'_{opp})=\frac{v'_{opp}}{k'_{opp}}=\frac{|\overline{y}'_{opp}-\overline{p}'_{opp}|}{|\overline{r}'_{opp}-\overline{p}'_{opp}|}=O(\frac{1}{R})$ (LM)

We have, for vectors $\{\overline{u},\overline{v},\overline{w}\}$, that;

$|\overline{u}-\overline{w}|\geq|\overline{u}-\overline{v}|-|\overline{w}-\overline{v}|$

so that;

$|\overline{r}'-\overline{p}'|\geq|\overline{r}'-\overline{z}_{\overline{y}}|-|\overline{p}'-\overline{z}_{\overline{y}}|$

$|\overline{r}'_{opp}-\overline{p}'_{opp}|\geq|\overline{r}'_{opp}-\overline{z}_{\overline{y}}|-|\overline{p}'_{opp}-\overline{z}_{\overline{y}}|$

$=|\overline{r}'-\overline{z}_{\overline{y}}+O(\frac{1}{R})|-|\overline{p}'_{opp}-\overline{z}_{\overline{y}}|$

and, moreover;

$|\overline{r}'|-|\overline{z}_{\overline{y}}|\leq|\overline{r}'-\overline{z}_{\overline{y}}|=|\overline{r}'_{opp}-\overline{z}_{\overline{y}}+O(\frac{1}{R})|\leq|\overline{r}'|+|\overline{z}_{\overline{y}}|$

so that;

$|\overline{r}'-\overline{z}_{\overline{y}}|=O(R)$, $|\overline{r}'_{opp}-\overline{z}_{\overline{y}}|=O(R)$

where $R=|\overline{r}'|$, and, using (LM), $\alpha=O(\frac{1}{R})$, $\alpha'=O(\frac{1}{R})$. Then, it is clear that that the maximal distance between points $\overline{q}'$ on the arc $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ and the orthogonal projections $pr^2(\overline{q}')$ onto the plane P' is at most $\alpha's=O(\frac{1}{R})$, and similarly, the maximal distance between points $\overline{q}'_{opp}$ on the arc $\delta B(\overline{r}',=ct+|\overline{r}-\overline{r}'_{opp}|)\cap B(\overline{0},s)$ and the orthogonal projections $pr^2(\overline{q}'_{opp})$ onto the plane P'_{opp} is at most $\alpha'_{opp}s=O(\frac{1}{R})$. Similarly, as the orthogonal distances between P' and P'_{opp} is $|\overline{y}'-\overline{y}'_{opp}|=O(\frac{1}{R})$, we can, for sufficiently large R, choose $\{\overline{x},\overline{y},\overline{x}_{opp},\overline{y}_{opp}\}$ compatibly, such that, uniformly;

$|\overline{q}'-\overline{q}'_{opp}|=O(\frac{1}{R})=\epsilon(R)$

for $\{\overline{q}',\overline{q}'_{opp}\}$ defined by coordinates $\theta=\theta_{opp}$, $\phi=\phi_{opp}$ with $0\leq\theta\leq max(\theta_{max},\theta_{max,opp})$, where;

$\theta_{max}=max_{0\leq\phi\leq 2\pi}\theta(\overline{q}')=O(\frac{1}{R})$

for $\overline{q}'$ in $B(\overline{0},s)\cap\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, with coordinates $\{\theta,\phi\}$, and;

$\theta_{max,opp}=max_{0\leq\phi\leq 2\pi}\theta_{opp}(\overline{q}'_{opp})=O(\frac{1}{R}$

for $\overline{q}'_{opp}$ in $B(\overline{0},s)\cap\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)$, with coordinates $\{\theta_{opp},\phi_{opp}\}$

It follows that, for sufficiently large R, using the surface measure $dS=r^2sin(\theta)$, the fact (TT) and $r^2(1-cos(\frac{1}{r}))=O(1)$, and footnote

5, for sufficiently large r;

$$|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

$$\leq 2\epsilon(R)|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(ct+|\overline{r}-\overline{r}'_{opp}|)^2\int_0^{max(\theta_{max},\theta_{max,opp})}sin(\theta)d\theta$$

$$=2\epsilon(R)|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(ct+|\overline{r}-\overline{r}'_{opp}|)^2(1-cos(max(\theta_{max},\theta_{max,opp})))$$

$$\leq C\epsilon(R)$$

$$\leq \frac{D}{|\overline{r}'|}$$

where $\{C,D\}\subset\mathcal{R}_{>0}$.

It follows from (P), for sufficiently large $r(\epsilon)$, following the method of (ii), that;

$$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$$

$$\leq\frac{Ms^2}{\pi\epsilon_0c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0c^3}\frac{D}{|\overline{r}'|}\frac{1}{|(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})||\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}|}$$

$$\leq\frac{E_1}{|\overline{r}'|^3}$$

where $E_1\in\mathcal{R}_{>0}$.

(ii). Using the facts that $|\frac{\partial\rho}{\partial t}|_0|\leq M$ on $B(\overline{0},s)$, the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ is at most $2\pi s^2$, $\overline{r}'_{opp}=2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})$, we have, for sufficiently large $R=|\overline{r}'$, that;

$$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$$

$$\leq\frac{1}{4\pi\epsilon_0c}\frac{2\pi Ms^2}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}+\frac{1}{4\pi\epsilon_0c}\frac{2\pi Ms^2}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}$$

$$=\frac{Ms^2}{8\pi c\epsilon_0(ct-|\overline{r}-\overline{r}'|)^2|\overline{r}-\overline{r}'|}+\frac{Ms^2}{8\pi c\epsilon_0(-ct-|\overline{r}_1+\overline{r}'|)^2|\overline{r}_1+\overline{r}'|}$$

$$=\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}-\overline{r}'|^3|\frac{ct}{|\overline{r}-\overline{r}'|}+1|^2}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3|(\frac{-ct}{|\overline{r}_1+\overline{r}'|}-1)|^2}$$

$\leq \frac{Ms^2}{4\pi c\epsilon_0|\overline{r}-\overline{r}'|^3}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3}$

$\leq \frac{3Ms^2}{8\pi c\epsilon_0|\overline{r}'|^3}$

$=\frac{E_2}{|\overline{r}'|^3}$

where $\overline{r}_1=\overline{r}-2\overline{z}_{\overline{y}}+O(\frac{1}{R})$, $E_2\in\mathcal{R}_{>0}$.

(iii). We have that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$=|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}(-ct+|\overline{r}-\overline{r}'|)\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{z}(\overline{y}))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}(ct+|\overline{r}-\overline{r}'_{opp}|)\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot$
$(\overline{z}_{opp}(\overline{y})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'|)}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot\overline{z}(\overline{y})dS(\overline{y})|$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot\overline{z}_{opp}(\overline{y})dS(\overline{y})|$

(NN)

Letting $\overline{z}_0=\frac{(\overline{y}'-\overline{r}')}{-ct+|\overline{r}-\overline{r}'|}$, so that $|\overline{z}_0|=1$, R the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$, using Lemma 0.49, following the method of (i), we have that, for sufficiently large R;

$|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot\overline{z}(\overline{y})dS(\overline{y})|$

$=|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})+\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot$

$\overline{z_0}dS(\overline{y})|$

$\leq|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})|+|[\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)]\cdot$

$\overline{z_0}dS(\overline{y})|$

$\leq Rmax_{\overline{y}\in B(\overline{0},s)}|D(\frac{\partial\rho}{\partial t})(\overline{y},0)||\overline{z}(\overline{y})-\overline{z_0}|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})\cdot\overline{z_0}|$

$\leq RMmax_{\overline{y}\in B(\overline{0},s)}|\overline{z}(\overline{y})-\overline{z_0}|+|\overline{z_0}||\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RM|(1-cos(\theta_{max}),sin(\theta_{max})|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})$

$-\int_{P'}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|+|\int_{P'}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$=\sqrt{2}RM(1-cos(\theta_{max}))^{\frac{1}{2}}+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})-\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RMF\theta_{max}+O(\frac{1}{R})$

$\leq\frac{2sH}{-ct+|\overline{r}-\overline{r}'|}+\frac{W}{|\overline{r}'|}$

$=\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|\overline{r}'|}$

where $\{F,G,W,H,A_1,B_1\}\subset\mathcal{R}_{>0}$. Similarly, using P'_{opp}, there exist $\{A_2,B_2\}\subset\mathcal{R}_{>0}$, such that

$|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot\overline{z}(\overline{y})dS(\overline{y})|\leq\frac{A_2}{ct+|\overline{r}-\overline{r}'_{opp}|}+\frac{B_2}{|1+\overline{r}'_{opp}|}$

$=\frac{A_2}{ct+|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}+\frac{B_2}{|2\overline{z}_{\overline{y}}-\overline{r}'|+O(\frac{1}{R})}$

so that, from (NN), following the method of (ii)

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}')}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{ct+|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}+\frac{B_2}{|1+2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})|})$

$=\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|(t-\frac{|\overline{r}-\overline{r}'|}{c})||\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$

$+\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c}||\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{ct+|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}+\frac{B_2}{|1+2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})|})$

$\leq\frac{E_3}{|\overline{r}'|^3}$

where $E_3 \in \mathcal{R}_{>0}$ $((i),(ii),(iii))$

$\square$

Definition 0.53. *For the blow up circles* $\{S_{1,a},S_{2,a}\}$, *we define the corresponding shifted asymptotic cones* $\{SCone(S_{1,a}),SCone(S_{2,a})\}$ *by;*

$SCone(S_{1,a})=\bigcup_{\overline{y}\in S_{1,a}} l_{\overline{0},\overline{y},sh}$

$SCone(S_{2,a})=\bigcup_{\overline{y}\in S_{2,a}} l_{\overline{0},\overline{y},sh}$

Fix base points $\overline{y}_{1,a}\in S_{1,a}$ *and* $\overline{y}_{2,a}\in S_{2,a}$, *the circles having centres* $\{\overline{c}_{1,a},\overline{c}_{2,a}\}$ *with radii* $\{r_{1,a},r_{2,a}\}$ *and points on the circle* $\{\overline{z}_{1,a},\overline{z}_{2,a}\}$, *such that* $l_{\overline{c}_{i,a},\overline{y}_{i,a}}$ *and* $l_{\overline{c}_{i,a},\overline{z}_{i,a}}$ *are perpendicular for* $1\leq i\leq 2$ *then we can define parameterisations* $\beta_1:[0,2\pi)\rightarrow S_{1,a}$, $\beta_2:[0,2\pi)\rightarrow S_{2,a}$, *by;*

$\beta_i(\gamma)=\overline{c}_{i,a}+r_{i,a}(\overline{y}_{i,a}-\overline{c}_{i,a})cos(\gamma)+r_{i,a}(\overline{z}_{i,a}-\overline{c}_{i,a})sin(\gamma)$

We define the maps $\{\theta_1,\theta_2\}$, $\theta_i:\mathcal{R}\times(0,2\pi)\rightarrow SCone(S_{i,a})$, $1\leq i\leq 2$, *by;*

$\theta_i(r,\gamma)=\overline{z}_{\beta_i(\gamma)}+\frac{r}{w}\beta_i(\gamma)$

where, for $\overline{y}\in S_{i,a}$, $\overline{u}_{\overline{y}}$ *has modulus* $|\frac{-b(\overline{y})}{r_2\gamma(\overline{y})}|$ *with* $\overline{u}_{\overline{y}}\in S_{1,\overline{y},a}$ *perpendicular to* $\overline{y}\in l_{\overline{0},\overline{y}}$.

Lemma 0.54. *Cancellation along the shifted asymptotic cone and* $V_w(\overline{x})$

Proof. Using the notation above, we have that, for $i\in\{1,2\}$

(i). $\theta_i(0,\gamma)=\overline{u}_{\beta_i(\gamma)}$

(ii). $\theta_i(r,\gamma)_{opp}=\theta_i(-r,\gamma)+O(\frac{1}{r})$, for sufficiently large $r>0$, ([17]).

[17] As, by the above, if $\overline{r}'=\theta_i(r,\gamma)$, then;

$\overline{r}'_{opp}=-(\overline{r}'-\overline{v}_{\beta_i(\gamma)})+\overline{w}_{\beta_i(\gamma)}+\overline{v}_{\beta_i(\gamma)}+O(\frac{1}{R})$;

$=-\overline{r}'+(2\overline{v}_{\beta_i(\gamma)}+\overline{w}_{\beta_i(\gamma)})+O(\frac{1}{r})$

so that $|\overline{r}'-\overline{z}_{\beta_i(\gamma)}|=|\overline{r}'_{opp}-\overline{z}_{\beta_i(\gamma)}|+O(\frac{1}{r})$

where $\overline{z}_{\beta_i(\gamma)}=\frac{1}{2}(2\overline{v}_{\beta_i(\gamma)}+\overline{w}_{\beta_i(\gamma)})$ and $\overline{z}_{\beta_i(\gamma)}\in l_{\overline{0},\beta_i(\gamma),sh}$.

(iii). There exist $R_i \subset \mathcal{R}_{>0}$ with θ_i diffeomorphisms outside $[-R_i, R_i] \times [0, 2\pi)$, with the partial derivatives uniformly bounded.

(iv). $Im(\theta_1|_{\mathcal{R}\setminus[-R_1,R_1]\times[0,2\pi)} \cap Im(\theta_2|_{\mathcal{R}\setminus[-R_2,R_2]\times[0,2\pi)}) = \emptyset$

(v). For $r_2 > r_1 > R_i$, $|\theta_i(r_2, \gamma) - \theta_i(r_1, \gamma)| = r_2 - r_1$, and for $r_2 < r_1 < -R_i$, $|\theta_i(r_2, \gamma) - \theta_i(r_1, \gamma)| = r_1 - r_2$

It follows from $(iii), (v)$ that, for $1 \leq i \leq 2$, the pullbacks;

$$\theta_i|^*_{\mathcal{R}\setminus[-R_1,R_1]\times[0,2\pi)}(dLeb|_{SCone(S_{i,a})}) = |\frac{\partial\theta_1}{\partial r} \times \frac{\partial\theta_1}{\partial\gamma}|drd\gamma = f(r, \gamma)drd\gamma$$

has the property that $f(r, \gamma)$ has order $O(r)$, uniformly in γ and $f(r, \gamma) = f(-r, \gamma)$, for $r \in \mathcal{R}_{>0}$. For $R \in \mathcal{R}_{>0}$, with $R > R_i$, we can define the regions $S_{R,i} \subset \mathcal{R} \times [\alpha, \beta)$, by;

$$S_{R,i} = \{(r', \gamma) : R_i \leq |r'| \leq R, \gamma \in [\alpha, \beta)\}$$

with corresponding regions $\theta_i(S_{R,i}) \subset SCone(S_{i,a})$

Then, by the calculation above, using fact (ii), Lemma 0.52 and the mean value theorem, letting;

$$H^+(\overline{r}') = (\frac{1}{4\pi\epsilon_0}\frac{\dot{\rho}(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|})_1$$

$$H^-(\overline{r}') = (\frac{1}{4\pi\epsilon_0}\frac{\dot{\rho}(\overline{r}', -t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|})_1$$

where by t_r we mean $t - \frac{|\overline{r}-\overline{r}'|}{c}$ and by $-t_r$ we mean $-t - \frac{|\overline{r}-\overline{r}'|}{c}$, we have that, for $r > R_i$;

$$|\theta_1^* H^+(r, \gamma) + \theta_1^* H^-(-r, \gamma)|$$

$$= |H^+(\overline{r}') + H^-(\overline{r}'_{opp} + O(\tfrac{1}{r}))|$$

$$\leq |H^+(\overline{r}') + H^-(\overline{r}'_{opp})| + |H^-(\overline{r}'_{opp} + O(\tfrac{1}{r})) - H^-(\overline{r}'_{opp})|$$

$$\leq \tfrac{C}{r^3} + |H^-(\overline{r}'_{opp} + O(\tfrac{1}{r})) - H^-(\overline{r}'_{opp})|$$

$$= \tfrac{C}{r^3} + |DH^-(\overline{r}'_1) \centerdot O(\tfrac{1}{r})|$$

$\leq \frac{C}{r^3} + \frac{E}{r} | \bigtriangledown (H^-)(\overline{r}'_1)|$ (YY)

where $|\overline{r}'_1 - \overline{r}'_{opp}| = O(\frac{1}{r})$

We have that;

$| \bigtriangledown (H^-)(\overline{r}'_1)| \leq \sqrt{3} max_{1\leq i\leq 3}(\frac{\partial H^-}{\partial r'_i})|_{\overline{r}'_1}$

and;

$\frac{\partial H^-}{\partial r'_i}|_{\overline{r}'_1} = \frac{1}{4\pi\epsilon_0}[[(\frac{\partial \dot{\rho}}{\partial r'_i}(\overline{r}', -t_r) + \frac{\partial^2 \rho}{\partial^2 t}(\overline{r}', -t_r)\frac{(r_i - r'_i)}{c|\overline{r}-\overline{r}_i|})]\frac{r_1 - r'_1}{|\overline{r}-\overline{r}'|^2} - \frac{\dot{\rho}(\overline{r}', -t_r)}{|\overline{r}-\overline{r}'|^2} + \frac{2\dot{\rho}(\overline{r}', -t_r)|r_1 - r'_1|^2}{|\overline{r}-\overline{r}'|^3}]|_{\overline{r}'_1}$

so that, using the fact that $|\dot{\rho}| \leq M$ for some $M \in \mathcal{R}_{>0}$;

$|\frac{\partial H}{\partial r'_i}|_{\overline{r}'_1}| \leq [\frac{|\frac{\partial \dot{\rho}}{\partial r'_i}(\overline{r}', -t_r)| + |\frac{1}{c}\frac{\partial^2 \rho}{\partial^2 t}(\overline{r}', -t_r)|}{4\pi\epsilon_0 |\overline{r}-\overline{r}'|} + \frac{M}{4\pi\epsilon_0|\overline{r}-\overline{r}'|^2} + \frac{\dot{\rho}(\overline{r}', -t_r)}{2\pi\epsilon_0|\overline{r}-\overline{r}'|}]|_{\overline{r}'_1}$

We have that ρ obeys the wave equation $\bigtriangledown^2(\rho) + \frac{1}{c^2}\frac{\partial^2 \rho}{\partial^2 t} = 0$, determined by the initial conditions $\{\rho_0, (\frac{\partial \rho}{\partial t})_0\}$, so that $\dot{\rho}$ obeys the same wave equation determined by the initial conditions $\{(\frac{\partial \rho}{\partial t})_0, -c^2 \bigtriangledown^2 (\rho_0)\}$, $\frac{\partial^2 \rho}{\partial t^2}$ obeys the wave equation determined by the initial conditions $\{-c^2 \bigtriangledown^2 (\rho_0), -c^2 \bigtriangledown^2 (\frac{\partial \rho_0}{\partial t})\}$, $\frac{\partial \dot{\rho}}{\partial r'_i}$, $1 \leq i \leq 3$, obeys the wave equation determined by the initial conditions $\{\frac{\partial^2 \rho_0}{\partial t \partial r'_i}, -c^2 \bigtriangledown^2 (\frac{\partial \rho_0}{\partial r_i})\}$.

Using Kirchoff's formula, it follows that there exist $\{D_{1i}, E_{1i}, D_2, E_2, D_3, E_3\} \subset \mathcal{R}_{>0}$, for $1 \leq i \leq 3$, such that, for sufficiently large $|\overline{r}'|$;

$|\frac{\partial \dot{\rho}}{\partial r'_i}|_{\overline{r}', -t_r} \leq \frac{D_{1i}}{-t - \frac{|\overline{r}-\overline{r}'|}{c}} \leq \frac{E_{1i}}{|\overline{r}'|}$

$|\frac{\partial^2 \rho}{\partial t^2}|_{\overline{r}', -t_r} \leq \frac{D_2}{-t - \frac{|\overline{r}-\overline{r}'|}{c}} \leq \frac{E_2}{|\overline{r}'|}$

$|\dot{\rho}|_{\overline{r}', -t_r} \leq \frac{D_3}{-t - \frac{|\overline{r}-\overline{r}'|}{c}} \leq \frac{E_3}{|\overline{r}'|}$

so that, for sufficiently large $|\overline{r}'|$, there exists $\{G, H, K_i\} \subset \mathcal{R}_{>0}$, for $1 \leq i \leq 3$;

$|\frac{\partial H}{\partial r'_i}|_{\overline{r}'_1}| \leq [\frac{1}{4\pi\epsilon_0|\overline{r}-\overline{r}'|}(\frac{E_{1i}}{|\overline{r}'|} + \frac{E_2}{|\overline{r}'|} + \frac{M}{|\overline{r}-\overline{r}'|^2} + \frac{2E_3}{|\overline{r}'|})]|_{\overline{r}'_1}$

$\leq [\frac{G}{4\pi\epsilon_0|\overline{r}'|}(\frac{E_{1i}}{|\overline{r}'|}+\frac{E_2}{|\overline{r}'|}+\frac{MH}{|\overline{r}'|^2}+\frac{2E_3}{|\overline{r}'|})]|_{\overline{r}'_1}$

$\leq \frac{K_i}{|\overline{r}'|^2}|_{\overline{r}'_1}$

$= \frac{K_i}{|\overline{r}'_1|^2}$

and, for some $\{X,Y,Z\}\subset\mathcal{R}_{>0}$;

$|\bigtriangledown(H)(\overline{r}'_1)|\leq\frac{\sqrt{3}max_{1\leq i\leq 3}K_i}{|\overline{r}'_1|^2}$

$=\frac{\sqrt{3}max_{1\leq i\leq 3}K_i}{|\overline{r}'_{opp}+O(\frac{1}{r})|^2}$

$\leq\frac{X}{|\overline{r}'_{opp}|^2}$

$\leq\frac{Y}{|\overline{r}'|^2}$

$\leq\frac{Z}{r^2}$

so that, from (YY)

$|\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma)|$

$\leq\frac{C}{r^3}+\frac{E}{r}\frac{Z}{r^2}$

$=\frac{C+EZ}{r^3}$ $(*)$

We also have;

$|f(r,\gamma)|\leq Dr$

$|(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)|\leq\frac{(C+F)D}{r^2}$

where $\{D,F\}\subset\mathcal{R}_{>0}$, $F=EZ$, so that;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}(H^++H^-)(\overline{r}')d\overline{r}'$

$=lim_{R\rightarrow\infty,R>R_i}\int_{S_{R,i}}(\theta_1^*(H^++H^-))(r,\gamma)f(r,\gamma)drd\gamma$

$=lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}[\int_{R_i}^R\theta_1^*H^+(r,\gamma)f(r,\gamma)dr+\int_{-R}^{-R_i}\theta_1^*H^-(r,\gamma)f(r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}[\int_{R_i}^{R}\theta_1^*H^+(r,\gamma)f(r,\gamma)dr+\int_{-R}^{-R_i}\theta_1^*H^-(-r,\gamma)f(-r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}\int_{R_i}^{R}(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)drd\gamma$

$= \int_{[0,2\pi)}\int_{R_i}^{\infty}(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)drd\gamma$

where, letting $G(\gamma)=\int_{R_i}^{\infty}(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)dr$;

$|G(\gamma)|\leq\int_{R_i}^{\infty}\frac{CD}{r^2}dr=[\frac{-CD}{r}]_{R_i}^{\infty}$

$=\frac{CD}{R_i}$

so that;

$lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}\int_{R_i}^{\infty}(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)drd\gamma=\int_{[\alpha,\beta)}G(\gamma)d\gamma$

exists and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{[0,\pi)}\int_{R_i}^{\infty}(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)drd\gamma|\leq\frac{CD(\beta-\alpha)}{R_i}$

It follows;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}(H^++H^-)(\overline{r}')d\overline{r}'$

exists, and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}(H^++H^-)(\overline{r}')d\overline{r}'|\leq\frac{CD(\beta-\alpha)}{R_i}$

as well. (UU)

Idea for $V_w(\overline{x})$, using calculation $(*)$ above;

With the same notation as above, for sufficiently large R, letting $\overline{r}''\in V_w(\overline{x})$, with $\overline{r}'=pr^*(\overline{r}'')$, with pr^* the orthogonal projection of $S_{1,\overline{y},a}$ onto the asymptotic line $l_{\overline{0},\overline{y},sh}$, $\overline{r}'_{opp}$ the opposite point to $\overline{r}'$ and $\overline{r}''_{opp}$ the nearest point to $\overline{r}'_{opp}$ on $V_w(\overline{x})\cap S_{1,\overline{y},a}$. Let $dV_{\overline{y}}$ be the restriction of Lebesgue measure to $V_w(\overline{x})\cap S_{1,\overline{y},a}$, $dZ_{\overline{y}}$ the restriction of Lebesgue measure to $l_{\overline{0},\overline{y},sh}=SCone_{1,a}\cap S_{1,\overline{y},a}$.

Using the notation above, we have that;

$||\frac{-b(0)}{\theta r_2\gamma(0)}|-R|=\epsilon_{\overline{y}}+O(\frac{1}{R})$

$\overline{r}''=\overline{r}'+O(\frac{1}{R})$

$\overline{r}'=\frac{R\overline{y}}{w}+\overline{v}_{\overline{y}}$

so that;

$\overline{r}''(\theta)=\frac{R\overline{y}}{w}+\overline{v}_{\overline{y}}+O(\frac{1}{R})$

$=(\frac{-b(0)}{\theta r_2\gamma(0)w}+\frac{\epsilon_{\overline{y}}}{w}+O(\frac{1}{R}))\overline{y}+\overline{v}_{\overline{y}}+O(\frac{1}{R})$

$=(\frac{-b(0)}{\theta r_2\gamma(0)w}+\frac{\epsilon_{\overline{y}}}{w})\overline{y}+\overline{v}_{\overline{y}}+O(\frac{1}{R})$

$=(\frac{-b(0)}{\theta r_2\gamma(0)w}+\frac{\epsilon_{\overline{y}}}{w})\overline{y}+\overline{v}_{\overline{y}}+\delta(\theta)$

where $\delta(\theta)=O(\frac{1}{R})$ is analytic in θ, so that $|\delta'(\theta)|\leq N$, for some $N\in\mathcal{R}_{>0}$. It follows that;

$\frac{d\overline{r}''}{d\theta}=\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\delta'(\theta)$

It follows that, using Newton's expansion;

$\frac{|\frac{d\overline{r}''(\theta)}{d\theta}|}{|pr^*(\frac{d\overline{r}''(\theta)}{d\theta})|}=\frac{|\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\delta'(\theta)|}{|pr^*(\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w})+pr^*(\delta'(\theta))|}$

$=\frac{|\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\delta'(\theta)|}{|\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\overline{v}_{\overline{y}}+pr^*(\delta'(\theta))|}$

$=\frac{|\frac{b(0)\overline{y}}{r_2\gamma(0)w}+\theta^2\delta'(\theta)|}{|\frac{b(0)\overline{y}}{r_2\gamma(0)w}+\theta^2(\overline{v}_{\overline{y}}+pr^*(\delta'(\theta)))|}$

$=\frac{(1+\theta^2\frac{r_2^2\gamma(0)^2w^2|\delta'(\theta)|^2}{b(0)^2|\overline{y}|^2})^{\frac{1}{2}}}{(1+\theta^2\frac{r_2^2\gamma(0)^2w^2|\overline{v}_{\overline{y}}+pr^*(\delta'(\theta))|^2}{b(0)^2|\overline{y}|^2})^{\frac{1}{2}}}$

$=1+O(\theta^2)$

$=1+O(\frac{1}{R^2})$ (SS)

so that;

$dV_{\overline{y}}(\overline{r}'')=dZ_{\overline{y}}(\overline{r}')+O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}')$

and, similarly;

$dV_{\overline{y}}(\overline{r}''_{opp}) = dZ_{\overline{y}}(\overline{r}'_{opp}) + O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}'_{opp})$

By the above, we have that;

$(i).\ \overline{r}'' = \overline{r}' + O(\frac{1}{R})$

$(ii).\ \overline{r}''_{opp} = \overline{r}'_{opp} + O(\frac{1}{R})$

$(iii).\ H^+(\overline{r}'') = H^+(\overline{r}') + O(\frac{1}{R^3})$

$(iv).\ H^-(\overline{r}''_{opp}) = H^-(\overline{r}'_{opp}) + O(\frac{1}{R^3})$

$(v).\ dV_{\overline{y}}(\overline{r}'') = dZ_{\overline{y}}(\overline{r}') + O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}')$

$(vi).\ dV_{\overline{y}}(\overline{r}''_{opp}) = dZ_{\overline{y}}(\overline{r}'_{opp}) + O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}'_{opp})$

$(vii).\ H^+(\overline{r}') + H^-(\overline{r}'_{opp}) = O(\frac{1}{R^3})$

$(viii).\ dZ_{\overline{y}}(\overline{r}') = dZ_{\overline{y}}(\overline{r}'_{opp}) = O(R)$

Then, using $(i) - (viii)$;

$H^+(\overline{r}'')dV_{\overline{y}}(\overline{r}'') + H^-(\overline{r}''_{opp})dV_{\overline{y}}(\overline{r}''_{opp})$

$= [H^+(\overline{r}') + O(\frac{1}{R^3})]dV_{\overline{y}}(\overline{r}'') + [H^-(\overline{r}'_{opp}) + O(\frac{1}{R^3})]dV_{\overline{y}}(\overline{r}''_{opp})$

$= [H^+(\overline{r}')+O(\frac{1}{R^3})][dZ_{\overline{y}}(\overline{r}')+O(\frac{1}{R^2})dZ(\overline{r}')]+[H^-(\overline{r}'_{opp})+O(\frac{1}{R^3})][dZ_{\overline{y}}(\overline{r}'_{opp})$

$+O(\frac{1}{R^2})dZ(\overline{r}'_{opp})]$

$= H^+(\overline{r}')dZ_{\overline{y}}(\overline{r}')+H^-(\overline{r}'_{opp})dZ_{\overline{y}}(\overline{r}'_{opp})+H^+(\overline{r}')O(\frac{1}{R^2})O(R)+H^-(\overline{r}'_{opp})O(\frac{1}{R^2})O(R)$

$+O(\frac{1}{R^3})O(R) + O(\frac{1}{R^3})O(\frac{1}{R^2})O(R) + O(\frac{1}{R^3})O(R) + O(\frac{1}{R^3})O(\frac{1}{R^2})O(R)$

$= H^+(\overline{r}')dZ_{\overline{y}}(\overline{r}') + H^-(\overline{r}'_{opp})dZ_{\overline{y}}(\overline{r}'_{opp}) + O(\frac{1}{R^2})$

$= O(\frac{1}{R^3})O(R) + O(\frac{1}{R^2})$

$= O(\frac{1}{R^2})$

With the same notation as above, let dV be the restriction of Lebesgue measure to $V_w(\overline{x})$, dZ the restriction of Lebesgue measure to $SCone_{1,a}$.

Choose a parametrisation $\overline{\beta}:[0,2\pi)\rightarrow S_{1,a}$. Following the calculation (SS) above, we have that, for $t\in[0,2\pi)$;

$||\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))}|-R|=\epsilon_{\overline{\beta}(t)}+O(\frac{1}{R})$

$\overline{r}''=\overline{r}'+O(\frac{1}{R})$

$\overline{r}'=\frac{R\overline{\beta}(t)}{w}+\overline{v}_{\overline{\beta}(t)}$

so that;

$\overline{r}''(\theta)=\frac{R\overline{\beta}(t)}{w}+\overline{v}_{\overline{\beta}(t)}+O(\frac{1}{R})$

$=(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w}+O(\frac{1}{R}))\overline{\beta}(t)+\overline{v}_{\overline{\beta}(t)}+O(\frac{1}{R})$

$=(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w})\overline{\beta}(t)+\overline{v}_{\overline{\beta}(t)}+O(\frac{1}{R})$

$=(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w})\overline{\beta}(t)+\overline{v}_{\overline{\beta}(t)}+\delta(\theta,t)$

where $\delta(\theta,t)=O(\frac{1}{R})$, uniformly in t, and is analytic in θ and t, so that $max(|\frac{\partial\delta}{\partial\theta}|,|\frac{\partial\delta}{\partial t}|)\leq N$, for some $N\in\mathcal{R}_{>0}$. It follows that;

$\frac{\partial\overline{r}''}{\partial\theta}=\frac{b(\overline{\beta}(t))\overline{\beta}(t)}{\theta^2 r_2\gamma(\overline{\beta}(t))w}+\frac{\partial\delta(\theta,t)}{\partial\theta}$

$=\frac{A_1(t)}{\theta^2}\overline{\beta}(t)+\frac{\partial\delta'(\theta,t)}{\partial\theta}$

where $A_1(t)=\frac{b(\overline{\beta}(t))}{r_2\gamma(\overline{\beta}(t))w}$

$\frac{\partial\overline{r}''}{\partial t}=(\frac{(-b\circ\overline{\beta})'(t)}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{b(\overline{\beta}(t))(\gamma\circ\overline{\beta})'(t)}{\theta r_2(\gamma\circ\overline{\beta})^2(t)w}+\frac{(\epsilon\circ\overline{\beta})'(t)}{w})\overline{\beta}(t)+(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w})\overline{\beta}'(t)$

$+(\overline{v}\circ\overline{\beta})'(t)+\frac{\partial\delta(\theta,t)}{\partial t}$

$=(\frac{A_2(t)}{\theta}+A_3(t))\overline{\beta}(t)+(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}'(t)+(\overline{v}\circ\overline{\beta})'(t)+\frac{\partial\delta(\theta,t)}{\partial t}$

where;

$A_2(t)=\frac{(-b\circ\overline{\beta})'(t)}{r_2\gamma(\overline{\beta}(t))w}+\frac{b(\overline{\beta}(t))(\gamma\circ\overline{\beta})'(t)}{r_2(\gamma\circ\overline{\beta})^2(t)w}$

$A_3(t)=\frac{(\epsilon\circ\overline{\beta})'(t)}{w}$

$A_4(t)=\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}$

$A_5(t)=\frac{\epsilon_{\overline{\beta}(t)}}{w}$

so that $\{A_1,A_2,A_3,A_4,A_5\}$ are analytic and bounded on the interval $[0,2\pi]$. We have, for $t\in[0,2\pi)$, that $(\overline{v}\circ\overline{\beta})(t)\centerdot\overline{b}(t)=0$, $pr^*((\overline{v}\circ\overline{\beta})(t))=\overline{0}$, so that $pr^*((\overline{v}\circ\overline{\beta})'(t))=\overline{0}$. Similarly, $pr^*(\overline{\beta}(t))=\overline{\beta}(t)$ so that $pr^*(\overline{\beta}'(t))=\overline{\beta}'(t)$.

It follows that;

$\frac{\partial\overline{r}''(\theta)}{\partial\theta}\times\frac{\partial\overline{r}''(\theta)}{\partial t}=\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O(\frac{1}{\theta^2},t)$

$pr^*(\frac{\partial\overline{r}''(\theta)}{\partial\theta})=\frac{A_1(t)}{\theta^2}\overline{\beta}(t)+pr^*(\frac{\partial\delta'(\theta,t)}{\partial\theta})$

$=\frac{A_1(t)}{\theta^2}\overline{\beta}(t)+O(1,t)$

$pr^*(\frac{\partial\overline{r}''(\theta)}{\partial t})=(\frac{A_2(t)}{\theta}+A_3(t))\overline{\beta}(t)+(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}'(t)+pr^*((\overline{v}\circ\overline{\beta})')(t)$

$+pr^*(\frac{\partial\delta(\theta,t)}{\partial t})$

$=(\frac{A_2(t)}{\theta}+A_3(t))\overline{\beta}(t)+(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}'(t)+O(1,t)$

$pr^*(\frac{\partial\overline{r}''(\theta)}{\partial\theta})\times pr^*(\frac{\partial\overline{r}''(\theta)}{\partial t})=\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\frac{1}{\theta^2},t)$

It follows that, using Newton's expansion;

$\frac{|\frac{\partial\overline{r}''(\theta)}{\partial\theta}\times\frac{\partial\overline{r}''(\theta)}{\partial t}|}{|pr^*(\frac{\partial\overline{r}''(\theta)}{d\theta})\times pr^*(\frac{\partial\overline{r}''(\theta)}{dt})|}=\frac{|\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O(\frac{1}{\theta^2},t)|}{|\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\frac{1}{\theta^2},t)|}$

$=\frac{|(A_1(t)A_4(t)+\theta A_1(t)A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O(\theta,t)|}{|(A_1(t)A_4(t)+\theta A_1(t)A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\theta,t)|}$

$=\frac{|A_1(t)A_4(t)\overline{\beta}(t)\times\overline{\beta}'(t)+O(\theta,t)|}{|A_1(t)A_4(t)\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\theta,t)|}$

$=\frac{(1+O''(\theta,t))^{\frac{1}{2}}}{(1+O'''(\theta,t))^{\frac{1}{2}}}$

$=(1+\frac{1}{2}O''(\theta,t)+O''''(\theta^2,t))(1-\frac{1}{2}O'''(\theta,t)+O'''''(\theta^2,t))$

$= 1 + O(\theta, t)$

$= 1 + O(\frac{1}{R}, t)$ (SSS)

so that;

$dV(\overline{r}'') = dZ(\overline{r}') + O(\frac{1}{R}, t)dZ(\overline{r}')$

and, similarly;

$dV(\overline{r}''_{opp}) = dZ(\overline{r}'_{opp}) + O(\frac{1}{R}, t)dZ(\overline{r}'_{opp})$

As above, using (SSS) now for $(v), (vi)$, we have that;

(i). $\overline{r}'' = \overline{r}' + O(\frac{1}{R}, t)$

(ii). $\overline{r}''_{opp} = \overline{r}'_{opp} + O(\frac{1}{R}, t)$

(iii). $H^+(\overline{r}'') = H^+(\overline{r}') + O(\frac{1}{R^3}, t)$

(iv). $H^-(\overline{r}''_{opp}) = H^-(\overline{r}'_{opp}) + O(\frac{1}{R^3}, t)$

(v). $dV(\overline{r}'') = dZ(\overline{r}') + O(\frac{1}{R}, t)dZ(\overline{r}')$

(vi). $dV(\overline{r}''_{opp}) = dZ(\overline{r}'_{opp}) + O(\frac{1}{R}, t)dZ(\overline{r}'_{opp})$

(vii). $H^+(\overline{r}') + H^-(\overline{r}'_{opp}) = O(\frac{1}{R^3}, t)$

$(viii)$. $dZ(\overline{r}') = dZ(\overline{r}'_{opp}) = O(R, t)$

Then, using $(i) - (viii)$;

$H^+(\overline{r}'')dV(\overline{r}'') + H^-(\overline{r}''_{opp})dV(\overline{r}''_{opp})$

$= [H^+(\overline{r}') + O(\frac{1}{R^3}, t)]dV(\overline{r}'') + [H^-(\overline{r}'_{opp}) + O(\frac{1}{R^3}, t)]dV(\overline{r}''_{opp})$

$= [H^+(\overline{r}')+O(\frac{1}{R^3}, t)][dZ(\overline{r}')+O(\frac{1}{R}, t)dZ(\overline{r}')]+[H^-(\overline{r}'_{opp})+O(\frac{1}{R^3}, t)][dZ(\overline{r}'_{opp})$

$+O(\frac{1}{R}, t)dZ(\overline{r}'_{opp})]$

$= H^+(\overline{r}')dZ(\overline{r}')+H^-(\overline{r}'_{opp})dZ(\overline{r}'_{opp})+H^+(\overline{r}')O(\frac{1}{R}, t)O(R)+H^-(\overline{r}'_{opp})O(\frac{1}{R}, t)O(R, t)$

$+O(\frac{1}{R^3},t)O(R,t)+O(\frac{1}{R^3},t)O(\frac{1}{R},t)O(R,t)+O(\frac{1}{R^3},t)O(R,t)+O(\frac{1}{R^3},t)O(\frac{1}{R},t)O(R,t)$

$= H^+(\overline{r}')dZ(\overline{r}') + H^-(\overline{r}'_{opp})dZ(\overline{r}'_{opp}) + O(\frac{1}{R^2},t)$

$= O(\frac{1}{R^3},t)O(R,t) + O(\frac{1}{R^2},t)$

$= O(\frac{1}{R^2},t)$

.... Look at argument of (UU) again, $drd\gamma$, $(0 \leq \gamma < 2\pi)$. Final integration over $0 \leq w \leq s$, exclude discrete case, use Lemma 0.50.

□

Lemma 0.55. *Let $\{\overline{r},\overline{y}\} subset \mathcal{R}^3$, let $l \subset \mathcal{R}^3$ be a line, with $\{\overline{p},\overline{p}'\} \subset l$ and $\overline{p} \neq \overline{p}'$. Then if $\overline{x}_\lambda = \overline{p} + \lambda(\overline{p}' - \overline{p})$, we have that;*

$$lim_{\lambda\rightarrow\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|) = -lim_{\lambda\rightarrow-\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|)$$

Proof. By rotating and translating coordinates (x,y,z), which preserves distance, we may assume that l is the line $y = z = 0$, $\overline{p} = \overline{0}$, $\overline{p}' = (x_0,0,0)$, $\overline{y} = (y_1,y_2,0)$ and $\overline{r} = (r_1,r_2,r_3)$. Then, using Newton's expansion;

$|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|$

$= |(\lambda x_0,0,0) - (y_1,y_2,0)| - |(\lambda x_0,0,0) - (r_1,r_2,r_3)|$

$= [(\lambda x_0 - y_1)^2 + y_2^2]^{\frac{1}{2}} - [(\lambda x_0 - r_1)^2 + r_2^2 + r_3^2]^{\frac{1}{2}}$

$= [\lambda^2 x_0^2 - 2\lambda x_0 y_1 + y^2]^{\frac{1}{2}} - [\lambda^2 x_0^2 - 2\lambda x_0 r_1 + r^2]^{\frac{1}{2}}$

$= |\lambda x_0|[1 - \frac{2y_1}{\lambda x_0} + \frac{y^2}{\lambda^2 x_0^2}]^{\frac{1}{2}} - |\lambda x_0|[1 - \frac{2r_1}{\lambda x_0} + \frac{r^2}{\lambda^2 x_0^2}]^{\frac{1}{2}}$

$= |\lambda x_0|(1 - \frac{y_1}{\lambda x_0} + O(\frac{1}{\lambda^2})) - |\lambda x_0|(1 - \frac{r_1}{\lambda x_0} + O(\frac{1}{\lambda^2}))$

$= -\frac{sign(\lambda)y_1}{x_0} + \frac{sign(\lambda)r_1}{x_0} + O(\frac{1}{\lambda})$

where $y = |\overline{y}$ and $r = |\overline{r}|$, so that;

$lim_{\lambda\rightarrow\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|) = -\frac{y_1}{x_0} + \frac{r_1}{x_0}$

$lim_{\lambda\to-\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|) = \frac{y_1}{x_0} - \frac{r_1}{x_0}$

$= -lim_{\lambda\to\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|)$

□

Definition 0.56. *For $f \in C^\infty(\mathcal{R}^4)$ and $h \in \mathcal{R}$, we define the time shift f^h by $f^h(\overline{x}, t) = f(\overline{x}, t+h)$. For a field $\overline{f}$, with $\overline{f} = (f_1, f_2, f_3)$ and $f_i \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, we define $\overline{f}^h = (f_1^h, f_2^h, f_3^h)$.*

Lemma 0.57. *Let $(\rho, \overline{J})$ be a charge and current configuration with $\rho \in C^\infty(\mathcal{R}^4)$, $\overline{J} = (j_1, j_2, j_3)$, and $j_i \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, such that $(\rho, \overline{J})$ satisfies the continuity equation. Then, for $h \in \mathcal{R}_{>0}$, the time shifts $(\rho^h, \overline{J}^h)$ satisfy the continuity equation and so do the sums $(\rho+\rho^h, \overline{J}+\overline{J}^h)$. If for $h \in \mathcal{R}_{>0}$, there exists electric and magnetic fields $(\overline{E}_h, \overline{B}_h)$ such that $(\rho+\rho^h, \overline{J}+\overline{J}^h, \overline{E}_h, \overline{B}_h)$ satisfy Maxwell's equations, then there exist fields $\overline{E}$ and $\overline{B}$ such that $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfy Maxwell's equations.*

Proof. By the hypotheses, we have for $\{h_1, h_2\} \subset \mathcal{R}_{>0}$, with $h_2 > h_1$ that there exist pairs $(\overline{E}_{h_1}, \overline{B}_{h_1})$ and $(\overline{E}_{h_2}, \overline{B}_{h_2})$ such that $(\rho+\rho^{h_1}, \overline{J}+\overline{J}^{h_1}, \overline{E}_{h_1}, \overline{B}_{h_1}$ and $(\rho+\rho^{h_2}, \overline{J}+\overline{J}^{h_2}, \overline{E}_{h_2}, \overline{B}_{h_2}$ satisfy Maxwell equations, so that, taking the difference, $(\rho^{h_1}-\rho^{h_2}, \overline{J}^{h_1}-\overline{J}^{h_2}, \overline{E}_{h_1}-\overline{E}_{h_2}, \overline{B}_{h_1}-\overline{B}_{h_2})$ satisfy Maxwell's equations, $(*)$. Then $h_2 - h_1 > 0$, so that, by the hypotheses, there exist $(\overline{E}_{h_2-h_1}, \overline{B}_{h_2-h_1})$ such that $(\rho + \rho^{h_2-h_1}, \overline{J} + \overline{J}^{h_2-h_1}, \overline{E}_{h_2-h_1}, \overline{B}_{h_2-h_1})$ satisfy Maxwell's equations, $(**)$. As is easily checked, if $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfy Maxwell's equations, then, for $h \in \mathcal{R}$, $(\rho^h, \overline{J}^h, \overline{E}^h, \overline{B}^h)$ satisfy Maxwell's equations, so that, from $(**)$;

$$(\rho^{h_1} + \rho^{h_2-h_1+h_1}, \overline{J}^{h_1} + \overline{J}^{h_2-h_1+h_1}, \overline{E}^{h_1}_{h_2-h_1}, \overline{B}^{h_1}_{h_2-h_1})$$

$$= (\rho^{h_1} + \rho^{h_2}, \overline{J}^{h_1} + \overline{J}^{h_2}, \overline{E}^{h_1}_{h_2-h_1}, \overline{B}^{h_1}_{h_2-h_1})\ (***)$$

satisfies Maxwell's equations. Then adding the equations $(*), (***)$, we obtain that;

$$(2\rho^{h_1}, 2\overline{J}^{h_1}, \overline{E}_{h_1} - \overline{E}_{h_2} + \overline{E}^{h_1}_{h_2-h_1}, \overline{B}_{h_1} - \overline{B}_{h_2} + \overline{B}^{h_1}_{h_2-h_1})$$

satisfies, Maxwell's equation and;

$$(\rho^{h_1}, \overline{J}^{h_1}, \tfrac{1}{2}(\overline{E}_{h_1} - \overline{E}_{h_2} + \overline{E}^{h_1}_{h_2-h_1}), \tfrac{1}{2}(\overline{B}_{h_1} - \overline{B}_{h_2} + \overline{B}^{h_1}_{h_2-h_1}))$$

satisfies Maxwell's equations. Again, by the observation above, it follows that;

$$(\rho^{h_1-h_1},\overline{J}^{h_1-h_1},\tfrac{1}{2}(\overline{E}_{h_1}^{-h_1}-\overline{E}_{h_2}^{-h_1}+\overline{E}_{h_2-h_1}^{h_1-h_1}),\tfrac{1}{2}(\overline{B}_{h_1}^{-h_1}-\overline{B}_{h_2}^{-h_1}+\overline{B}_{h_2-h_1}^{h_1-h_1}))$$

$$(\rho,\overline{J},\tfrac{1}{2}(\overline{E}_{h_1}^{-h_1}-\overline{E}_{h_2}^{-h_1}+\overline{E}_{h_2-h_1}),\tfrac{1}{2}(\overline{B}_{h_1}^{-h_1}-\overline{B}_{h_2}^{-h_1}+\overline{B}_{h_2-h_1}))$$

satisfies Maxwell's equations, as required. □

Lemma 0.58. *Let $(\rho_w,\overline{J}_w)$ for $w\neq c$, be the smooth charge and current configurations defined above, satisfying the continuity equation. Then the causal fields $(\overline{E}_w,\overline{B}_w)$ defined by Jefimenko's equations exist for $w\neq c$, with $(\rho_w,\overline{J}_w,\overline{E}_w,\overline{B}_w)$ satisfying Maxwell's equations. Moreover $lim_{w\rightarrow c}\overline{E}_w$ and $\lim_{w\rightarrow c}\overline{B}_w$ exist and define fields $(\overline{E}_c,\overline{B}_c$ such that $(\rho_c,\overline{J}_c,\overline{E}_c,\overline{B}_c)$ satisfy Maxwell's equations.*

Proof. The first claim will be proved later, the second claim follows from a result in [14]. For $h\in\mathcal{R}_{>0}$, we have that $(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h)$ satisfies the continuity equation, $w\neq c$. By the observation in the previous lemma, $(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h,\overline{E}_w+\overline{E}_w^h,\overline{B}_w+\overline{B}_w^h)$ satisfies Maxwell's equations, and is defined by Jefimenko's equations relative to $(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h)$. By the main proof, (choosing the initial conditions at $\frac{t+h}{2}$, between t and $t+h$) we have that $lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^h)=\overline{E}_{c,h}$ and $lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^h)=\overline{B}_{c,h}$ exist, so that (more proof required);

$$lim_{w\rightarrow c}(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h,\overline{E}_w+\overline{E}_w^h,\overline{B}_w+\overline{B}_w^h)$$

$$=(\rho_c+\rho_c^h,\overline{J}_c+\overline{J}_c^h,\overline{E}_{c,h},\overline{B}_{c,h})$$

satisfies Maxwell's equations. By Lemma 0.57, for $\{h_1,h_2\}\subset\mathcal{R}_{>0}$, with $h_1<h_2$, we have that;

$$(\rho_c,\overline{J}_c,\tfrac{1}{2}(\overline{E}_{c,h_1}^{-h_1}-\overline{E}_{c,h_2}^{-h_1}+\overline{E}_{c,h_2-h_1}),\tfrac{1}{2}(\overline{B}_{c,h_1}^{-h_1}-\overline{B}_{c,h_2}^{-h_1}+\overline{B}_{c,h_2-h_1}))$$

$$=(\rho_c,\overline{J}_c,\tfrac{1}{2}(lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^{h_1})^{-h_1}-lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^{h_2})^{-h_1}$$

$$+lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^{h_2-h_1})),\tfrac{1}{2}(lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^{h_1})^{-h_1}$$

$$-lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^{h_2})^{-h_1}+lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^{h_2-h_1})))$$

$$=(\rho_c,\overline{J}_c,lim_{w\rightarrow c}\overline{E}_w,lim_{w\rightarrow c}\overline{B}_w)$$

satisfies Maxwell's equations, as required.

□

Lemma 0.59. *Cartesian method*

If $g : \mathcal{R}^3 \rightarrow \mathcal{R}$ is analytic for $|\overline{x}| > r$, where $r \in \mathcal{R}_{>0}$, analytic at infinity, of very moderate decrease, and continuous, with $\{\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\}$ analytic for $|\overline{x}| > r$ and analytic at infinity, we can define, for $\overline{k} \in \mathcal{R}^3$, with $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$;

$$\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_1 \rightarrow \infty} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_1}^{r_1} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} g(\overline{x}) e^{-i\overline{k}.\overline{x}} dx_1 dx_2 dx_3$$

Moreover, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, we have that;

$$\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r \rightarrow \infty} \int_{0 \leq \theta \leq \pi, -\pi \leq \phi \leq \pi} g(r, \theta, \phi) e^{-ik_1 r sin(\theta) cos(\phi)}$$

$$e^{-ik_2 r sin(\theta) sin(\phi)} e^{-ik_3 r cos(\theta)} r^2 sin(\theta) dr d\theta d\phi$$

Proof. Let C_r be the cube defined by $C_r = \{(x, y, z) \in \mathcal{R}^3 : |x| \leq r, |y| \leq r, |z| \leq r\}$, then, as g is analytic for $|\overline{x}| > r$, we have that g is analytic on $\mathcal{R}^3 \setminus C_r$, with global power series expansion $\sum_{(i,j,k) \in \mathcal{Z}^3_{\geq 0}} a_{ijk} x^i y^j z^k$, convergent on $\mathcal{R}^3 \setminus C_r$. As g is continuous, it is bounded on C_r and we can define;

$$f(\overline{k}) = \int_{\overline{x} \in C_r} g(\overline{x}) e^{-i\overline{k}.\overline{x}} d\overline{x}$$

For $(x, y) \in \mathcal{R}^2$, we have that $g_{x,y}(z)$ is analytic for $|z| > r$, analytic at infinity, and of very moderate decrease. In particularly, by Lemma 0.46, using the fact that $g_{x,y}$ is also of very moderate decrease, $g_{x,y}$ is eventually monotone, and for $k_3 \neq 0$, we can define;

$$g_3(x, y, k_3) = lim_{r_3 \rightarrow \infty} \int_{r < |z| < r_3} g_{x,y}(z) e^{-ik_3 z} dz$$

$$= lim_{r_3 \rightarrow \infty} \int_{r < |z| < r_3} (\sum_{(i,j,k) \in \mathcal{Z}^3_{\geq 0}} a_{ijk} x^i y^j z^k)(\sum_{m=0}^{\infty} \frac{(-ik_3 z)^m}{m!}) dz$$

$$= lim_{r_3 \rightarrow \infty} \int_{r < |z| < r_3} (\sum_{(i,j,k,m) \in \mathcal{Z}^4_{\geq 0}} \frac{(-ik_3)^m a_{ijk}}{m!} x^i y^j z^{k+m}) dz$$

$$= lim_{r_3 \rightarrow \infty} \sum_{(i,j,k,m) \in \mathcal{Z}^4_{\geq 0}} \frac{(-ik_3)^m a_{ijk}}{m!} x^i y^j \int_{r < |z| < r_3} z^{k+m} dz$$

$$= lim_{r_3 \rightarrow \infty} \sum_{(i,j,k,m) \in \mathcal{Z}^4_{\geq 0}} \frac{(-ik_3)^m a_{ijk}}{m!} x^i y^j ([\frac{z^{k+m+1}}{k+m+1}]_{-r_3}^{-r} + [\frac{z^{k+m+1}}{k+m+1}]_r^{r_3})$$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m a_{ijk}}{m!(k+m+1)}x^i y^j(r_3^{k+m+1}-r^{k+m+1})$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m r_3^{k+m+1}a_{ijk}}{m!(k+m+1)}x^i y^j$

$-\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m a_{ijk}r^{k+m+1}}{m!(k+m+1)}x^i y^j$

$=\sum_{(i,j)\in\mathcal{Z}^2_{\geq 0}}(b_{ij}-c_{ij})x^i y^j$

where;

$b_{ij} = lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m r_3^{k+m+1}a_{ijk}}{m!(k+m+1)}$

$c_{ij} = \sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k+m+1 odd}\frac{2(-ik_3)^m a_{ijk}r^{k+m+1}}{m!(k+m+1)}$

so that $g_3(x,y,k_3)$ is analytic for $(x,y)\in\mathcal{R}^2$, in particularly continuous.

For $(x_0 : y_0)\in P^1(\mathcal{R})$, we have that $(x_0 : y_0 : 1)\in P^2(\mathcal{R})$ and, as g is analytic at infinity, there exists $\epsilon_{x_0,y_0,1}$, such that $g(\frac{x_0}{x},\frac{y_0}{y},\frac{1}{z})$ is defined by a convergent power series $\sum_{(i,j,k)\in\mathcal{Z}^3}d_{ijk}x^i y^j z^k$ in the region $x^2+y^2+z^2<\epsilon^2_{x_0,y_0,1}$. Without loss of generality, assuming that $x_0\neq 0$, $y_0\neq 0$, as g is analytic for $|\frac{x_0}{x}|>r$, $|\frac{y_0}{y}|>r$, $|\frac{1}{z}|>r$, $|x|<\frac{|x_0|}{r}$, $|y|<\frac{|y_0|}{r}$, $|z|<\frac{1}{r}$, by uniqueness of power series, we can replace the region $x^2+y^2+z^2<\epsilon^2_{x_0,y_0,1}$, by the region $|x|<\frac{|x_0|}{r}$, $|y|<\frac{|y_0|}{r}$, $|z|<\frac{1}{r}$. Then;

$g_3(\frac{x_0}{x},\frac{y_0}{y},k_3) = lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3}g(\frac{x_0}{x},\frac{y_0}{y},w)e^{-ik_3 w}dw$

$= lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3}g(\frac{x_0}{x},\frac{y_0}{y},\frac{1}{z})e^{\frac{-ik_3}{z}}-\frac{dz}{z^2}dz$ $(z=\frac{1}{w}, z\neq 0)$

$= lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3}(\sum_{(i,j,k)\in\mathcal{Z}^3_{\geq 0}}d_{ijk}x^i y^j z^k)(\sum_{m=0}^{\infty}\frac{-(-ik_3)^m}{z^{m+2}m!})dz$

$= lim_{r_3\rightarrow\infty}\int_{r<|z|<r_3}(\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{-(-ik_3)^m d_{ijk}}{m!}x^i y^j z^{k-m-2})dz$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0}}\frac{-(-ik_3)^m d_{ijk}}{m!}x^i y^j\int_{r<|z|<r_3}z^{k-m-2}dz$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1}\frac{-(-ik_3)^m d_{ijk}}{m!}x^i y^j([\frac{z^{k-m-1}}{k-m-1}]^{-r}_{-r_3}+[\frac{z^{k-m-1}}{k-m-1}]^{r_3}_{r})$

$+lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k=m+1}\frac{-(-ik_3)^m d_{ijk}}{m!}x^i y^j([-ln(z)]^{r_3}_{r}+[ln(z)]^{r_3}_{r})$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1}\frac{-(-ik_3)^m d_{ijk}}{m!}x^i y^j([\frac{z^{k-m-1}}{k-m-1}]^{-r}_{-r_3}+[\frac{z^{k-m-1}}{k-m-1}]^{r_3}_r)$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1odd}\frac{-2(-ik_3)^m d_{ijk}}{m!(k-m-1)}x^i y^j(r_3^{k-m-1}-r^{k-m-1})$

$= lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1odd}\frac{-2(-ik_3)^m r_3^{k-m-1}d_{ijk}}{m!(k-m-1)}x^i y^j$

$-\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1odd}\frac{-2(-ik_3)^m d_{ijk}r^{k-m-1}}{m!(k-m-1)}x^i y^j$

$= \sum_{(i,j)\in\mathcal{Z}^2_{\geq 0}}(k_{ij}-l_{ij})x^i y^j$

where;

$k_{ij} = lim_{r_3\rightarrow\infty}\sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1odd}\frac{-2(-ik_3)^m r_3^{k-m-1}d_{ijk}}{m!(k-m-1)}$

$l_{ij} = \sum_{(i,j,k,m)\in\mathcal{Z}^4_{\geq 0},k\neq m+1,k-m-1odd}\frac{-2(-ik_3)^m d_{ijk}r^{k-m-1}}{m!(k-m-1)}$

We can then take $\epsilon_{x_0,y_0} = \frac{min(\frac{|x_0|}{r},\frac{|y_0|}{r})}{\sqrt{2}}$, so that as $(x_0:y_0)\in P^2(\mathcal{R})$ was arbitrary, $g_3(x,y,k_3)$ is analytic at infinity.

As g is of very moderate decrease, we have that;

$|g_{xy}(z)| = |g(x,y,z)| \leq \frac{C}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

$= \frac{|z|}{(x^2+y^2+z^2)^{\frac{1}{2}}}\frac{C}{|z|}$

$\leq \frac{C}{|z|}$ (A)

for $(x^2+y^2+z^2)^{\frac{1}{2}} > |z| > s$. As $\frac{\partial g}{\partial z}$ is analytic for $|\overline{x}| > r$ and analytic at infinity, it has finitely many zeroes, so that $g_{xy}(z)$ is eventually monotone in the interval $|z| > E$, for some $E\in\mathcal{R}_{>0}$, uniformly in (x,y), (B), and we can achieve both $(A),(B)$, for $|z| > v = max(s,E)$. Without loss of generality, we can assume that $v > E > r$. We also have that, for $(x^2+y^2+z^2)^{\frac{1}{2}} \geq (x^2+y^2)^{\frac{1}{2}} > s$;

$|g_{xy}(z)| \leq \frac{C}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

$= \frac{C}{(x^2+y^2)^{\frac{1}{2}}}\frac{(x^2+y^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

$\leq \frac{C}{(x^2+y^2)^{\frac{1}{2}}}$

Then, by a simple generalisation of Lemma 0.35, for $|(x,y)| \geq s$, we have that;

$|g_3(x,y,k_3)| = |lim r_3 \rightarrow \infty \int_{r<|z|<r_3} g(x,y,z)e^{-ik_3 z}dz|$

$\leq \frac{4Cv}{|(x,y)|} + \frac{6C\pi}{|(x,y)|k_3|}$

$= \frac{W}{|(x,y)|}$ (F)

where $W = 4Cv + \frac{6C\pi}{|k_3|}$, so that $g_3(x,y,k_3)$ is of very moderate decrease.

As $g_3(x,y,k_3$ is analytic for $(x,y) \in \mathcal{R}^2$ and analytic at infinity, so is $\frac{\partial g_3}{\partial y}$, so that, for fixed $x \in \mathcal{R}$, $g_{3,x,k_3}(y)$ is eventually monotone and of very moderate decrease, so that, for $k_2 \neq 0$, we can define;

$g_2(x,k_2,k_3) = lim_{r_2 \rightarrow \infty} \int_{-r_2}^{r_2} g_3(x,y,k_3)e^{-ik_2 y}dy$

As $g_3(x,y,k_3$ is analytic for $(x,y) \in \mathcal{R}^2$ and analytic at infinity, using $(1:1) \in P^1(\mathcal{R})$, $g_3(\frac{1}{x},\frac{1}{y},k_3)$ is defined by a convergent power series $\sum_{(i,j)\in \mathcal{Z}^2_{\geq 0}} s_{ij}x^i y^j$, valid for $(x,y) \in \mathcal{R}^2$, so that, for $x_0 \neq 0$;

$g_2(x_0,k_2,k_3) = lim_{r_2 \rightarrow \infty} \int_{-r_2}^{r_2} g_3(x_0,w,k_3)e^{-ik_2 w}dw$

$= lim_{r_2 \rightarrow \infty} \int_{-r_2}^{r_2} g_3(\frac{1}{x},\frac{1}{y},k_3)e^{\frac{-ik_2}{y}} - \frac{1}{y^2}dy$ $(w = \frac{1}{y}$, $x_0 = \frac{1}{x})$

$= lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \int_{\epsilon<|y|<r_2} (\sum_{(i,j)\in \mathcal{Z}^2_{\geq 0}} s_{ij}x^i y^j)(\sum_{m=0}^{\infty} \frac{-(-ik_2)^m}{y^{m+2}m!})dy$

$= lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \int_{\epsilon<|y|<r_2} (\sum_{(i,j,m)\in \mathcal{Z}^3_{\geq 0}} \frac{-(-ik_2)^m s_{ij}}{m!} x^i y^{j-m-2})dy$

$= lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \sum_{(i,j,m)\in \mathcal{Z}^3_{\geq 0}} \frac{-(-ik_2)^m s_{ij}}{m!} x^i \int_{\epsilon<|y|<r_2} y^{j-m-2}dy$

$= lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \sum_{(i,j,m)\in \mathcal{Z}^3_{\geq 0}, j \neq m+1} \frac{-(-ik_2)^m s_{ij}}{m!} x^i([\frac{y^{j-m-1}}{j-m-1}]_{-r_2}^{-\epsilon} + [\frac{y^{j-m-1}}{j-m-1}]_{\epsilon}^{r_2})$

$+lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \sum_{(i,j,m)\in \mathcal{Z}^3_{\geq 0}, j=m+1} \frac{-(-ik_3)^m s_{ij}}{m!} x^i([-ln(z)]_{\epsilon}^{r_2} + [ln(z)]_{\epsilon}^{r_2})$

$= lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \sum_{(i,j,m)\in \mathcal{Z}^3_{\geq 0}, j \neq m+1} \frac{-(-ik_3)^m s_{ij}}{m!} x^i([\frac{y^{j-m-1}}{j-m-1}]_{-r_2}^{-\epsilon} + [\frac{y^{j-m-1}}{j-m-1}]_{\epsilon}^{r_2})$

$= lim_{r_2 \rightarrow \infty, \epsilon \rightarrow 0} \sum_{(i,j,m)\in \mathcal{Z}^3_{\geq 0}, j \neq m+1, j-m-1 odd} \frac{-2(-ik_2)^m s_{ij}}{m!(j-m-1)} x^i(r_2^{j-m-1} - \epsilon^{j-m-1})$

$= lim_{r_2\rightarrow\infty}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m r_2^{j-m-1}s_{ij}}{m!(j-m-1)}x^i$

$-lim_{\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m s_{ij}\epsilon^{k-m-1}}{m!(j-m-1)}x^i$

$= \sum_{i\in\mathcal{Z}_{\geq 0}}(\alpha_i-\beta_i)x^i$

$= \sum_{i\in\mathcal{Z}_{\geq 0}}(\alpha_i-\beta_i)x^i$

$= \sum_{i\in\mathcal{Z}_{\geq 0}}(\alpha_i-\beta_i)(\frac{1}{x_0})^i$

where;

$\alpha_i = lim_{r_2\rightarrow\infty}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m r_2^{j-m-1}s_{ij}}{m!(j-m-1)}$

$\beta_i = lim_{\epsilon\rightarrow 0}\sum_{(i,j,m)\in\mathcal{Z}^3_{\geq 0},j\neq m+1,j-m-1odd}\frac{-2(-ik_2)^m s_{ij}\epsilon^{k-m-1}}{m!(j-m-1)}$

It follows that $g_2(x,k_2,k_3)$ is analytic at infinity, and, as $\frac{1}{x_0}$ is analytic for $x_0\neq 0$, and the composition of analytic functions is analytic, $g_2(x,k_2,k_3)$ is analytic for $x\neq 0$.

By the same reasoning as above, we have that $g_2(x,k_2,k_3)$ is of very moderate decrease, and using the fact that $\frac{dg_2}{dx}$ is analytic for $x\neq 0$, and analytic at infinity , using Lemma 0.46, $g_2(x,k_2,k_3)$ is eventually monotone, so , for $k_3\neq 0$, we can define;

$g_1(k_1,k_2,k_3) = lim_{r_1\rightarrow\infty}\int_{-r_1}^{r_1} g_2(x,k_2,k_3)e^{-ik_1x}dx$

For $|z|<r$, $x\in\mathcal{R}$, by the usual arguments, we can define;

$h_2(x,z,k_2) = lim_{r_2\rightarrow\infty}\int_{r<|y|<r_2} g(x,y,z)e^{-ik_2y}dy$

As above, as g is analytic in the region $|y|>r$ and analytic at infinity, we can show that h_2 is analytic for $x\neq 0$ and $z\neq 0$, analytic at infinity, and of very moderate decrease. By the usual arguments, we can then define, for $|z|<r$;

$h_1(z,k_1,k_2) = lim_{r_1\rightarrow\infty}\int_{|x|<r_1} g(x,y,z)e^{-ik_1x}dx$

and show that h_1 is analytic for $0<|z|<r$, and smooth at 0 (extra argument here), in particular, bounded. Then , for $k_3\neq 0$, let;

$h_3(k_1,k_2,k_3) = \int_{|z|<r} h_1(z,k_1,k_2)e^{-ik_3z}dz$

For $|z| < r$, $|y| < r$, define;

$s_1(y,z,k_1) = lim_{r_1\rightarrow\infty}\int_{r<|x|<r_1} g(x,y,z)e^{-ik_1x}dx$

As above, as g is analytic in the region $|x| > r$ and analytic at infinity, we can show that s_1 is analytic for $y \neq 0$ and $z \neq 0$, analytic at infinity, and of very moderate decrease. We can also show that s_1 is smooth along on the locus $((y = 0 \cup z = 0) \cap (|y| < r \cap |z| < r)) \subset \mathcal{R}^2$ (extra argument here). Then, by the usual arguments, we can define;

$s_{2,3}(k_1,k_2,k_3) = \int_{|y|<r,|z|<r} s_1(y,z,k_1)e^{-ik_2y}e^{-ik_3z}dydz$

Let $m(k_1,k_2,k_3) = h_3(k_1,k_2,k_3) + s_{2,3}(k_1,k_2,k_3)$, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$. Then, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, it is clear, totalling the volumes, that we have;

$\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}(f(\overline{k}) + g_1(\overline{k}) + m(\overline{k}))$

$\square$

Lemma 0.60. *Let g and all its partial derivatives $\{\frac{\partial^{(i_1,i_2,i_3)}g}{\partial^{i_1}x\partial^{i_2}y\partial^{i_3}z} : 0 \leq i_1+i_2+i_3 \leq 4\}$ satisfy the hypotheses of the previous lemma. Then, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, with $|k_1|, |k_2|, |k_3|$, sufficiently large, there exists constants $C_{i_1,i_2,i_3} \in \mathcal{R}_{>0}$, with;*

$|\mathcal{F}(\frac{\partial^{(i_1,i_2,i_3)}g}{\partial^{i_1}x\partial^{i_2}y\partial^{i_3}z})(\overline{k})| \leq \frac{C_{i_1,2,i_3}}{|k_1||k_2||k_3|}$

and $D \in \mathcal{R}_{>0}$, with;

$|\mathcal{F}(g)(\overline{k})| \leq \frac{D}{|k_1||k_2||k_3||\overline{k}|^4}$

We have that, for $r > 0$, $\mathcal{F}(g)|_{B(\overline{0},r)} \in L^1(B(\overline{0},r))$, $\mathcal{F}(g)|_V \in L^1(V)$, $\mathcal{F}(g)|_{V_i} \in L^1(V_i)$, for $1 \leq i \leq 3$, $\mathcal{F}(g)|_{V_{ij}} \in L^1(V_{ij})$, $1 \leq i < j \leq 3$, where;

$V = \{(k_1,k_2,k_3) : |k_1| \geq E_1, |k_2| \geq E_2, |k_3| \geq E_3\}$.

$V_i = \{(k_1,k_2,k_3) : |k_i| < E_i, |k_l| \geq E_l, l \neq i, 1 \leq l \leq 3\}$.

$V_{ij} = \{(k_1, k_2, k_3) : |k_i| < E_i, |k_j| < E_j, |k_l| \geq E_l, l \neq i, l \neq j,$

$1 \leq l \leq 3\}.$

In particularly, $\mathcal{F}(g) \in L^1(\mathcal{R}^3)$.

Proof. For the first claim, let;

$a_3(x, y, k_3) = lim_{r_3 \rightarrow \infty} \int_{-r_3}^{r_3} g(x, y, z)e^{-ik_3 z} dz$

for $k_3 \neq 0$. (Then for fixed x, k_3, $a_3(x, y, k_3)$ is of very moderate decrease in y and oscillatory for sufficiently large y.

Then, we can define;

$a_2(x, k_2, k_3) = lim_{r_2 \rightarrow \infty} \int_{-r_2}^{r_2} a_3(x, y, k_3)e^{-ik_2 y} dy$

for $k_2 \neq 0$. (For, fixed k_2, k_3, $a_2(x, k_2, k_3)$ is of very moderate decrease in y and oscillatory, for sufficiently large x).

so we can define, for $k_1 \neq 0$;

$a_1(k_1, k_2, k_3) = lim_{r_1 \rightarrow \infty} \int_{-r_1}^{r_1} a_2(x, k_2, k_3)e^{-ik_1 y} dy$

$\mathcal{F}(g)(\overline{k}) = \frac{1}{(2\pi)}^{\frac{3}{2}} a_1(k_1, k_2, k_3)$

Using the end of the proof of Lemma 0.35, we can show that that there exists $C \in \mathcal{R}_{>0}$, independent of x, y, with;

$|a_3(x, y, k_3)| \leq \frac{C||g||_\infty}{|k_3|}$

for sufficiently large $|k_3| \geq C_3$. Similarly, for sufficiently large $|k_2| \geq C_2$;

$|a_2(x, k_2, k_3)| \leq \frac{C||a_3|_{|k_3| \geq C_3}||_\infty}{|k_2}$

$\leq \frac{C^2||g||_\infty}{|k_2||k_3|}$

and, for sufficiently large $|k_1| \geq C_1$;

$|a_1(k_1,k_2,k_3)| \leq \frac{C||a_2|_{|k_2|\geq C_2,|k_3|\geq C_3}||_\infty}{|k_1}$

$\leq \frac{C^2||a_3|_{|k_3|\geq C_3}||_\infty}{|k_2||k_3|}$

$\leq \frac{C^3||g||_\infty}{|k_1||k_2||k_3|}$

so that, for $|k_1| \geq C_1|$, $|k_2| \geq C_2$, $|k_3| \geq C_3$;

$||\mathcal{F}(g)(\overline{k})||_\infty \leq \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{C^3||g||_\infty}{|k_1||k_2||k_3|}$

$= \frac{C_{0,0,0}}{|k_1||k_2||k_3|}$

where $C_{0,0,0} = \frac{1}{(2\pi)^{\frac{3}{2}}}C^3||g||_\infty$

Similarly, for $|k_1|, |k_2|, |k_3|$ sufficiently large, we can find constants $C_{i_1,i_2,i_3} \in \mathcal{R}_{>0}$, for $i_1+i_2+i_3 \geq 4$, such that;

$|\mathcal{F}(\frac{\partial^{(i_1,i_2,i_3)}g}{\partial^{i_1}x\partial^{i_2}y\partial^{i_3}z})(\overline{k})| \leq \frac{C_{i_1,2,i_3}}{|k_1||k_2||k_3|}$

For the second claim, we have, for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, using repeated integration by parts, that;

$\mathcal{F}(\frac{\partial^4 g}{\partial x^4} + \frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial x^2\partial y^2} + 2\frac{\partial^4 g}{\partial x^2\partial z^2} + 2\frac{\partial^4 g}{\partial y^2\partial z^2})(\overline{k})$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_1\rightarrow\infty} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty} \int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(\frac{\partial^4 g}{\partial x^4}+\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial x^2\partial y^2}$

$+2\frac{\partial^4 g}{\partial x^2\partial z^2} + 2\frac{\partial^4 g}{\partial y^2\partial z^2})e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$= (k_1^4+k_2^4+k_3^4+2k_1^2k_2^2+2k_1^2k_3^2+2k_2^2k_3^2)\frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_1\rightarrow\infty} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty}$

$\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3} g(x,y,z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$= |\overline{k}|^4\mathcal{F}(g)(\overline{k})$

so that, using the first claim, for sufficiently large $|k|_1, |k|_2, |k|_3$;

$|\mathcal{F}(g)(\overline{k})| \leq \frac{\mathcal{F}(g)(\frac{\partial^4 g}{\partial x^4}+\frac{\partial^4 g}{\partial y^4}+\frac{\partial^4 g}{\partial z^4}+2\frac{\partial^4 g}{\partial x^2\partial y^2}+2\frac{\partial^4 g}{\partial x^2\partial z^2}+2\frac{\partial^4 g}{\partial y^2\partial z^2})(\overline{k})}{|\overline{k}|^4}$

$\leq \frac{C_{4,0,0}+C_{0,4,0}+C_{0,0,4}+2C_{2,2,0}+2C_{2,0,2}+2C_{0,2,2}}{|k_1||k_2||k_3||\overline{k}|^4}$

$= \frac{D}{|k_1||k_2||k_3||\overline{k}|^4}$

where $D = C_{4,0,0} + C_{0,4,0} + C_{0,0,4} + 2C_{2,2,0} + 2C_{2,0,2} + 2C_{0,2,2}$.

For the next claim, we have as g is of very moderate decrease, that $|g| \leq \frac{D}{|x|}$, for $|x| \geq s$, and, as g is continuous, that $|g| \leq C$, for $|\overline{x}| \leq s$. Using polar coordinates (R, θ, ϕ), we have;

$\int_{\mathcal{R}^3} |g|^4 d\overline{x}$

$= \int_{B(\overline{0},s)} |g|^4 d\overline{x} + \int_{\mathcal{R}^3 \setminus B(\overline{0},s)} |g|^4 d\overline{x}$

$\leq \frac{4C^4\pi s^3}{3} + \int_{\mathcal{R}^3 \setminus B(\overline{0},s)} \frac{D}{|x|^4} d\overline{x}$

$\frac{\leq \frac{4C^4\pi s^3}{3} + \int_s^\infty |DR^2 sin(\theta)}{R^4|dR}$

$\leq \frac{4C^4\pi s^3}{3} + D\int_s^\infty \frac{dR}{R^2}$

$= \frac{4C^4\pi s^3}{3} + \frac{D}{s}$

so that $g \in L^4(\mathcal{R}^3)$. Letting $p = 4$, and $\frac{1}{p} + \frac{1}{q} = 1$, so that $q = \frac{4}{3}$, and generalising the Haussdorff-Young inequality, see [18], we have that $\mathcal{F}(g) \in L^{\frac{4}{3}}(\mathcal{R}^3)$, and we can find $F \in \mathcal{R}_{>0}$, with;

$||\mathcal{F}(g)||_{L^{\frac{4}{3}}(\mathcal{R}^3)} \leq F||g||_{L^4(\mathcal{R}^3)}$

$\leq F(\frac{4C^4\pi s^3}{3} + \frac{D}{s})^{\frac{1}{2}}$

By Holders's inequality, we have that for $r > 0$, $\mathcal{F}(g)|_{B(\overline{0},r)} \in L^1(B(\overline{0}, r))$, and;

$||\mathcal{F}(g)(\overline{k})||_{L^1(B(\overline{0},r))}$

$\leq ||\mathcal{F}(g)(\overline{k})||_{L^{\frac{4}{3}}(B(\overline{0},r))} ||1||_{L^4(B(\overline{0},r))}$

$\leq F(\frac{4C^4\pi s^3}{3} + \frac{D}{s})^{\frac{1}{2}} (\frac{4\pi r^3}{3})^{\frac{1}{2}}$

Using the second claim, we have that there exist constants $\{E_1, E_2, E_3\} \subset \mathcal{R}_{>0}$, such that, for $|k_1| \geq E_1, |k_2| \geq E_2, |k_3| \geq E_3$;

$|\mathcal{F}(g)(\overline{k})| \leq \frac{D}{|k_1||k_2||k_3||\overline{k}|^4}$

$\leq \frac{D}{E_1E_2E_3|\overline{k}|^4}$

$= \frac{F}{|\overline{k}|^4}$

where $F = \frac{D}{E_1E_2E_3|}$. Then, using polar coordinates, $k_1' = rsin(\theta)cos(\phi)$, $k_2' = rsin(\theta)sin(\phi)$, $k_3' = rcos(\theta)$, $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$;

$\int_{k_1\geq E_1,k_2\geq E_2,k_3\geq E_3} |\mathcal{F}(g)(\overline{k})|d\overline{k}$

$= \int_{k_1'\geq 0,k_2'\geq 0,k_3'\geq 0} |\mathcal{F}(g)(k_1'+E_1,k_2'+E_2,k_3'+E_3)|d\overline{k}'$

$\leq \int_{k_1'\geq 0,k_2'\geq 0,k_3'\geq 0} \frac{F}{|(k_1'+E_1,k_2'+E_2,k_3'+E_3)|^4}d\overline{k}'$

$= \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_0^\infty \frac{F}{|(k_1'+E_1,k_2'+E_2,k_3'+E_3)|^4}r^2sin(\theta)drd\theta d\phi$

$= \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_0^\infty \frac{F}{|(rsin(\theta)cos(\phi)+E_1,rsin(\theta)sin(\phi)+E_2,rcos(\theta)+E_3)|^4}r^2sin(\theta)drd\theta d\phi$

$\leq \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_0^{r_0} \frac{F}{|(rsin(\theta)cos(\phi)+E_1,rsin(\theta)sin(\phi)+E_2,rcos(\theta)+E_3)|^4}r^2sin(\theta)drd\theta d\phi$

$+ \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_{r_0}^\infty \frac{Fr^2}{r^4}\frac{1}{|(sin(\theta)cos(\phi)+\frac{E_1}{r},sin(\theta)sin(\phi)+\frac{E_2}{r},cos(\theta)+\frac{E_3}{r})|^4}drd\theta d\phi$

$\leq \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_0^{r_0} \frac{Fr_0^2}{|(E_1,E_2,E_3)|^4}$

$+ \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_0^{r_0} \frac{F}{r^2}\frac{1}{(1+\frac{2sin(\theta)cos(\phi)E_1}{r}+\frac{2sin(\theta)sin(\phi)E_2}{r}+\frac{2cos(\theta)E_3}{r}+\frac{E_1^2}{r^2}+\frac{E_2^2}{r^2}+\frac{E_3^2}{r^2})^2}drd\theta d\phi$

$\leq r_0(\frac{\pi}{2})^2\frac{Fr_0^2}{|(E_1,E_2,E_3)|^4}$

$+ \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_{r_0}^\infty \frac{F}{r^2}\frac{1}{(1+\frac{2sin(\theta)cos(\phi)E_1}{r}+\frac{2sin(\theta)sin(\phi)E_2}{r}+\frac{2cos(\theta)E_3}{r}+\frac{E_1^2}{r^2}+\frac{E_2^2}{r^2}+\frac{E_3^2}{r^2})^2}drd\theta d\phi$

$\leq \frac{\pi^2r_0^3F}{4(E_1^2+E_2^2+E_3^2)^2} + \int_{0\leq\theta\leq\frac{\pi}{2},0\leq\phi\leq\frac{\pi}{2}} \int_{r_0}^\infty \frac{4F}{r^2}drd\theta d\phi$

$= \frac{\pi^2r_0^3F}{4(E_1^2+E_2^2+E_3^2)^2} + \frac{4F(\frac{\pi}{2})^2}{r_0}$

$= \frac{\pi^2r_0^3F}{4(E_1^2+E_2^2+E_3^2)^2} + \frac{F\pi^2}{r_0}$

for $r_0 \geq 12max(E_1,E_2,E_3)$.

Similarly, repeating the calculation for all the finitely many connected regions in $|k_1| \geq E_1, |k_2| \geq E_2, |k_3| \geq E_3$, we obtain that

$\mathcal{F}(g)|_V \in L^1(V)$, where;

$V = \{(k_1, k_2, k_3) : |k_1| \geq E_1, |k_2| \geq E_2, |k_3| \geq E_3\}$.

Using the same argument as above, for $a_2(x, k_2, k_3)$, we have, without loss of generality, that for $|k_2| \geq E_2$ and for $|k_3| \geq E_3$, there exists $D \in \mathcal{R}_{>0}$, with;

$|\mathcal{F}(\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2})(x, k_2, k_3)| \leq \frac{D||\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2}||_\infty}{|k_2||k_3|}$

We have, for $k_2 \neq 0$, $k_3 \neq 0$, that;

$\mathcal{F}(\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2})(x, k_2, k_3)$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} (\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2}) e^{-ik_2 y} e^{-ik_3 z} dydz$

$= (k_2^4 + k_3^4 + 2k_2^2 k_3^2) \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty}$

$\int_{-r_2}^{r_2} \int_{-r_3}^{r_3} g(x, y, z) e^{-ik_2 y} e^{-ik_3 z} dydz$

$= |(k_2, k_3)|^4 \mathcal{F}(g)(x, k_2, k_3)$

so that, for $x \in \mathcal{R}$, $|k_2| \geq E_2$, $|k_3| \geq E_3$;

$|\mathcal{F}(g)(x, k_2, k_3)| \leq \frac{D||\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2}||_\infty}{|k_2||k_3||(k_2, k_3)|^4}$

$\leq \frac{D||\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2}||_\infty}{C_2 C_3 |(k_2, k_3)|^4}$

$= \frac{E}{|(k_2, k_3)|^4}$

where $E = \frac{D||\frac{\partial^4 g}{\partial y^4} + \frac{\partial^4 g}{\partial z^4} + 2\frac{\partial^4 g}{\partial y^2 \partial z^2}||_\infty}{C_2 C_3}$.

As above, we have that;

$\int_{k_2 \geq E_2, k_3 \geq E_3} |\mathcal{F}(g)(x, k_2, k_3)| dk_2 dk_3$

$= \int_{k_2' \geq 0, k_3' \geq 0} |\mathcal{F}(g)(k_2' + E_2, k_3' + E_3)| dk_2' dk_3'$

$\leq \int_{k_2' \geq 0, k_3' \geq 0} \frac{E}{|k_2' + E_2, k_3' + E_3)|^4} dk_2' dk_3' < \infty$

so clearly, for $x \in \mathcal{R}$, $\mathcal{F}(g)(x,k_2,k_3) \in L^1(S)$, where $S = \{(k_2,k_3) \in \mathcal{R}^2, |k_2| \geq E_2, |k_3| \geq E_3\}$. Let;

$$\theta(x) = \int_{|k_2|\geq E_2,|k_3|\geq E_3} \mathcal{F}(g)(x,k_2,k_3)dk_2dk_3$$

As above, we have for sufficiently large x, $\theta(x)$ is non oscillatory and of very moderate decrease.

Interchanging limits, we have that;

$$\int_{V_1} \mathcal{F}(g)(k_1,k_2,k_3)dk_1dk_2dk_3$$

$$= \int_{|k_1|<E_1} \int_{|k_2|\geq E_2,|k_3|\geq E_3} \mathcal{F}(g)(k_1,k_2,k_3)dk_1dk_2dk_3$$

$$= \int_{|k_1|<E_1} \int_{|k_2|\geq E_2,|k_3|\geq E_3} (lim_{r_1\rightarrow\infty} \int_{-r_1}^{r_1} \mathcal{F}(g)(x,k_2,k_3)e^{-ik_1x}dx)dk_1dk_2dk_3$$

$$= \int_{|k_1|<E_1} lim_{r_1\rightarrow\infty} \int_{-r_1}^{r_1} (\int_{|k_2|\geq E_2,|k_3|\geq E_3} \mathcal{F}(g)(x,k_2,k_3)dk_2dk_3)e^{-ik_1x}dxdk_1$$

$$= \int_{|k_1|<E_1} (lim_{r_1\rightarrow\infty} \int_{-r_1}^{r_1} \theta(x)e^{-ik_1x}dx)dk_1$$

$$= \int_{|k_1|<E_1} \mathcal{F}_1(\theta)(k_1)dk_1$$

where F_1 is the Fourier transform for non-oscillatory functions of very moderate decrease in one variable. As above, we have that $\mathcal{F}_1(\theta) \in L^2(\mathcal{R})$, so that $\mathcal{F}_1(\theta)|_{|k_1|<E_1} \in L^1(|k_1| < E_1)$. It follows that;

$$\int_{|k_1|<E_1} \mathcal{F}_1(\theta)(k_1)dk_1 < \infty$$

and $\mathcal{F}(g)(k_1,k_2,k_3) \in L^1(V_1)$. Similarly, we can show that;

$$\mathcal{F}(g)(k_1,k_2,k_3) \in (\bigcap_{1\leq i\leq 3} L^1(V_i) \cap \bigcap_{1\leq i<j\leq 3} L^1(V_{ij}))$$

As $\mathcal{R}^3 \setminus (\bigcup_{1\leq i\leq 3} V_i \cup \bigcup_{1\leq i<j\leq 3} V_{ij}) = C_{E_1,E_2,E_3}$, where;

$$C_{E_1,E_2,E_3} = \{(k_1,k_2,k_3) \in \mathcal{R}^3 : |k_1| < E_1, |k_2| < E_2, |k_3| < E_3\}$$

and $C_{E_1,E_2,E_3} \subset B(\bar{0},r)$, where $r = max(E_1,E_2,E_3)$, we have that $\mathcal{F}(g)(k_1,k_2,k_3) \in L^1(C_{E_1,E_2,E_3})$ and $\mathcal{F}(g)(k_1,k_2,k_3) \in L^1(\mathcal{R}^3)$.

□

Definition 0.61. *Let* $f \in C^{14}(\mathcal{R}^2)$ *with* $\frac{\partial^{i_1+i_2}f}{\partial x^{i_1}\partial y^{i_2}}$ *bounded for* $0 \leq i_1 + i_2 \leq 14$. *Let* $C_n = \{(x,y) \in \mathcal{R}^2 : |x| \leq n, |y| \leq n\}$. *Then we* *define an inflexionary approximation sequence* $\{f_m : m \in \mathcal{N}\}$ *by the requirements;*

(i). $f_m \in C^{14}(\mathcal{R}^2)$

(ii). $f_m|_{C_m} = f|_{C_m}$

(iii) $f_m|_{(\mathcal{R}^2 \setminus C_{m+\frac{1}{m}})} = 0$

(iv). *For* $|x| \leq m$, *for* $0 \leq i \leq 13$;

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m)} = \frac{\partial^i f}{\partial y^i}|_{(x,m)}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m)} = \frac{\partial^i f}{\partial y^i}|_{(x,-m)}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m+\frac{1}{m})} = 0$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m-\frac{1}{m})} = 0$

(v). *For* $|x| \leq m$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,m)} > 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,m)} < 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m}} \leq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,-m)} > 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,-m)} < 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m}} \leq 0$

(vi). *For* $0 \leq |y| \leq m + \frac{1}{m}$, $0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial x^i}|_{(x,y)} = \frac{\partial^i f_m}{\partial x^i}|_{(m,y)}$, $m \leq x \leq m + \frac{1}{m}$

$\frac{\partial^i f_m}{\partial x^i}|_{(x,y)} = \frac{\partial^i f_m}{\partial x^i}|_{(-m,y)}$, $-m - \frac{1}{m} \leq x \leq -m$

$\frac{\partial^i f_m}{\partial x^i}|_{(m+\frac{1}{m},y)} = 0$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m-\frac{1}{m},y)} = 0$

(vii) *For* $m \leq |x| \leq m + \frac{1}{m}$, $0 \leq |y| \leq m + \frac{1}{m}$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(m,y)} > 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y}} \geq 0$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(m,y)} < 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y}} \leq 0$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(-m,y)} > 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y}} \geq 0$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(-m,y)} < 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y}} \leq 0$

where;

$V_{x,m} = \{(x,y) \in \mathcal{R}^2 : y \in (m, m+\frac{1}{m})\}$

$V_{x,-m} = \{(x,y) \in \mathcal{R}^2 : y \in (-m-\frac{1}{m}, -m)\}$

$H_{m,y} = \{(x,y) \in \mathcal{R}^2 : x \in (m, m+\frac{1}{m})\}$

$H_{-m,y} = \{(x,y) \in \mathcal{R}^2 : x \in (-m-\frac{1}{m}, -m)\}$

Definition 0.62. *Let* $f \in C^{14}(\mathcal{R}^3)$ *with* $\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}$ *bounded for* $0 \leq i_1 + i_2 + i_3 \leq 14$*. Let* $W_n = \{(x,y,z) \in \mathcal{R}^3 : |x| \leq n, |y| \leq n, |z| \leq n\}$*. Then we define an inflexionary approximation sequence* $\{f_m : m \in \mathcal{N}\}$ *by the requirements;*

(i). $f_m \in C^{14}(\mathcal{R}^3)$

(ii). $f_m|_{W_m} = f|_{W_m}$

(iii) $f_m|_{(\mathcal{R}^3 \setminus W_{m+\frac{1}{m}})} = 0$

(iv). *For* $0 \leq |y| \leq m, 0 \leq |z| \leq m$*, for* $0 \leq i \leq 13$*;*

$\frac{\partial^i f_m}{\partial x^i}|_{(m,y,z)} = \frac{\partial^i f}{\partial x^i}|_{(m,y,z)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m,y,z)} = \frac{\partial^i f}{\partial x^i}|_{(-m,y,z)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(m+\frac{1}{m},y,z)} = 0$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m-\frac{1}{m},y,z)} = 0$

(v). *For* $0 \leq |y| \leq m, 0 \leq |z| \leq m$

$if\ \frac{\partial^{14}f}{\partial x^{14}}|_{(m,y,z)} > 0,\ \frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{m,y,z}} \geq 0$

$if\ \frac{\partial^{14}f}{\partial y^{14}}|_{(m,y,z)} < 0,\ \frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{m,y,z}} \leq 0$

$if\ \frac{\partial^{14}f}{\partial y^{14}}|_{(-m,y,z)} > 0,\ \frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{-m,y,z}} \geq 0$

$if\ \frac{\partial^{14}f}{\partial y^{14}}|_{(-m,y,z)} < 0,\ \frac{\partial^{14}f_m}{\partial x^{14}}|_{H_{-m,y,z}} \leq 0$

(vi). *For* $0 \leq |x| \leq m + \frac{1}{m}$ $0 \leq |z| \leq m$, $0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial y^i}|_{(x,m,z)},\ m \leq y \leq m + \frac{1}{m}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial y^i}|_{(x,-m,z)},\ -m - \frac{1}{m} \leq y \leq -m$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m+\frac{1}{m},z)} = 0$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m-\frac{1}{m},z)} = 0$

(vii) *For* $0 \leq |x| \leq m + \frac{1}{m}$, $0 \leq |z| \leq m$

$if\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{(x,m,z)} > 0,\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{V_{x,m,z}} \geq 0$

$if\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{(x,m,z)} < 0,\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{V_{x,m,z}} \leq 0$

$if\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{(x,-m,z)} > 0,\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{V_{x,-m,z}} \geq 0$

$if\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{(x,-m,z)} < 0,\ \frac{\partial^{14}f_m}{\partial y^{14}}|_{V_{x,-m,z}} \leq 0$

$(viii)$. *For* $0 \leq |x| \leq m + \frac{1}{m}$ $0 \leq |y| \leq m + \frac{1}{m}$, $0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial z^i}|_{(x,y,m)},\ m \leq z \leq m + \frac{1}{m}$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m)},\ -m - \frac{1}{m} \leq z \leq -m$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m+\frac{1}{m})} = 0$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m-\frac{1}{m})} = 0$

(ix) *For* $0 \leq |x| \leq m + \frac{1}{m}$, $0 \leq |y| \leq m + \frac{1}{m}$

if $\frac{\partial^{14} f_m}{\partial z^{14}}|_{(x,y,m)} > 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,m}} \geq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,m)} < 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,m}} \leq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,-m)} > 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,-m}} \geq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,-m)} < 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,-m}} \leq 0$

where;

$H_{m,y,z} = \{(x, y, z) \in \mathcal{R}^3 : x \in (m, m + \frac{1}{m})\}$

$H_{-m,y,z} = \{(x, y, z) \in \mathcal{R}^3 : x \in (-m - \frac{1}{m}, -m)\}$

$V_{x,m,z} = \{(x, y, z) \in \mathcal{R}^3 : y \in (m, m + \frac{1}{m})\}$

$V_{x,-m,z} = \{(x, y, z) \in \mathcal{R}^3 : y \in (-m - \frac{1}{m}, -m)\}$

$D_{x,y,m} = \{(x, y, z) \in \mathcal{R}^3 : z \in (m, m + \frac{1}{m})\}$

$D_{x,y,-m} = \{(x, y, z) \in \mathcal{R}^3 : z \in (-m - \frac{1}{m}, -m)\}$

Lemma 0.63. *If $[a, b] \subset \mathcal{R}$, with a, b finite, and $\{g, g_1, g_2\} \subset C^\infty([a, b])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m}] \times [a, b])$, with the property that;*

$h(m, y) = g(y)$, $\frac{\partial h}{\partial x}|_{(m,y)} = g_1(y)$, $\frac{\partial^2 h}{\partial x^2}|_{(m,y)} = g_2(y)$, $y \in [a, b]$, *(i)*

$h(m + \frac{1}{m}, y) = \frac{\partial h}{\partial x}(m + \frac{1}{m}, y) = \frac{\partial^2 h}{\partial x^2}(m + \frac{1}{m}, y) = 0$, $y \in [a, b]$, *(ii)*

$|h|_{[m,m+\frac{1}{m}]\times[a,b]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^3 h}{\partial x^3}(m, y) > 0$, $\frac{\partial^3 h}{\partial x^3}(x, y) > 0$, for $x \in [m, m + \frac{1}{m}]$, and if $\frac{\partial^3 h}{\partial x^3}(m, y) < 0$, $\frac{\partial^3 h}{\partial x^3}(x, y) < 0$, for $x \in [m, m + \frac{1}{m}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^3 h}{\partial x^3}|_{(x,y)}|dx = |g_2(y)|$

Moreover, for $i \in \mathcal{N}$, $\frac{\partial^i h}{\partial y^i}$ has the property that;

$\frac{\partial^i h}{\partial y^i}(m,y) = g^{(i)}(y),\ \frac{\partial^{i+1} h}{\partial y^i \partial x}|_{(m,y)} = g_1^{(i)}(y),\ \frac{\partial^{i+2} h}{\partial y^i \partial x^2}|_{(m,y)} = g_2^{(i)}(y)$

$y \in [a,b]$, $(i)'$

$\frac{\partial^i h}{\partial y^i}(m+\frac{1}{m},y) = \frac{\partial^{i+1} h}{\partial y^i \partial x}(m+\frac{1}{m},y) = \frac{\partial^{i+2} h}{\partial y^i \partial x^2}(m+\frac{1}{m},y) = 0$

$y \in [a,b]$, $(ii)'$

$|\frac{\partial^i h}{\partial y^i}|_{[m,m+\frac{1}{m}]\times[a,b]}| \leq C_i$

for some $C_i \in \mathcal{R}_{>0}$, *independent of* m *sufficiently large, and, if* $\frac{\partial^{i+3} h}{\partial y^i \partial x^3}(m,y) > 0$, $\frac{\partial^{i+3} h}{\partial y^i \partial x^3}(x,y) > 0$, *for* $x \in [m, m+\frac{1}{m}]$, *and if* $\frac{\partial^{i+3} h}{\partial y^i \partial x^3}(m,y) < 0$, $\frac{\partial^{i+3} h}{\partial y^i \partial x^3}(x,y) < 0$, *for* $x \in [m, m+\frac{1}{m}]$, $(**)$. *In particularly;*

$\int_m^{m+\frac{1}{m}} |\frac{\partial^{i+3} h}{\partial y^i \partial x^3}|_{(x,y)}|dx = |g_2^{(i)}(y)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.30, replacing the constant coefficients $\{a_0, a_1, a_2\} \subset \mathcal{R}$ with the data $\{g(y), g_1(y), g_2(y)\}$. The properties $(i), (ii)$ are then clear. Noting that $[a,b]$ is a finite interval and $\{g, g_1, g_2\} \subset C^\infty([a,b])$, by continuity, there exists a constant D, with $max(|g(y)|, |g_1(y)|, |g_2(y)| : y \in [a,b]) \leq D$, so, as in the proof of Lemma 0.30, we can use the bound $C = 16D + 7D + D = 24D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.30, for sufficiently large m, again using the fact that the data $\{g(y), g_1(y), g_2(y) : y \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^i h}{\partial y^i}$, for $i \in \mathcal{N}$, we are just differentiating the coefficients which are linear in the data $\{g(y), g_1(y), g_2(y)\}$, so we obtain a function which fits the data $\{g^{(i)}(y), g_1^{(i)}(y), g_2^{(i)}(y)\}$ and $(i)', (ii)'$ follow. Noting that, for $i \in \mathcal{N}$, $\{g^{(i)}, g_1^{(i)}, g_2^{(i)}\} \subset C^\infty([a,b])$, again by continuity, there exists constants D_i, with $max(|g^{(i)}(y)|, |g_1^{(i)}(y)|, |g_2^{(i)}(y)| : y \in [a,b]) \leq D_i$, so, again, as in the proof of Lemma 0.30, we can use the bound $C_i = 16D_i + 7D_i + D_i = 24D_i$, for $m > 1$. The proof of $(**)$ follows uniformly in y, for each $i \in \mathcal{N}$, as in the proof of Lemma 0.30, for sufficiently large m, again using the fact that the data $\{g^{(i)}(y), g_1^{(i)}(y), g_2^{(i)}(y) : y \in [a,b]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.64. *Conjecture*

Fix $n \in \mathcal{N}$, *with* $n \geq 3$. *If* $m \in \mathcal{R}_{>0}$ *is sufficiently large,* $\{a_i : 0 \leq i \leq n-1\} \subset \mathcal{R}$, *there exists* $h \in \mathcal{R}[x]$ *of degree* $2n-1$, *with the property* *that;*

$h^{(i)}(m) = a_i,\ 0 \leq i \leq n-1\ (i)$

$h^{(i)}(m+\frac{1}{m}) = 0,\ 0 \leq i \leq n-1\ (ii)$

$|h|_{[m,m+\frac{1}{m}]}| \leq C$

for some $C \in \mathcal{R}_{>0}$*, independent of* m *sufficiently large, and, if* $h^{(n)}(m) > 0$, $h^{(n)}(x)|_{[m,m+\frac{1}{m}]} > 0$*, if* $h^{(n)}(m) < 0$, $h^{(n)}|_{[m,m+\frac{1}{m}]} < 0$*. In particularly;*

$\int_m^{m+\frac{1}{m}} |h^{(n)}(x)|dx = |a_{n-1}|$, ([18])

Proof. We sketch a proof based on the special case $n = 3$, which was shown in Lemma 0.30, leaving the details to the reader, ([19]). We have that $h(x) = (x-(m+\frac{1}{m}))^n p(x)$ where $p(x)$ is a polynomial satisfies condition (ii). Computing the derivatives $h^{(i)}(m)$, for $0 \leq i \leq n-1$, we obtain n linear equations involving the unknowns $p^{(i)}(m)$, $0 \leq i \leq n-1$, of the form;

$\sum_{k=0}^{i} \frac{d_{ik}p^{(k)}(m)}{m^{n-i+k}} = a_i,\ (0 \leq i \leq n-1)\ (*)$

---

[18] If $a_0 > 0$, $a_1 > 0$, there does not exist a smooth function h on the interval $(m, m+\frac{1}{m})$, with $h(m) = a_0$, $h'(m) = a_1$, $h(m+\frac{1}{m}) = 0$, $h'(m+\frac{1}{m}) = 0$, such that $h'' > 0$ or $h'' < 0$. To see this, if $h'' > 0$, using the MVT, we have that $h'(x) > h'(m) > 0$, for $x \in (m, m+\frac{1}{m})$, contradicting the fact that $h'(m+\frac{1}{m}) = 0$. If $h'' < 0$, and $h'(x)$ has no roots in the interval $(m, m+\frac{1}{m})$, then as $h'(m) > 0$, $h'(x) > 0$ on $(m, m+\frac{1}{m})$, and h is increasing on $(m, m+\frac{1}{m})$, so that $h(m+\frac{1}{m}) > h(m) = a_0 > 0$, contradicting the fact that $h(m+\frac{1}{m}) = 0$. Otherwise, if $h'(x)$ has a root in the interval $(m, m+\frac{1}{m})$, as $h'' < 0$, it attains a maximum at $x_0 \in (m, m+\frac{1}{m})$. Using the MVT again, we must have that for $y \in (x_0, m+\frac{1}{m})$, $h'(y) < h'(x_0) = 0$, so that $h'(m+\frac{1}{m}) < 0$, contradicting the fact that $h'(m+\frac{1}{m}) = 0$.

[19] One step requires the verification that for a computable polynomial r_n of degree $n-1$, $r_n(1) \neq 0$, which is highly unlikely on generic grounds and the fact that $r_3(1) \neq 1$, although $r_2(1) = 1$, see footnote 18. The geometric idea is that allowing for inflexionary type curves, where we can have points $x_{0,i} \in (m, m+\frac{1}{m})$ for which $h^{(i)}(x_{0,i}) = 0$, where $2 \leq i \leq n-1$, the end conditions can be satisfied while still having $h^{(n)}|_{(m,m+\frac{1}{m})} > 0$ or $h^{(n)}|_{(m,m+\frac{1}{m})} < 0$. However, you still need to do a concrete calculation, which in the case of verifying the conjecture for all $n \in \mathcal{N}$, $n \geq 3$, would involve finding the exact pattern in the coefficients obtained in the proof of Lemma 0.30. We actually only need the result for some $n \geq 14$ in the rest of this paper.

which we can solve for $p^{(i)}(m)$, $0 \leq i \leq n-1$, using the fact that the matrix $(d_{ik})_{0\leq i\leq n-1, 0\leq k\leq i}$ is lower triangular and $|d_{ii}| = 1$, for $0 \leq i \leq n-1$. Then we can take;

$p(x) = \sum_{i=0}^{n-1} p^{(i)}(m)(x-m)^i$

so that h has degree $n+(n-1) = 2n-1$. It is clear from $(*)$, that we have;

$p^{(i)}(m) = \sum_{k=0}^{i} c_{ik} a_{i-k} m^{n+k}$, $(0 \leq i \leq n-1)$

where $(c_{ik})_{0\leq i\leq n-1, 0\leq k\leq i}$ is a real matrix, so that $p(x)$ has the form;

$p(x) = \sum_{i=0}^{n-1} v_i x^i$ $(**)$

where;

$v_{n-1-i} = \sum_{k=0}^{n-1} r_{ik} m^{n+k} + \sum_{l=0}^{i} s_{il} m^{2n-1+l}$, $(0 \leq i \leq n-1)$

for real matrices $(r_{ik})_{0\leq i\leq n-1, 0\leq k\leq n-1}$ and $(s_{il})_{0\leq i\leq n-1, 0\leq l\leq i}$.

It is then clear, using the product rule and $(**)$, that;

$h^{(n)}(x) = \sum_{k=0}^{n-1} w_k x^k$

where $w_k = z_k a_0 m^{3n-2-k} + O(m^{3n-3-k})$, $(0 \leq k \leq n-1)$

By homogeneity, it is then clear that the real roots of $h^{(n)}(x)$ are of the form $t_{s_0} m + O(1)$, where $t_{s_0} \in \mathcal{R}$, $1 \leq s_0 \leq n-1$, and t_{s_0} satisfies a polynomial $r(x)$ of degree $n-1$, which is effectively computable for given n. We can exclude any roots in the interval $[m, m+\frac{1}{m}]$, for sufficiently large m, provided $t_{s_0} \neq 1$, for $1 \leq s_0 \leq n-1$, which we can check by showing that $r(1) \neq 0$. We have that;

$|h|_{(m,m+\frac{1}{m})}| = |(x-(m+\frac{1}{m}))^n p(x)|$

$\leq \frac{1}{m^n} |\sum_{i=0}^{n-1} p^{(i)}(m)(x-m)^i|$

$\leq \frac{1}{m^n} \sum_{i=0}^{n-1} \frac{|p^{(i)}(m)|}{m^i}$

$\leq \sum_{i=0}^{n-1}\sum_{k=0}^{i}|c_{ik}|a_{i-k}|\frac{m^{n+k}}{m^{n+i}}$

$\leq \sum_{i=0}^{n-1}\sum_{k=0}^{i}|c_{ik}|a_{i-k}| = C$, $(m > 1)$

The last claim is just the FTC.

□

Lemma 0.65. *If $[a,b] \subset \mathcal{R}$, with a,b finite, $n \geq 3$, and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m}] \times [a,b])$, with the property that;*

$\frac{\partial^{(j)}h}{\partial x^j}|_{(m,y)} = g_j(y)$, $y \in [a,b]$, (i)

$\frac{\partial h^j}{\partial x^j}(m+\frac{1}{m}, y) = 0$, $y \in [a,b]$, (ii)

$|h|_{[m,m+\frac{1}{m}]\times[a,b]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^n h}{\partial x^n}(m,y) > 0$, $\frac{\partial^n h}{\partial x^n}(x,y) > 0$, for $x \in [m, m+\frac{1}{m}]$, and if $\frac{\partial^n h}{\partial x^n}(m,y) < 0$, $\frac{\partial^n h}{\partial x^n}(x,y) < 0$, for $x \in [m, m+\frac{1}{m}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^n h}{\partial x^n}|_{(x,y)}|dx = |g_{n-1}(y)|$

Moreover, for $i \in \mathcal{N}$, $\frac{\partial^i h}{\partial y^i}$ has the property that;

$\frac{\partial^{i+j}h}{\partial x^j \partial y^i}(m,y) = g_j^{(i)}(y)$, $y \in [a,b]$, $(i)'$

$\frac{\partial^{i+j}h}{\partial x^j \partial y^i}(m+\frac{1}{m}, y) = 0$, $y \in [a,b]$, $(ii)'$

$|\frac{\partial^i h}{\partial y^i}|_{[m,m+\frac{1}{m}]\times[a,b]}| \leq C_i$

for some $C_i \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(m,y) > 0$, $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(x,y) > 0$, for $x \in [m, m+\frac{1}{m}]$, and if $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(m,y) < 0$, $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(x,y) < 0$, for $x \in [m, m+\frac{1}{m}]$, $(**)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^{i+n}h}{\partial y^i \partial x^n}|_{(x,y)}|dx = |g_{n-1}^{(i)}(y)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.64, replacing the constant coefficients $\{a_j : 0 \leq j \leq n-1\} \subset \mathcal{R}$ with the data $\{g_j(y) : 0 \leq j \leq n-1\}$. The properties

$(i),(ii)$ are then clear. Noting that $[a,b]$ is a finite interval and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, by continuity, there exists a constant D, with $max(|g_j(y)| : 0 \leq j \leq n-1, y \in [a,b]) \leq D$, so, as in the proof of Lemma 0.30, we can use the bound $C = \sum_{0 \leq j \leq n-1} L_j D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.30, for sufficiently large m, again using the fact that the data $\{g_j(y) : 0 \leq j \leq n-1, y \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^i h}{\partial y^i}$, for $i \in \mathcal{N}$, we are just differentiating the coefficients which are linear in the data $\{g_j(y) : 0 \leq j \leq n-1\}$, so we obtain a function which fits the data $\{g_j^{(i)}(y) : 0 \leq j \leq n-1\}$ and $(i)', (ii)'$ follow. Noting that, for $i \in \mathcal{N}$, $\{g_j^{(i)} : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, again by continuity, there exist constants D_i, with $max(|g_j^{(i)}(y)| : 0 \leq j \leq n-1, y \in [a,b]) \leq D_i$, so, again, as in the proof of Lemma 0.30, we can use the bound $C_i = \sum_{0 \leq j \leq n-1} L_j D_i$, for $m > 1$. The proof of $(**)$ follows uniformly in y, for each $i \in \mathcal{N}$, as in the proof of Lemma 0.30, for sufficiently large m, again using the fact that the data $\{g_j^{(i)}(y) : 0 \leq j \leq n-1, y \in [a,b]\}$ is bounded. The last claim is again just the FTC. $\square$

Lemma 0.66. *If $[a,b] \subset \mathcal{R}$, $[c,d] \subset \mathcal{R}$, with a,b,c,d finite, $n \geq 3$, and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b] \times [c,d])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m}] \times [a,b] \times [c,d])$, with the property that;*

$\frac{\partial^{(j)} h}{\partial x^j}|_{(m,y,z)} = g_j(y,z)$, $(y,z) \in [a,b] \times [c,d]$, (i)

$\frac{\partial h^j}{\partial x^j}(m+\frac{1}{m}, y, z) = 0$, $(y,z) \in [a,b] \times [c,d]$, (ii)

$|h|_{[m,m+\frac{1}{m}] \times [a,b] \times [c,d]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^n h}{\partial x^n}(m,y,z) > 0$, $\frac{\partial^n h}{\partial x^n}(x,y,z) > 0$, for $x \in [m, m+\frac{1}{m}]$, and if $\frac{\partial^n h}{\partial x^n}(m,y,z) < 0$, $\frac{\partial^n h}{\partial x^n}(x,y,z) < 0$, for $x \in [m, m+\frac{1}{m}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m}} |\frac{\partial^n h}{\partial x^n}|_{(x,y,z)}| dx = |g_{n-1}(y,z)|$

Moreover, for $(i,k) \subset \mathcal{N}^2$, $0 \leq j \leq n-1$, $\frac{\partial^{i+k} h}{\partial y^i \partial z^k}$, has the property that;

$\frac{\partial^{i+j+k} h}{\partial x^j \partial y^i \partial z^k}(m,y,z) = \frac{\partial^{i+k} g_j}{\partial y^i \partial z^k}(y,z)$, $(y,z) \in [a,b] \times [c,d]$, $(i)'$

$\frac{\partial^{i+j+k}h}{\partial x^j \partial y^i \partial z^k}(m+\frac{1}{m}, y, z) = 0$, $(y,z) \in [a,b] \times [c,d]$, $(ii)'$

$|\frac{\partial^{i+k}h}{\partial y^i \partial z^k}|_{[m,m+\frac{1}{m}]\times[a,b]\times[c,d]}| \leq C_{i,k}$

for some $C_{i,k} \in \mathcal{R}_{>0}$*, independent of* m *sufficiently large, and, if* $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(m,y,z) > 0$, $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(x,y,z) > 0$*, for* $x \in [m, m+\frac{1}{m}]$*, and if* $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(m,y) < 0$, $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(x,y,z) < 0$*, for* $x \in [m, m+\frac{1}{m}]$*,* $(**)$*. In particularly;*

$\int_m^{m+\frac{1}{m}} |\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}|_{(x,y,z)}|dx = |\frac{\partial^{i+k}g_{n-1}}{\partial y^i \partial z^k}(y,z)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.64, replacing the constant coefficients $\{a_j : 0 \leq j \leq n-1\} \subset \mathcal{R}$ with the data $\{g_j(y,z) : 0 \leq j \leq n-1\}$. The properties $(i), (ii)$ are then clear. Noting that $[a,b] \times [c,d]$ is compact and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b] \times [c,d])$, by continuity, there exists a constant D, with $max(|g_j(y,z)| : 0 \leq j \leq n-1, (y,z) \in [a,b] \times [c,d]) \leq D$, so, as in the proof of Lemma 0.64, we can use the bound $C = \sum_{0 \leq j \leq n-1} L_j D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.64, for sufficiently large m, again using the fact that the data $\{g_j(y,z) : 0 \leq j \leq n-1, (y,z) \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^{i+k}h}{\partial y^i \partial z^k}$, for $(i, j \in \mathcal{N}^2$, we are just differentiating the coefficients which are linear in the data $\{g_j(y,z) : 0 \leq j \leq n-1\}$, so we obtain a function which fits the data $\{\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k}(y,z) : 0 \leq j \leq n-1\}$ and $(i)', (ii)'$ follow. Noting that, for $(i,k) \in \mathcal{N}^2$, $\{\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k} : 0 \leq j \leq n-1\} \subset C^\infty([a,b] \times [c,d])$, again by continuity, there exist constants $D_{i,k}$, with $max(|\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k}(y,z)| : 0 \leq j \leq n-1, y \in [a,b] \times [c,d]) \leq D_{i,k}$, so, again, as in the proof of Lemma 0.64, we can use the bound $C_{i,k} = \sum_{0 \leq j \leq n-1} L_j D_{i,k}$, for $m > 1$. The proof of $(**)$ follows uniformly in (y,z), for each $(i,k) \in \mathcal{N}^2$, as in the proof of Lemma 0.64, for sufficiently large m, again using the fact that the data $\{\frac{\partial^{i+k}g_j}{\partial Y^i \partial z^k}(y) : 0 \leq j \leq n-1, (y,z) \in [a,b] \times [c,d]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.67. *For* $f \in C^{27}(\mathcal{R}^2)$ *with* $\frac{\partial^{i_1+i_2}f}{\partial x^{i_1} \partial y^{i_2}}$ *bounded by some constant* $F \in \mathcal{R}_{>0}$*, for* $0 \leq i_1 + i_2 \leq 27$*. Then for sufficiently large* m*, there exists an inflexionary approximation sequence* $\{f_m : m \in \mathcal{N}\}$*, with the property that;*

$max(\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial x^{14}}|dxdy, \int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial y^{14}}|dxdy) \leq Gm^2$

for some $G \in \mathcal{R}_{>0}$, for sufficiently large m.

Proof. Define $f_m = f$ on C_m, so that (ii) of Definition 0.61 is satisfied. Using two applications of Lemma 0.65 with $n = 14$, changing to a vertical rather than horizontal orientation, and the fact that, for $0 \leq i \leq 13$, $|x| \leq m$, $\frac{\partial^i f}{\partial y^i}|_{(x,m)}$ and $\frac{\partial^i f}{\partial y^i}|_{(x,-m)}$ define smooth functions on $[-m, m]$, we can extend f_m to $R = \{(x, y) : |x| \leq m, m \leq |y| \leq m+\frac{1}{m}\}$, such that $f_m|R_1$ satisfies conditions $(iv), (v)$ of Definition 0.61, where $R_1 = \{(x, y) : |x| \leq m, 0 \leq |y| \leq m+\frac{1}{m}\}$. Again, using two applications of Lemma 0.65 with $n = 14$, and the original horizontal orientation, and the fact that, for $0 \leq i \leq 13$, $0 \leq |y| \leq m + \frac{1}{m}$, $\frac{\partial^i f_m}{\partial x^i}|_{(m,y)}$ and $\frac{\partial^i f}{\partial x^i}|_{(-m,y)}$ define smooth functions on $[-m - \frac{1}{m}, m + \frac{1}{m}]$, we can extend f_m to $S = \{(x, y) : m \leq |x| \leq m + \frac{1}{m}, 0 \leq |y| \leq m + \frac{1}{m}\}$, such that $f_m|C_{m+\frac{1}{m}}$ satisfies conditions $(vi), (vii)$ of Definition 0.61. Conditions $(i), (iii)$ are then clear. We then have, using (iii), that;

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial x^{14}}|dxdy = \int_{C_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial x^{14}}|dxdy$

$= \int_{|x|\leq m,|y|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdy + \int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdy + \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdy$

$+ \int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdy$

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial y^{14}}|dxdy = \int_{C_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m,|y|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdy + \int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdy + \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdy$

$+ \int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdy \ (*)$

We then have the following cases, using the second clause in Lemma 0.65 repeatedly with the appropriate orientations;

Case 1;

$\int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14} f_m}{\partial x^{14}}|dxdy$

$= \int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14} f}{\partial x^{14}}|dxdy \leq Fm^2$

$\int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdy$

$=\int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14}f}{\partial y^{14}}|dxdy\leq Fm^2$

Case 2;

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdy$

$=\int_{|x|\leq m}(\int_{|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dy)dx$

$\leq\frac{2}{m}\int_{|x|\leq m}C_{14}dx$

$\leq 2m\frac{2}{m}C_{14}$

$=4C_{14}$

Case 3;

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdy$

$=\int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dy$

$=\int_{|y|\leq m}(|\frac{\partial^{13}f}{\partial x^{13}}|_{(m,y)}+|\frac{\partial^{13}f}{\partial x^{13}}|_{(-m,y)})dy$

$\leq 4mF$

Case 4.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdy$

$=\int_{m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dy$

$=\int_{m\leq|y|\leq m+\frac{1}{m}}(|\frac{\partial^{13}f_m}{\partial x^{13}}|_{(m,y)}+|\frac{\partial^{13}f_m}{\partial x^{13}}|_{(-m,y)}dy$

$\leq\int_{m\leq y\leq m+\frac{1}{m}}C_{13,1}dy+\int_{-m-\frac{1}{m}\leq -m}C_{13,2}dy$

$\leq\frac{max(C_{13,1},C_{13,2})}{m}$ (the constants $\{C_{13,1},C_{13,2}\}$ coming from the two applications of Lemma 0.65 at the two boundaries)

Case 5;

$\int_{|x|\leq m, m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dy)dx$

$= \int_{|x|\leq m}(|\frac{\partial f}{\partial y^{13}}|_{(x,m)} + |\frac{\partial f(x,y)}{\partial y^{13}}|_{(x,-m)}dx)$

$\leq 4mF$

Case 6;

$\int_{|y|\leq m, m\leq|x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dx)dy$

$\leq \frac{1}{m}\int_{|y|\leq m}(|\sum_{i=0}^{13} D_i|\frac{\partial^i\partial^{14} f}{\partial y^{14}\partial x^i}|(m,y) + |\sum_{i=0}^{13} D_i|\frac{\partial^i\partial^{14} f}{\partial y^{14}\partial x^i}|(-m,y))dy$

$\leq \frac{2}{m}(2m)F(\sum_{i=0}^{13} D_i)$

$= 4F(\sum_{i=0}^{13} D_i)$

Case 7.

$\int_{m\leq|x|\leq m+\frac{1}{m}, m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|x|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dx)dy$

$\leq \frac{1}{m}\int_{m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14} f_m}{\partial x^i\partial y^{14}}|_{(m,y)} + L_{i,14}|\frac{\partial^{i+14} f_m}{\partial x^i\partial y^{14}}|_{(-m,y)})dy$

$= \frac{1}{m}\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+13} f}{\partial x^i\partial y^{13}}|_{(m,m)}|+|\frac{\partial^{i+13} f}{\partial x^i\partial y^{13}}|_{(m,-m)}|+|\frac{\partial^{i+13} f}{\partial x^i\partial y^{13}}|_{(-m,m)}|+|\frac{\partial^{i+13} f}{\partial x^i\partial y^{13}}|_{(-m,-m)}|)$

$\leq \frac{4F(\sum_{i=0}^{13} L_{i,14})}{m}$ (the constants $L_{i,14}$,$0\leq i\leq 13$ coming from the proof of Lemma 0.65)

Combining the seven cases and $(*)$, we obtain, for sufficiently large m, that;

$\int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial x^{14}}|dxdy \leq Fm^2 + 4C_{14} + 4mF + \frac{max(C_{13,1},C_{13,2})}{m} \leq Gm^2$

$\int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial y^{14}}|dxdy \leq Fm^2 + 4mF + 4F(\sum_{i=0}^{13} D_i) + \frac{4F(\sum_{i=0}^{13} L_{i,14})}{m} \leq Gm^2$

□

Lemma 0.68. *For $f \in C^{40}(\mathcal{R}^3)$ with $\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}$ bounded by some constant $F \in \mathcal{R}_{>0}$, for $0 \leq i_1+i_2+i_3 \leq 40$. Then for sufficiently large m, there exists an inflexionary approximation sequence $\{f_m : m \in \mathcal{N}\}$, with the property that;*

$$max(\int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial x^{14}}|dxdydz, \int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial y^{14}}|dxdydz, \int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial z^{14}}|dxdydz) \leq Gm^3$$

for some $G \in \mathcal{R}_{>0}$, for sufficiently large m.

Proof. Define $f_m = f$ on W_m, so that (ii) of Definition 0.62 is satisfied. Using two applications of Lemma 0.66 with $n = 14$, with a horizontal orientation, and the fact that, for $0 \leq i \leq 13$, $0 \leq |y| \leq m$, $0 \leq |z| \leq m$ $\frac{\partial^i f}{\partial x^i}|_{(m,y,z)}$ and $\frac{\partial^i f}{\partial x^i}|_{(-m,y,z)}$ define smooth functions on $[-m,m]^2$, we can extend f_m to $A_1 = \{(x,y,z) : m \leq |x| \leq m+\frac{1}{m}, 0 \leq |y| \leq m, 0 \leq |z| \leq m\}$, such that $f_m|A_2$ satisfies conditions $(iv),(v)$ of Definition 0.62, where $A_2 = \{(x,y,z) : 0 \leq |x| \leq m+\frac{1}{m}, 0 \leq |y| \leq m, 0 \leq |z| \leq m\}$. Again, using two applications of Lemma 0.66 with $n = 14$ again, this time with a vertical orientation, and the fact that, for $0 \leq i \leq 13$, $0 \leq |x| \leq m+\frac{1}{m}$, $0 \leq |z| \leq m$, $\frac{\partial^i f_m}{\partial y^i}|_{(x,m,z)}$ and $\frac{\partial^i f_m}{\partial y^i}|_{(x,-m,z)}$ define smooth functions on $[-m-\frac{1}{m}, m+\frac{1}{m}] \times [-m,m]$, we can extend f_m to $A_3 = \{(x,y,z) : 0 \leq |x| \leq m+\frac{1}{m}, m \leq |y| \leq m+\frac{1}{m}, 0 \leq |z| \leq m\}$, such that $f_m|A_4$ satisfies conditions $(vi),(vii)$ of Definition 0.62, where $A_4 = \{(x,y,z) : 0 \leq |x| \leq m+\frac{1}{m}, 0 \leq |y| \leq m+\frac{1}{m}, 0 \leq |z| \leq m\}$. Again, using two applications of Lemma 0.66 with $n = 14$ again, this time with a lateral orientation, and the fact that, for $0 \leq i \leq 13$, $0 \leq |x| \leq m+\frac{1}{m}$, $0 \leq |y| \leq m+\frac{1}{m}$, $\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m)}$ and $\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m)}$ define smooth functions on $[-m-\frac{1}{m}, m+\frac{1}{m}]^2$, we can extend f_m to $W_{m+\frac{1}{m}}$ such that $f_m|_{W_{m+\frac{1}{m}}}$ satisfies conditions $(viii),(ix)$ of Definition 0.62.

Conditions $(i),(iii)$ are then clear. We then have, using (iii), that;

(a). $\int_{\mathcal{R}^3} |\frac{\partial f_m}{\partial x^{14}}|dxdydz = \int_{W_{m+\frac{1}{m}}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$= \int_{|x|\leq m,|y|\leq m,|z|\leq m} |\frac{\partial f_m}{\partial x^{14}}|dxdydz + \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+ \int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m} |\frac{\partial f_m}{\partial x^{14}}|dxdydz + \int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},|z|\leq m} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+ \int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz + \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+ \int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz + \int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

(b). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz=\int_{W_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

(c). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz=\int_{W_{m+\frac{1}{m}}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$(*)$

We then have the following cases, using the second clause in Lemma 0.66 repeatedly with the appropriate orientations;

Case 1;

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14} f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14} f}{\partial x^{14}}|dxdydz\leq Fm^3$

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14} f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14} f}{\partial y^{14}}|dxdydz\leq Fm^3$

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14} f}{\partial z^{14}}|dxdydz\leq Fm^3$

Case 2;

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dydz$

$=\int_{|y|\leq m}(|\frac{\partial^{13}f}{\partial x^{13}}|_{(m,y,z)}+|\frac{\partial^{13}f}{\partial x^{13}}|_{(-m,y,z)})dydz$

$\leq 2(2m)^2F$

$=8m^2F$

Case 3;

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dx)dydz$

$\leq\frac{1}{m}\int_{|y|\leq m,|z|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(m,y,z)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(-m,y,z))dydz$

$\leq\frac{2}{m}(2m)^2F(\sum_{i=0}^{13}D_i)$

$=8mF(\sum_{i=0}^{13}D_i)$

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dx)dydz$

$\leq\frac{1}{m}\int_{|y|\leq m,|z|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial z^{14}\partial x^i}|(m,y,z)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial z^{14}\partial x^i}|(-m,y,z))dydz$

$\leq\frac{2}{m}(2m)^2F(\sum_{i=0}^{13}D_i)$

$=8mF(\sum_{i=0}^{13}D_i)$

Case 4.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|z|\leq m}(\int_{|y|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dy)dxdz$

$\leq\frac{2}{m}\int_{|x|\leq m,|z|\leq m}C_{14}dx$

$= (2m)^2 \frac{2}{m} C_{14,0}$

$= 8mC_{14,0}$

$\int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial z^{14}}| dxdydz$

$= \int_{|x|\leq m, |z|\leq m} (\int_{|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial z^{14}}| dy) dxdz$

$\leq \frac{2}{m} \int_{|x|\leq m, |z|\leq m} C_{0,14} dx$

$= (2m)^2 \frac{2}{m} C_{0,14}$

$= 8mC_{0,14}$

Case 5.

$\int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial y^{14}}| dxdydz$

$= \int_{|x|\leq m, |z|\leq m} (\int_{m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}| dy) dxdz$

$= \int_{|x|\leq m, |z|\leq m} (|\frac{\partial f}{\partial y^{13}}|_{(x,m,z)} + |\frac{\partial f}{\partial y^{13}}|_{(x,-m,z)}) dxdz)$

$\leq 2(2m)^2 F$

$= 8m^2 F$

Case 6.

$\int_{m\leq |x|\leq m+\frac{1}{m}, m\leq |y|\leq m+\frac{1}{m}, |z|\leq m} |\frac{\partial^{14} f_m}{\partial x^{14}}| dxdydz$

$= \int_{m\leq |x|\leq m+\frac{1}{m}, |z|\leq m} (\int_{m\leq |y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial x^{14}}| dy) dxdz$

$\leq \frac{1}{m} \int_{m\leq |x|\leq m+\frac{1}{m}, |z|\leq m} (\sum_{i=0}^{13} L_{i,14} |\frac{\partial^{i+14} \partial^{14} f_m}{\partial y^i \partial x^{14}}|_{(x,m,z)} + L_{i,14} |\frac{\partial^{i+14} \partial^{14} f_m}{\partial y^i \partial x^{14}}|_{(x,-m,z)}) dxdz$

$= \frac{1}{m} \int_{|z|\leq m} (\sum_{i=0}^{13} L_{i,14} (|\frac{\partial^{i+13} \partial^{14} f}{\partial y^i \partial x^{13}}|_{(m,m,z)}| + |\frac{\partial^{i+13} \partial^{14} f}{\partial y^i \partial x^{13}}|_{(m,-m,z)}| + |\frac{\partial^{i+13} \partial^{14} f}{\partial y^i \partial x^{13}}|_{(-m,m,z)}|$

$+ |\frac{\partial^{i+13} \partial^{14} f}{\partial y^i \partial x^{13}}|_{(-m,-m,z)}|)) dz$

$\leq (2m) \frac{4F(\sum_{i=0}^{13} L_{i,14})}{m}$

$= 8F(\sum_{i=0}^{13} L_{i,14})$

(the constants $L_{i,14}$,$0 \leq i \leq 13$ coming from the proof of Lemma 0.65)

Case 7.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},|z|\leq m} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dy)dxdz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|z|\leq m}(|\frac{\partial^{13} f_m}{\partial y^{13}}|_{(x,m,z)} + |\frac{\partial^{13} f_m}{\partial y^{13}}|_{(x,-m,z)})dxdz$

$\leq \int_{m\leq x\leq m+\frac{1}{m},|z|\leq m} C_{13,1}dxdz + \int_{-m-\frac{1}{m}\leq -m,|z|\leq m} C_{13,2}dxdz$

$\leq (2m)\frac{max(C_{13,1},C_{13,2})}{m}$

$= 2max(C_{13,1}, C_{13,2})$

(the constants $\{C_{13,1}, C_{13,2}\}$ coming from the two applications of Lemma 0.65 at the two boundaries)

Case 8.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},|z|\leq m} |\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dy)dxdz$

$\leq \frac{1}{m} \int_{m\leq|x|\leq m+\frac{1}{m},|z|\leq m}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14} f_m}{\partial y^i \partial z^{14}}|_{(x,m,z)}+L_{i,14}|\frac{\partial^{i+14} f_m}{\partial y^i \partial z^{14}}|_{(x,-m,z)})dxdz$

$\leq \frac{1}{m^2} \int_{|z|\leq m}(\sum_{i=0}^{13} \sum_{j=0}^{13} L_{i,14}L_{j,i,14}(|\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(m,m,z)}|+|\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(m,-m,z)}|$

$+|\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(-m,m,z)}| + |\frac{\partial^{i+j+14} f}{\partial x^j \partial y^i \partial z^{14}}|_{(-m,-m,z)}|))dz$

$\leq (2m)\frac{4F(\sum_{i=0}^{13} \sum_{j=0}^{13} L_{i,14}L_{j,i,14})}{m^2}$

$= \frac{8F}{m}(\sum_{i=0}^{13} \sum_{j=0}^{13} L_{i,14}L_{j,i,14})$

(the constants $L_{i,14}, L_{j,i,14}$,$0 \leq i \leq 13, 0 \leq j \leq 13$ coming from two applications of the proof of Lemma 0.66)

Case 9.

$\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{2}{m}\int_{|x|\leq m,|y|\leq m}(E_{14,0})$

$=(2m)^2\frac{2}{m}E_{14,0}$

$=8mE_{14,0}$

$\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{2}{m}\int_{|x|\leq m,|y|\leq m}(E_{0,14})$

$=(2m)^2\frac{2}{m}E_{0,14}$

$=8mE_{0,14}$

(the constants $E_{0,14}$, $E_{14,0}$ coming from an application of Lemma 0.66 with a different orientation)

Case 10.

$\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial f_m}{\partial z^{14}}|dz)dxdy$

$=\int_{|x|\leq m,|y|\leq m}(|\frac{\partial f}{\partial z^{13}}|(x,y,m)+|\frac{\partial f}{\partial z^{13}}|(x,y,m))dxdy$

$\leq 2(2m)^2F$

$=8m^2F$

Case 11.

$\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m,m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{m\leq |x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dz)dxdy$

$\leq \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,m)+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,m)+L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dx)dy$

$= \int_{|y|\leq m}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(m,y,m) + \sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(-m,y,m)$

$+ \sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(m,y,m) + \sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(-m,y,-m))dy$

$\leq (2m)(4F)(\sum_{i=0}^{13} L_{i,14})$

$= 8mF(\sum_{i=0}^{13} L_{i,14})$

Case 12.

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial^{14}f_m}{\partial y^{14}}|dz)dxdy$

$\leq \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial y^i\partial x^{14}}|(x,y,-m))dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dx)dy$

$= \int_{|y|\leq m}(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(m,y,m)$

$+ \sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(-m,y,m)$

$+ \sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(m,y,-m)$

$+ \sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(-m,y,-m))dy$

$\leq (2m)(4F)(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14})$

$= 8mF(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14})$

Case 13.

$\int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m,m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m}} |\frac{\partial^{14}f_m}{\partial z^{14}}|dz)dxdy$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m}}(|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dx)dy$

$\leq \int_{|y|\leq m}(\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(m,y,m)+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(-m,y,m)$

$+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(m,y,-m)+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f}{\partial x^i\partial z^{13}}|(-m,y,-m))$

$\leq (2m)(4F)(\sum_{i=0}^{13}L_{i,13})$

$= 8mF(\sum_{i=0}^{13}L_{i,13})$

Cases 14-16 are similar to cases 11-13, interchanging the orders of integration, with case 14 corresponding to case 12, case 15 corresponding to case 11 and case 16 corresponding to case 13, so that;

Case 14.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$\leq 8mF(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14})$

Case 15.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$\leq 8mF(\sum_{i=0}^{13}L_{i,14})$

Case 16.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$\leq 8mF(\sum_{i=0}^{13}L_{i,13})$

Case 17.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dxdy$

$=\frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m}}(\int_{m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,m)$

$+\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dy)dx$

$\leq \frac{1}{m^2}\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f_m}{\partial y^j\partial z^i\partial x^{14}}|(x,-m,-m))dx$

$=\frac{1}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+13}f}{\partial y^j\partial z^i\partial x^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,14})$

Case 18.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dxdy$

$=\frac{1}{m}\int_{|x|\leq m+\frac{1}{m}}(\int_{m\leq|y|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dy)dx$

$=\frac{1}{m}\int_{m\leq|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,m,m)$

$+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,-m,m)$

$+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,m,-m)$

$+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,-m,-m))dx$

$\leq \frac{1}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,14}L_{i,j,13})$

Case 19.

$\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m},m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(\int_{m\leq|z|\leq m+\frac{1}{m}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dz)dxdy$

$=\int_{m\leq|x|\leq m+\frac{1}{m},m\leq|y|\leq m+\frac{1}{m}}(|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dxdy$

$=\int_{m\leq|x|\leq m+\frac{1}{m}}(\int_{m\leq|y|\leq m+\frac{1}{m}}((|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dy)dx$

$\leq \frac{1}{m}\int_{|x|\leq m+\frac{1}{m}}(\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,m,m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,-m,m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,m,-m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,-m,-m))dx$

$\leq \frac{1}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^2}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13})$

It is then clear from $(*)$, summing the bounds from the individual cases 1-19, as at the end of the proof of Lemma 0.67, that there exists a constant $G\in\mathcal{R}_{>0}$ with;

$max(\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial x^{14}}|dxdydz, \int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz, \int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz)\leq Gm^3$

for sufficiently large m.

□

Lemma 0.69. *Let $\{f_m : m\in\mathcal{N}\}$ be an inflexionary sequence, then for $\overline{k}\neq 0$, sufficiently large m, we have that there exists $D\in\mathcal{R}_{>0}$, with;*

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{Dm^3}{|\overline{k}|^{14}}$

Moreover, for sufficiently large m, $\mathcal{F}(f_m)\in L^1(\mathcal{R}^3)$.

Proof. For $(k_1,k_2,k_3)\in\mathcal{R}^3$, using repeated integration by parts, and the fact that $f_m\in L1(\mathcal{R}^3)$, we have, for $m\in\mathcal{N}$;

$\mathcal{F}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}g}{\partial y^{14}}+\frac{\partial^{14}g}{\partial z^{14}})(\overline{k})$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}f_m}{\partial y^{14}}+\frac{\partial^{14}f_m}{\partial z^{14}})e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$=((ik_1)^{14}+(ik_2)^{14}+(ik_3)^{14})\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_m(x,y,z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$=(-k_1^{14}-k_2^{14}-k_3^{14})\mathcal{F}(f_m)(\overline{k})$

so that, for $\overline{k}\neq\overline{0}$;

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{|\mathcal{F}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}g}{\partial y^{14}}+\frac{\partial^{14}g}{\partial z^{14}})(\overline{k})|}{(k_1^{14}+k_2^{14}+k_3^{14})}$ (†)

We have, using the result of Lemma 0.68, for sufficiently large m, that;

$|\mathcal{F}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}g}{\partial y^{14}}+\frac{\partial^{14}g}{\partial z^{14}})(\overline{k})|$

$\frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{\mathcal{R}^3}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}f_m}{\partial y^{14}}+\frac{\partial^{14}f_m}{\partial z^{14}})e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz|$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(|\frac{\partial f_m}{\partial x^{14}}|+|\frac{\partial f_m}{\partial y^{14}}|+|\frac{\partial f_m}{\partial z^{14}}|)dxdydz$

$\leq \frac{3G}{(2\pi)^{\frac{3}{2}}}m^3$ $(\dagger\dagger)$

so that, combining $(\dagger)$ and $(\dagger\dagger)$, we have, for $\overline{k}\neq\overline{0}$, sufficiently large m;

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{3G}{(2\pi)^{\frac{3}{2}}}\frac{m^3}{(k_1^{14}+k_2^{14}+k_3^{14})}$ $(*)$

Using polar coordinates $k_1=rsin(\theta)cos(\phi)$, $k_2=rsin(\theta)sin(\phi)$, $k_3=rcos(\theta)$, $0\leq\theta\leq\pi$, $-\pi<\phi\leq\pi$, we have that;

$\frac{1}{(k_1^{14}+k_2^{14}+k_3^{14})}=\frac{1}{r^{14}}\frac{1}{\alpha(\theta,\phi)}$

where $\alpha(\theta,\phi)=sin^{14}(\theta)(cos^{14}(\phi)+sin^{14}(\phi))+cos^{14}(\theta)$

We have that, in the range $0\leq\theta\leq\pi$, $-\pi\leq\phi\leq\pi$, with $\theta\neq\frac{\pi}{2}$, $|\phi|\neq\frac{\pi}{2}$;

$\alpha(\theta,\phi)=0$

iff $tan^{14}(\theta)(1+tan^{14}(\phi))+\frac{1}{cos^{14}(\phi)}=0$

iff $tan^{14}(\theta)(1+tan^{14}(\phi))=-\frac{1}{cos^{14}(\phi)}$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta=\frac{\pi}{2}$, , $|\phi|\neq\frac{\pi}{2}$

$\alpha(\theta,\phi)=0$

iff $cos^{14}(\phi)+sin^{14}(\phi)=0$

iff $tan^{14}(\phi)=-1$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta \neq \frac{\pi}{2}$, , $|\phi| = \frac{\pi}{2}$

$\alpha(\theta, \phi) = 0$

iff $cos^{14}(\theta) + sin^{14}(\theta) = 0$

iff $tan^{14}(\theta) = -1$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta = \frac{\pi}{2}$, , $|\phi| = \frac{\pi}{2}$

$\alpha(\theta, \phi) = 0$

iff $1 = 0$

which is not the case. It follows that $\alpha(\theta, \phi) = 0$ has no solution in the range $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$. By continuity, compactness of $[0\,\pi] \times [-\pi, \pi]$ and the fact that $\alpha(\frac{\pi}{2}, \frac{\pi}{2}) = 1$, restricting the interval $[-\pi, \pi]$, there exists $\epsilon > 0$, with $\alpha(\theta, \phi) \geq \epsilon$, for $0 \leq \theta \leq \pi$, $-\pi < \phi \leq \pi$. In particularly;

$\frac{1}{(k_1^{14}+k_2^{14}+k_3^{14})} \leq \frac{1}{\epsilon r^{14}}$

$= \frac{1}{\epsilon |\bar{k}|^{14}}$

so that, from $(*)$;

$|\mathcal{F}(f_m)(\bar{k})| \leq \frac{3G}{(2\pi)^{\frac{3}{2}}} \frac{m^3}{\epsilon |\bar{k}|^{14}}$

$= \frac{Dm^3}{|\bar{k}|^{14}}$

where $D = \frac{3G}{\epsilon (2\pi)^{\frac{3}{2}}}$

For the final claim, we have, for $1 \leq i \leq 3$, $m \in \mathcal{N}$, as f_m is supported on $W_{m+\frac{1}{m}}$ and continuous, that $x_i f_m \in L^1(\mathcal{R}^3)$ and, differentiating under the integral sign;

$|\frac{\partial \mathcal{F}(f_m)(\bar{k})}{\partial k^i}| = |\frac{\partial}{\partial k^i}(\frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} f_m(\bar{x}) e^{-i\bar{k}.\bar{x}} d\bar{x})|$

$= |\frac{-i}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} x_i f_m(\bar{x}) e^{-i\bar{k}.\bar{x}} d\bar{x})|$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} |x_i f_m(\bar{x})| d\bar{x}$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} ||x_i f_m(\bar{x})||_1$

so that $\frac{\partial \mathcal{F}(f_m)(\bar{k})}{\partial k^i}$ is bounded, and, in particularly, $\mathcal{F}(f_m)$ is continuous, for $m \in \mathcal{N}$. It follows, using the first result, and polar coordinates, that, for $n > 1$, sufficiently large m;

$| \int_{\mathcal{R}^3} \mathcal{F}(f_m)(\bar{k}) d\bar{k}| \leq \int_{B(\bar{0},n)} |\mathcal{F}(f_m)(\bar{k})| d\bar{k} + \int_{\mathcal{R}^3 \setminus B(\bar{0},n)} |\mathcal{F}(f_m)(\bar{k})| d\bar{k}$

$\leq \frac{4C_n\pi^3}{3} + \int_{\mathcal{R}^3 \setminus B(\bar{0},n)} \frac{Dm^3}{|\bar{k}|^{14}}$

$\leq \frac{4C_n\pi^3}{3} + \int_0^\pi \int_{-\pi}^\pi \int_n^\infty \frac{Dm^3}{r^{14}} |r^2 sin(\theta)| dr d\theta d\phi$

$\leq \frac{4C_n\pi^3}{3} + 2D\pi^2 m^3 \int_n^\infty \frac{dr}{r^{12}}$

$\leq \frac{4C_n\pi^3}{3} + 2D\pi^2 m^3 [\frac{-1}{11r^{11}}]_n^\infty$

$= \frac{4C_n\pi^3}{3} + \frac{2D\pi^2 m^3}{11n^{11}}$

where $C_n = ||\mathcal{F}(f_m)|_{B(\bar{0},n)}||_\infty$, so that $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$.

□

Lemma 0.70. *Let $f \in C^{40}(\mathcal{R}^3)$, with $\frac{\partial^{i_1+i_2+i_3}}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}$ bounded for $0 \leq i_1 + i_2 + i_3 \leq 40$, f analytic for $|\bar{x}| > r$, where $r \in \mathcal{R}_{>0}$, and f analytic at infinity and of very moderate decrease. Then;*

$f(\bar{x}) = \mathcal{F}^{-1}(\mathcal{F}(f))(\bar{x})$, $(\bar{x} \in \mathcal{R}^3)$

where, for $g \in L^1(\mathcal{R}^3)$;

$\mathcal{F}^{-1}(g)(\bar{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} g(\bar{k}) e^{i\bar{k}.\bar{x}} d\bar{k}$

Proof. By Lemma 0.60, we have that $\mathcal{F}(f) \in L^1(\mathcal{R})$. Let $\{f_m : m \in \mathcal{N}\}$ be the approximating sequence, given by Lemma 0.67, then, for sufficiently large m, $f_m \in L^1(\mathcal{R})$ and $\mathcal{F}(f_m) \in L^1(\mathcal{R})$ by Lemma 0.69. It follows, see [5] or the method of [13], that for such m, $f_m = \mathcal{F}^{-1}(\mathcal{F}(f_m))$, $(***)$, By the proof of Lemma 0.59, we have that, for $\overline{k}$ with $min(|k_1|, |k_2|, |k_3|) > \epsilon > 0$, $|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq \frac{E_\epsilon}{m}$, (B). By the proof of Lemma 0.60, we have that $\mathcal{F}(f) - \mathcal{F}(f_m) \in L^{\frac{4}{3}}(\mathcal{R}^3)$, with $||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^{\frac{4}{3}}(\mathcal{R}^3)} \rightarrow 0$ as $m \rightarrow \infty$. In particularly, there exists a constant $H \in \mathcal{R}_{>0}$ with $||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^{\frac{4}{3}}(\mathcal{R}^3)} \leq H$, for sufficiently large m. We then have, using the Holder's inequality that, for $\epsilon > 0$, m sufficiently large;

$$||\mathcal{F}(f) - \mathcal{F}(f_m)||_{L^1(W_\epsilon)}$$

$$= ||(\mathcal{F}(f) - \mathcal{F}(f_m))|_{W_\epsilon} 1_{W_\epsilon}||_{L^1(W_\epsilon)}$$

$$\leq ||(\mathcal{F}(f) - \mathcal{F}(f_m))|_{W_\epsilon}||_{L^{\frac{4}{3}}(W_\epsilon)} ||1_{W_\epsilon}||_{L^4(W_\epsilon)}$$

$$\leq H||1_{W_\epsilon}||_{L^4(W_\epsilon)}$$

$$= 8H\epsilon^3$$

Letting $W_{i,\epsilon} = \{\overline{k} \in \mathcal{R}^3 : |k_i| < \epsilon\}$, $1 \leq i \leq 3$, and $V_\epsilon = \bigcup_{1 \leq i \leq 3} W_{i,\epsilon}$, we have that;

$$\mathcal{R}^3 \setminus V_\epsilon = \{\overline{k} \in \mathcal{R}^3 : min(|k_1|, |k_2|, |k_3|) > \epsilon\}$$

Using the notation of Lemma 0.60, we have that $W_{1,\epsilon} = W_\epsilon \cup V_{1,\epsilon} \cup V_{12,\epsilon} \cup V_{13,\epsilon}$, with ϵ replacing the parameters $\{E_1, E_2, E_3\}$. Using the method of Lemma 0.60, we can show that;

$$\theta_m(x, y) = \int_{|k_3| \geq \epsilon} \mathcal{F}(f - f_m)(x, y, k_3) dk_3$$

is non oscillatory and of very moderate decrease, with;

$$\int_{V_{12,\epsilon}} \mathcal{F}(f - f_m)(k_1, k_2, k_3) dk_1 dk_2 dk_3$$

$$\int_{|k_1| < \epsilon, |k_2| < \epsilon, |k_3| \geq \epsilon} \mathcal{F}(f - f_m)(k_1, k_2, k_3) dk_1 dk_2 dk_3$$

$$= \int_{|k_1 < \epsilon, |k_2| < \epsilon} \mathcal{F}(\theta_m) dk_1 dk_2$$

where $\mathcal{F}$ is the fourier transform for non oscillatory functions of very moderate decrease in 2 variables. As $\theta_m \in L^3(\mathcal{R}^2)$, $\mathcal{F}(\theta_m) \in L^{\frac{3}{2}}(\mathcal{R})$ by the Haussdorff-Young inequality, so that, by Holder's inequality;

$|\int_{V_{12,\epsilon}} \mathcal{F}(f-f_m)(k_1,k_2,k_3)dk_1dk_2dk_3|$

$\leq ||\mathcal{F}(\theta_m)||_{L^1(|k_1<\epsilon,|k_2|<\epsilon)}$

$\leq 4\epsilon^2||\mathcal{F}(\theta_m)||_{L^{\frac{3}{2}}(|k_1<\epsilon,|k_2|<\epsilon)}$

$\leq 4\epsilon^2||\theta_m||_{L^3(\mathcal{R}^2)}$

$\leq 4C_{12}D_{12}\epsilon^2$

$= E_{12}\epsilon^2$

where $C_{12} \in \mathcal{R}_{>0}$ is a uniform bound for $||\theta_m||_{L^3(\mathcal{R}^2)}$, D_{12} is the functional bound in the Haussdorff-Young inequality.

Similarly, we can show that;

$||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^1(V_{13,\epsilon})} \leq E_{13}\epsilon^2$

$||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^1(V_{1,\epsilon})} \leq E_1\epsilon$

so that;

$||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^1(W_{1,\epsilon})} \leq F_1\epsilon$, $(0<\epsilon<1)$

and, similarly;

$||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^1(W_{i,\epsilon})} \leq F_i\epsilon$, $(0<\epsilon<1)$

$||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^1(V_\epsilon)} \leq (F_1+F_2+F_3)\epsilon = F\epsilon$ $(0<\epsilon<1)$ (A)

Using the fact from Lemma 0.60, that $\mathcal{F}(f) \in L^1(\mathcal{R})$, for $\delta > 0$ arbitrary, we have that;

$\int_{\mathcal{R}^3\setminus B(\bar{0},n)} |\mathcal{F}(f)(\bar{k}|d\bar{k} < \delta$

for $n \in \mathcal{N}$, sufficiently large, $n \geq n_0$. Choosing $m \in \mathcal{N}$, with $m = [n^{\frac{10}{3}}]$, and using $(A), (B)$, Lemma 0.69we have, for $\overline{x} \in \mathcal{R}^3$, that;

$$|\mathcal{F}^{-1}(\mathcal{F}(f))(\overline{x}) - \mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x})| = |\mathcal{F}^{-1}(\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k}))|$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{B(\overline{0},n)}(\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$$

$$+\int_{\mathcal{R}^3 \setminus B(\overline{0},n)}(\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}|$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{B(\overline{0},n)}|\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k})|d\overline{k}$$

$$+\int_{\mathcal{R}^3 \setminus B(\overline{0},n)}|\mathcal{F}(f)(\overline{k})|d\overline{k} + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)}|\mathcal{F}(f_m)(\overline{k})|d\overline{k})$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{V_\epsilon \cap B(\overline{0},n)}|\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k})|d\overline{k} + \frac{4\pi n^3 E_\epsilon}{3m} + \delta + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)}\frac{Dm^3}{|k|^{14}}d\overline{k})$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{V_\epsilon}|\mathcal{F}(f)(\overline{k}) - \mathcal{F}(f_m)(\overline{k})|d\overline{k} + \frac{4\pi n^3 E_\epsilon}{3m} + \delta + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)}\frac{Dm^3}{|k|^{14}}d\overline{k})$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon + \frac{4\pi n^3 E_\epsilon}{3(n^{\frac{10}{3}}-1)} + \delta + \int_{\mathcal{R}^3 \setminus B(\overline{0},n)}\frac{Dn^{10}}{|k|^{14}}d\overline{k})$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon + \frac{4\pi E_\epsilon}{3n^{\frac{1}{3}}} + \delta + 2\pi^2\int_{r>n}\frac{Dn^{10}}{r^{14}}dr)$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon + \frac{4\pi E_\epsilon}{3n^{\frac{1}{3}}} + \delta + 2D\pi^2 n^{10}[\frac{-1}{13r^{13}}]_n^\infty)$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}(F\epsilon + \frac{4\pi E_\epsilon}{3n^{\frac{1}{3}}} + \delta + \frac{2D\pi^2}{13n^3})$$

$$< \frac{2\delta + F\epsilon}{(2\pi)^{\frac{3}{2}}}$$

for sufficiently large $n \geq n_0$, so that, as $\epsilon > 0$ and $\delta > 0$ were arbitrary, for $\overline{x} \in \mathcal{R}^3$;

$$lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x}) = \mathcal{F}^{-1}\mathcal{F}(f)(\overline{x}), \; (****)$$

and, by Definition 0.62, $(***)$, $(****)$;

$$f(\overline{x}) = lim_{m\rightarrow\infty}f_m(\overline{x}) = lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x}) = \mathcal{F}^{-1}\mathcal{F}(f)(\overline{x})$$

□

Definition 0.71. *We say that $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ is of moderate decrease n if $|f(\overline{x})| \leq \frac{C}{|\overline{x}|^n}$ for $|\overline{x}| > C$, $C \in \mathcal{R}_{>0}$, $n \geq 2$. We just say that f is of moderate decrease if f is of moderate decrease 2. We call $\{\theta, \phi\}$*

generic if $sin(\theta)cos(\phi)\neq 0$, $sin(\theta)sin(\phi)\neq 0$, $cos(\theta)\neq 0$

Lemma 0.72. *The results of Lemma 0.30 hold, replacing the intervals* $[m, m+\frac{1}{m}]$ *with* $[m, m+\frac{1}{m^2}]$ *and* $[m, m+\frac{1}{m^3}]$. *The generalisations of Lemmas and Definitions 0.61 to 0.69 also hold similarly, replacing* $\frac{1}{m}$ *by* $\frac{1}{m^2}$ *in the two dimensional case, and* $\frac{1}{m}$ *by* $\frac{1}{m^3}$ *in the three dimensional case. In particularly, we have that, for an inflexionary approximation sequence* $\{g_m : m\in\mathcal{N}\}$;

$$\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_m|d\overline{x}\leq\frac{E}{m}$$

for sufficiently large $m\in\mathcal{N}$, *where* $E\in\mathcal{R}_{>0}$.

Proof. In the proof of Lemma 0.30, observe that the coefficients of the polynomial p, depend only on the $\frac{1}{m}$ term, so we can obtain the new coefficients for p by substituting m^2 or m^3 for m. We then calculate in the $\frac{1}{m^3}$ case, that;

$$h'''(x) = (-360a_0m^{15}+O(m^{12}))x^2+(288a_0m^{18}+O(m^{16}))x$$

$$+(-36a_0m^{21}+O(m^{19}))$$

which has roots when;

$$x\simeq\frac{-288a_0+/-176a_0m^{18}+O(m^{16})}{-720a_0m^{15}+O(m^{12})}=O(m^3)+O(m)>0$$

Clearly, we can then assume that for sufficiently large m, $h'''(x)$ has no roots in the interval $[-m-\frac{1}{m^3}]\cup[m, m+\frac{1}{m^3}]$. For the final calculation, with $|h|_{[m+\frac{1}{m^3}]}$, we can replace m by m^3 throughout the proof, to get the same result, that $|h|_{[m+\frac{1}{m^3}]}\leq C$, independently of $m>1$. The case with m^2 replacing m is left to the reader, but we do not need it below. The rest of the Lemmas and Definitions 0.61 to 0.69 go through, once we have generalised the 1-dimensional case and the conjecture Lemma 0.64. In particularly, we obtain the result that for an inflexionary approximation sequence g_m in $\mathcal{R}^3$, $|g_m|_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}\leq C$, independently of m, so that, using the binomial theorem;

$$\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_m|d\overline{x}$$

$\leq Cvol([-m - \frac{1}{m^3}, m + \frac{1}{m^3}]^3 \setminus [-m, m]^3)$

$= 8C((m + \frac{1}{m^3})^3 - m^3)$

$8C(m^3 + \frac{3m^2}{m^3} + \frac{3m}{m^6} + \frac{1}{m^9} - m^3)$

$\leq \frac{E}{m}$

for m sufficiently large, where $E \in \mathcal{R}_{>0}$.

□

Lemma 0.73. *Let f satisfy the conditions of Lemma 0.70 with the extra assumption that $f \in C^{41}(\mathcal{R}^3)$, and the partial derivatives $\{\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k} : 1 \leq i+j+k \leq 41\}$ are of moderate decrease, and of moderate decrease $i+j+k+1$, then for $1 \leq i \leq 3$;*

$k_i \mathcal{F}(f)(\bar{k}) \in C^1(\mathcal{R}^3 \setminus (k_1 = 0 \cup k_2 = 0 \cup k_3 = 0))$

$lim_{\bar{k} \rightarrow 0, \bar{k} \notin (k_1=0 \cup k_2=0 \cup k_3=0)} k_i \mathcal{F}(f)(\bar{k}) = 0$

The same results hold for $k_i \mathcal{F}(\frac{\partial f}{\partial x_j})$, $1 \leq i \leq j \leq 3$, when $f \in C^{42}(\mathcal{R}^3)$.

Making a polar coordinate change, for $\{\theta, \phi\}$ generic, $r\mathcal{F}(f)_{\theta,\phi}(r) \in C^1(\mathcal{R}_{>0})$, $lim_{r \rightarrow 0} r\mathcal{F}(f)_{\theta,\phi}(r) = 0$, and similarly for $r\mathcal{F}(\frac{\partial f}{\partial x_j})$, $1 \leq j \leq 3$.

We have that $\mathcal{F}(f)(\bar{k}) \in L^1(\mathcal{R}^3)$ and $\{\frac{\mathcal{F}(\frac{\partial f}{\partial x_j})(\bar{k})}{|\bar{k}|} : 1 \leq j \leq 3\} \subset L^1(\mathcal{R}^3)$

For any given $\epsilon > 0$, there exists $\delta > 0$, for $1 \leq j \leq 3$, such that for a generic translation $\bar{l}$ with $l_1 \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$;

$max(|\int_0^\delta r\mathcal{F}_{\theta,\phi,\bar{l}}(\frac{\partial f}{\partial x_j})(r)dr|, |\int_0^\delta \frac{d}{dr}(r\mathcal{F}_{\theta,\phi,\bar{l}}(\frac{\partial f}{\partial x_j})(r))dr|) < \epsilon$

uniformly in $\{\theta, \phi\}$.

Proof. As $\frac{\partial f}{\partial x}$ is of moderate decrease and analytic at infinity, for fixed y, z, $f_{y,z}$ is of very moderate decrease and analytic at infinity, we have for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$;

$\mathcal{F}(\frac{\partial f}{\partial x})=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_1\rightarrow\infty}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}\frac{\partial f}{\partial x}(\overline{x})e^{-i\overline{k}.\overline{x}}dx_1dx_2dx_3$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{r_1\rightarrow\infty}\int_{-r_1}^{r_1}\frac{\partial f}{\partial x}(\overline{x})e^{-ik_1x_1}dx_1)e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{r_1\rightarrow\infty}([fe^{-ikx_1}]_{-r_1}^{r_1}+ik_1\int_{-r_1}^{r_1}f(\overline{x})e^{-ikx_1}dx_1)$

$e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$

$=ik_1\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{r_1\rightarrow\infty}\int_{-r_1}^{r_1}f(\overline{x})e^{-ikx_1}dx_1)e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$

$=ik_1\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_1\rightarrow\infty}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}f(\overline{x})e^{-i\overline{k}.\overline{x}}dx_1dx_2dx_3$

$=ik_1\mathcal{F}(f)(\overline{k})$ (TT)

the limit interchange being justified by the inversion theorem. It follows that, for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$, we have that;

$k_1\mathcal{F}(f)(\overline{k})=-i\mathcal{F}(\frac{\partial f}{\partial x})$

and similarly;

$k_i\mathcal{F}(f)(\overline{k})=-i\mathcal{F}(\frac{\partial f}{\partial x_i})$ (A), for $1\leq i\leq 3$ and $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$.

It follows that, using the fact that;

$F(x_1,k_2,k_3)=lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}\frac{\partial f}{\partial x}(x_1,x_2,x)e^{-ik_2x_2}e^{-ik_3x_3}dx_2dx_3$

is of moderate decrease, the DCT and the FTC, and the fact that $f_{y,z}$ is of very moderate decrease;

$lim_{\overline{k}\rightarrow 0,\overline{k}\notin(k_1=0\cup k_2=0\cup k_3=0)}k_1\mathcal{F}(f)(\overline{k})$

$-ilim_{\overline{k}\rightarrow 0,\overline{k}\notin(k_1=0\cup k_2=0\cup k_3=0)}\mathcal{F}(f)(\frac{\partial f}{\partial x})(\overline{k})$

$=\frac{-i}{(2\pi)^{\frac{3}{2}}}lim_{\overline{k}\rightarrow 0,\overline{k}\notin(k_1=0\cup k_2=0\cup k_3=0)}lim_{r_1\rightarrow\infty}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}\frac{\partial f}{\partial x}(\overline{x})e^{-i\overline{k}.\overline{x}}dx_1dx_2dx_3$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{k_2\rightarrow 0,k_3\rightarrow 0,k_2\neq 0,k_3\neq 0}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{k_1\rightarrow 0}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}(\overline{x})e^{-ik_1x_1}dx_1)$

$e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{k_2\rightarrow 0,k_3\rightarrow 0,k_2\neq 0,k_3\neq 0} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty} \int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}(\overline{x})dx_1)e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{k_2\rightarrow 0,k_3\rightarrow 0,k_2\neq 0,k_3\neq 0} lim_{r_2\rightarrow\infty} lim_{r_3\rightarrow\infty} \int_{-r_2}^{r_2}\int_{-r_3}^{r_3}([f]_{-\infty}^{\infty})e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$

$= 0$ (E)

Similarly;

$lim_{\overline{k}\rightarrow 0,\overline{k}\notin(k_1=0\cup k_2=0\cup k_3=0)} k_i\mathcal{F}(f)(\overline{k}) = 0$, $1\leq i\leq 3$

As $f\in C^{41}(\mathcal{R}^3)$, we have, by the product rule, that $x_i\frac{\partial f}{\partial x_j}\in C^{40}(\mathcal{R}^3)$, $1\leq i\leq j\leq 3$. As f is of very moderate decrease and;

$\{\frac{\partial f^{l+m+n}}{\partial x_1^l\partial x_2^m\partial x_3^m} : 1\leq l+m+n\leq 40\}$

are of very moderate decrease, we have, by repeated application of the product rule again, that;

$\{\frac{\partial^{l+m+n}x_i\frac{\partial f}{\partial x_j}}{\partial x_1^l\partial x_2^m\partial x_3^n} : 0\leq l+m+n\leq 40\}$, $1\leq i\leq j\leq 3$

are bounded. By Lemma 0.72, there exists an inflexionary approximation sequence g_m for $x\frac{\partial f}{\partial x}$ with the properties that;

(i) $g_m\in C^{14}(\mathcal{R}^3)$

(ii). $g_m|_{[-m,m]^3} = x\frac{\partial f}{\partial x}|_{[-m,m]^3}$

(iii). $\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_m(\overline{x})d\overline{x}\leq\frac{E}{m}$

(iv). $g_m|_{\mathcal{R}^3\setminus[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3} = 0$

By the construction of g_m, we have that $f_m = \frac{g_m}{x}$ is an approximation sequence for $\frac{\partial f}{\partial x}$, with the property that;

$(i)'$ $f_m\in C^{14}(\mathcal{R}^3)$

$(ii)'$. $f_m|_{[-m,m]^3} = \frac{\partial f}{\partial x}|_{[-m,m]^3}$

$(iii)'$. $\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|f_m(\overline{x})d\overline{x}\leq\frac{E'}{m}$

$(iv)'$. $f_m|_{\mathcal{R}^3\setminus[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3}=0$

Following through the proof of Lemma 0.70, as $\frac{\partial f}{\partial x}$ is of moderate decrease and, therefore, of very moderate decrease, we have that $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(\frac{\partial f}{\partial x})$ on compact subsets of $\mathcal{R}^3\setminus(k_1=0\cup k_2=0\cup k_3=0)$, so that $\mathcal{F}(\frac{\partial f}{\partial x})\in C(\mathcal{R}^3\setminus(k_1=0\cup k_2=0\cup k_3=0))$, As $x_ix_jf_m\in L^1(\mathcal{R}^3)$, for $1\leq i\leq j\leq 3$, we have that $\mathcal{F}(f_m)$ is twice differentiable, in particularly, $\mathcal{F}(f_m)\in C^1(\mathcal{R}^3)$. As f is analytic at infinity, so is $\frac{\partial f}{\partial x}$. Moreover, as $\frac{\partial f}{\partial x}$ is of moderate decrease, assuming without loss of generality $x_0\neq 0$;

$|\frac{\partial f}{\partial x}(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})|=|g_{x_0,y_0,z_0}(x,y,z)|$

$\leq\frac{C}{|(\frac{x_0}{x},\frac{y_0}{y},\frac{z_0}{z})|^2}$

$=\frac{Cx^2}{x_0^2+\frac{y_0x^2}{y^2}+\frac{z_0x^2}{z^2}}$

$\leq\frac{Cx^2}{x_0^2}$

so that $g_{x_0,y_0,z_0}\frac{x_0}{x}$ has a removable singularity at $x=0$, so that $x\frac{\partial f}{\partial x}$ is analytic at infinity. It follows that for $\{m,n\}\subset\mathcal{N}$, with $m\geq n$, differentiating under the integral sign, using the DCT, property (iii) of an inflexionary approximating sequence, and the fact that $x\frac{\partial f}{\partial x}$ is of moderate decrease and analytic at infinity, for $|k_1|\geq\epsilon_1>0$, $|k_2|\geq\epsilon_2>0$, $|k_3|\geq\epsilon_3>0$, we have that;

$|\frac{\partial\mathcal{F}(f_m)}{\partial k_1}-\frac{\partial\mathcal{F}(f_n)}{\partial k_1}|$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}|\frac{\partial}{\partial k_1}(\int_{\mathcal{R}^3}f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}-\frac{\partial}{\partial k_1}\int_{\mathcal{R}^3}f_n(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{\mathcal{R}^3}-ix_1f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}-\int_{\mathcal{R}^3}-ix_1f_n(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{\mathcal{R}^3}(g_m-g_n)(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_m(\overline{x})|d\overline{x}+\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_n(\overline{x})|d\overline{x}$

$+|\int_{[-m,m]^3\setminus[-n,n]^3}x_1\frac{\partial f}{\partial x_1}e^{-i\overline{k}.\overline{x}}d\overline{x}|)$

$\leq\frac{E}{m}+\frac{E}{n}+\frac{C(\overline{k})}{n}$ $(*)$

where $C(\overline{k})$ is uniformly bounded on the region $|k_1| \geq \epsilon_1 > 0$, $|k_2| \geq \epsilon_2 > 0$, $|k_3| \geq \epsilon_3 > 0$. It follows that the sequence $\{\frac{\partial\mathcal{F}(f_m)}{\partial k_1} : m \in \mathcal{N}\}$ is uniformly Cauchy on the region $|k_1| \geq \epsilon_1 > 0$, $|k_2| \geq \epsilon_2 > 0$, $|k_3| \geq \epsilon_3 > 0$, and converges uniformly. By considering inflexionary sequences for $y\frac{\partial f}{\partial x}$ and $z\frac{\partial f}{\partial x}$, we can similarly show that the sequences $\{\frac{\partial\mathcal{F}(f_m)}{\partial k_2} : m \in \mathcal{N}\}$ and $\{\frac{\partial\mathcal{F}(f_m)}{\partial k_3} : m \in \mathcal{N}\}$ are uniformly Cauchy on the region $|k_1| \geq \epsilon_1 > 0$, $|k_2| \geq \epsilon_2 > 0$, $|k_3| \geq \epsilon_3 > 0$, and converge uniformly. As $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(\frac{\partial f}{\partial x})$ on the regions $|k_1| \geq \epsilon_1 > 0$, $|k_2| \geq \epsilon_2 > 0$, $|k_3| \geq \epsilon_3 > 0$, it follows that $\mathcal{F}(\frac{\partial f}{\partial x}) \in C^1(\mathcal{R}^3 \setminus (k_1 = 0 \cup k_2 = 0 \cup k_3 = 0))$. The same result folds for $\mathcal{F}(\frac{\partial f}{\partial y})$ and $\mathcal{F}(\frac{\partial f}{\partial z})$, so by (A);

$$\{k_1\mathcal{F}(f)(\overline{k}), k_2\mathcal{F}(f)(\overline{k}), k_3\mathcal{F}(f)(\overline{k})\} \subset C^1(\mathcal{R}^3 \setminus (k_1 = 0 \cup k_2 = 0 \cup k_3 = 0))$$

(B)

It follows that, changing to polars;

$$\frac{\partial r\mathcal{F}(f)(\overline{k})}{\partial r} = (\frac{\partial}{\partial k_1}\frac{k_1}{r} + \frac{\partial}{\partial k_2}\frac{k_2}{r} + \frac{\partial}{\partial k_3}\frac{k_3}{r})(r\mathcal{F}(f)(\overline{k}))$$

$$= \frac{\partial k_1\mathcal{F}(f)(\overline{k})}{\partial k_1} + \frac{\partial k_2\mathcal{F}(f)(\overline{k})}{\partial k_2} + \frac{\partial k_3\mathcal{F}(f)(\overline{k})}{\partial k_3} \ (WW)$$

so that, for generic $\{\theta, \phi\}$, $r\mathcal{F}(f)(r)_{\theta,\phi} \in C^1(\mathcal{R}_{>0})$, by (B). Moreover;

$$lim_{r\rightarrow 0} r\mathcal{F}(f)(r)_{\theta,\phi}.$$

$$= lim_{\overline{k}(\theta,\phi)\rightarrow 0}\frac{r}{k_1} lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_1\mathcal{F}(f)(\overline{k})$$

$$= lim_{\overline{k}(\theta,\phi)\rightarrow 0}\frac{r}{k_2} lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_2\mathcal{F}(f)(\overline{k})$$

$$= lim_{\overline{k}(\theta,\phi)\rightarrow 0}\frac{r}{k_3} lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_3\mathcal{F}(f)(\overline{k})$$

$$= lim_{\overline{k}(\theta,\phi)\rightarrow 0} sign(k_1)(1 + \frac{k_2^2}{k_1^2} + \frac{k_3^2}{k_1^2}) lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_1\mathcal{F}(f)(\overline{k})$$

$$= lim_{\overline{k}(\theta,\phi)\rightarrow 0} sign(k_2)(1 + \frac{k_1^2}{k_2^2} + \frac{k_3^2}{k_2^2}) lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_2\mathcal{F}(f)(\overline{k})$$

$$= lim_{\overline{k}(\theta,\phi)\rightarrow 0} sign(k_3)(1 + \frac{k_1^2}{k_3^2} + \frac{k_2^2}{k_3^2}) lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_3\mathcal{F}(f)(\overline{k})$$

$$= 0$$

as the cases $max(|k_2|,|k_3|)\leq|k_1|$, $max(|k_1|,|k_3|)\leq|k_2|$ and $max(|k_1|,|k_2|)\leq|k_3|$ are exhaustive.

Clearly, we can repeat the above arguments for $\frac{\partial f}{\partial x_i}$, $1\leq i\leq 3$, and $f\in C^{42}(\mathcal{R}^3)$, using the fact that $\frac{\partial f}{\partial x_i}$ is of moderate decrease, in particularly of very moderate decrease, with the higher derivatives $\frac{\partial^{l+m+n}\frac{\partial f}{\partial x_i}}{\partial x^l y^m z^n}$ of moderate decrease $l+m+n+2$, in particularly of moderate decrease $l+m+n+1$.

For the next claim, we have, as f is of very moderate decrease, using polar coordinates, that;

$\int_{\mathcal{R}^3}|f|^{3+\epsilon}d\overline{x}$

$\leq\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\phi}\int_{\mathcal{R}_{>0}}|f|^{3+\epsilon}r^2drd\theta d\phi$

$\leq C+2\pi^2\int_{r_0}^{\infty}\frac{Dr^2}{r^{3+\epsilon}}dr$

$\leq C+2\pi^2\int_{r_0}^{\infty}[-r^{-\epsilon}]_{r_0}^{\infty}dr$

$=C+2\pi^2 r_0^{-\epsilon}<\infty$

so that $f\in L^{3+\epsilon}$, for $\epsilon>0$. By the Haussdorff-Young inequality, $\mathcal{F}(f)\in L^{\frac{3}{2}-\delta}(\mathcal{R}^3)$, for $\delta>0$, so that, due to the decay, $\mathcal{F}(f)\in L^1(\mathcal{R}^3)$, (R). A similar calculation show that, as $\frac{\partial f}{\partial x}$ is of moderate decrease 2, that $f\in L^{\frac{3}{2}+\epsilon}(\mathcal{R}^3)$, for $\epsilon>0$. Applying the Haussdorff-Young inequality, $\mathcal{F}(\frac{\partial f}{\partial x})\in L^{3-\delta}(\mathcal{R}^3)$, for $\delta>0$. In particular, due to the decay again, $\mathcal{F}(\frac{\partial f}{\partial x})\in L^2(\mathcal{R}^3)$. Locally, on $B(\overline{0},1)$, for $\delta>0$;

$\int_{B(\overline{0},1)}\frac{1}{|\overline{k}|^{3-\delta}}d\overline{k}$

$=\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\phi}\int_0^1\frac{r^2}{r^{3-\delta}}drd\theta d\phi$

$\leq 2\pi^2[r^{\delta}]_0^1$

$=2\pi^2<\infty$

so that $\frac{1}{|\overline{k}|}\in L^{3-\delta}(B(\overline{0},1))$, in particularly $\frac{1}{|\overline{k}|}\in L^2(B(\overline{0},1))$. As $\mathcal{F}(\frac{\partial f}{\partial x})\in L^2(B(\overline{0},1))$, by Holder's inequality, we obtain that $\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{|\overline{k}|}\in$

$L^1(B(\overline{0},1))$, and by the decay, we have that $\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{|\overline{k}|} \in L^1(\mathcal{R}^3)$. Similar arguments show that $\frac{\mathcal{F}(\frac{\partial f}{\partial x_i})(\overline{k})}{|\overline{k}|} \in L^1(\mathcal{R}^3)$, for $1 \leq i \leq 3$. We can also complete this argument with just the assumption that $\frac{\partial f}{\partial x}$ is of very moderate decrease. As by the argument (TT), for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$;

$$\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k}) = ik_1\mathcal{F}(f)(\overline{k})$$

so that;

$$\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{|\overline{k}|} = \frac{ik_1}{|\overline{k}|}\mathcal{F}(f)(\overline{k})$$

with, for $k_1 \neq 0$;

$$|\frac{|ik_1}{|\overline{k}|}| = |sign(k_1)||\frac{1}{(1+\frac{k_2}{k_1}^2+\frac{k_3}{k_1}^2)^{\frac{1}{2}}}| \leq 1$$

so that;

$$|\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{|\overline{k}|}| \leq |\mathcal{F}(f)(\overline{k})|$$

and, by (R), $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^3)$, so that $\frac{\mathcal{F}(\frac{\partial f}{\partial x})}{|\overline{k}|} \in L^1(\mathcal{R}^3)$. Similarly, for $1 \leq i \leq 3$, $\frac{\mathcal{F}(\frac{\partial f}{\partial x_i})}{|\overline{k}|} \in L^1(\mathcal{R}^3)$. We also have that, with just the assumption that $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is of very moderate decrease, $1 \leq i \leq j \leq 3$, using the argument (TT) twice, that for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$;

$$\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j}) = (ik_i)(ik_j)\mathcal{F}(f)(\overline{k})$$

$$= -k_i k_j \mathcal{F}(f)(\overline{k})$$

so that;

$$\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j})(\overline{k})}{|\overline{k}|^2} = \frac{-k_i k_j}{|\overline{k}|^2}\mathcal{F}(f)(\overline{k})$$

with, for $k_i \neq 0$, $k_j \neq 0$;

$$|\frac{|-k_i k_j}{|\overline{k}|^2}| = |sign(k_1)sign(k_2||\frac{1}{(1+\frac{k_2}{k_1}^2+\frac{k_3}{k_1}^2)^{\frac{1}{2}}}||\frac{1}{(1+\frac{k_1}{k_2}^2+\frac{k_3}{k_2}^2)^{\frac{1}{2}}}| \leq 1$$

so that;

$|\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j})(\overline{k})}{|\overline{k}|^2}| \leq |\mathcal{F}(f)(\overline{k})|$

and, by (R), $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^3)$, so that $\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j})(\overline{k})}{|\overline{k}|^2} \in L^1(\mathcal{R}^3)$.

The last claim follows from the fact that, for $\bar{l}$, with $l_1 \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$, the translation $\mathcal{F}_{\bar{l}}(\frac{\partial f}{\partial x_i})(\overline{k}) \in C^1(B(\overline{0}, \epsilon'))$, for some $\epsilon' > 0$. In particular, given $\epsilon > 0$, there exists $\delta > 0$, such that;

$max(|\int_0^\delta r\mathcal{F}_{\theta,\phi,\bar{l}}(\frac{\partial f}{\partial x_j})(r)dr|, |\int_0^\delta \frac{d}{dr}(r\mathcal{F}_{\theta,\phi,\bar{l}}(\frac{\partial f}{\partial x_j})(r))dr|) < \epsilon$

uniformly in $\{\theta, \phi\}$.

$\square$

Definition 0.74. *We say that a solution* $(\overline{E}, \overline{B})$ *to Maxwell's equations in vacuum is non oscillatory, if the components* $\{e_i, b_j\}$ *are of very moderate decrease, and analytic at infinity, with;*

$\frac{\partial^{l+m+m} e_i}{\partial x^l \partial y^m \partial z^n}$

$\frac{\partial^{l+m+m} b_i}{\partial x^l \partial y^m \partial z^n}$

of very moderate decrease, $1 \leq i \leq 3$, $l+m+n \geq 1$, *the components are analytic at infinity and* $\{e_i, b_j\}$ *are sufficiently differentiable, and using the fact that* $\overline{E} = \bigtriangledown \times \overline{E}_1$, $\overline{B} = \bigtriangledown \times \overline{B}_1$, *we can assume that the components of* $\{\overline{E}_1, \overline{B}_1\}$ *are of very moderate decrease and analytic at infinity. We also require that the components* $\{e_i', b_j'\}$ *of;*

$\{\frac{\partial \overline{E}}{\partial t}, \frac{\partial \overline{B}}{\partial t}\}$

or equivalently $\{c^2(\bigtriangledown \times \overline{B}), -\bigtriangledown \times \overline{E}\}$

or equivalently $\{c^2(\bigtriangledown \times \bigtriangledown \times \overline{B}_1), -\bigtriangledown \times \bigtriangledown \times \overline{E}_1\}$

are of moderate decrease.

Lemma 0.75. *There exists a solution* $(\overline{E}, \overline{B})$ *to Maxwell's equations in vacuum, with the property that for all* $t \in \mathcal{R}$, *the components of* $\overline{E}_t$

and $\overline{B}_t$ are smooth and have compact support. There exists a solution $(\overline{E}', \overline{B}')$ to Maxwell's equations in vacuum with $(\overline{E}', \overline{B}')$ non-oscillatory.

Proof. Choose smooth vector fields $\{\overline{e}, \overline{b}\}$ with compact support, then we have that;

$div(\bigtriangledown \times \overline{e}) = div(\bigtriangledown \times \overline{b}) = 0$

and $\overline{e}_1 = \bigtriangledown \times \overline{e}$, $\overline{b}_1 = \bigtriangledown \times \overline{b}$ are smooth and have compact support. Let;

$\overline{b}_2 = -(\bigtriangledown \times \overline{e}_1$

$\overline{e}_2 = c^2(\bigtriangledown \times \overline{b}_1)$

Then, by construction $\{\overline{e}_1, \overline{e}_2, \overline{b}_1, \overline{b}_2\}$ satisfy the equations;

(i). $div(\overline{e}_1) = 0$

(ii). $\bigtriangledown \times \overline{e}_1 = -\overline{b}_2$

(iii). $div(\overline{b}_1) = 0$

(iv). $\bigtriangledown \times \overline{b}_1 = \mu_0 \epsilon_0 \overline{e}_2$.

We have there exists a unique solution to the wave equations $\Box^2 \overline{E} = 0$ and $\Box^2 \overline{B} = \overline{0}$, with initial conditions $(\overline{e}_1, \overline{e}_2)$ and $(\overline{b}_1, \overline{b}_2)$ such that $\overline{E}_0 = \overline{e}_1$, $\frac{\partial \overline{E}}{\partial t}_0 = \overline{e}_2$, $\overline{B}_0 = \overline{b}_1$, $\frac{\partial \overline{B}}{\partial t}_0 - \overline{b}_2$. By Kirchoff's formula, we have that the components of $(\overline{E}, \overline{B})$ have compact support for all times t. Moreover $div(\overline{E}$ satisfies the wave equation with initial conditions $div(\overline{e}_1) = 0$ and $div(\overline{e}_2) = div(c^2(\bigtriangledown \times \overline{b}_1)) = 0$, so that, by uniqueness of the initial conditions, $div(\overline{E}) = 0$. Similarly, $div(\overline{B})$ satisfies the wave equation with initial conditions $div(\overline{b}_1) = 0$, and $div(\overline{b}_2) = div(- \bigtriangledown \times \overline{e}_1) = 0$, so that, by uniqueness again, $div(\overline{B}) = 0$ as well. We have that $\bigtriangledown \times \overline{E} + \frac{\partial \overline{B}}{\partial t}$ satisfies the wave equation with initial conditions $\bigtriangledown \times \overline{e}_1 + \overline{b}_2 = \overline{0}$ and;

$\bigtriangledown \times \overline{e}_2 + \frac{\partial^2 \overline{B}}{\partial^2 t}|_0$

$= \bigtriangledown \times \overline{e}_2 + c^2 \bigtriangledown^2 \overline{b}_1$

$= \bigtriangledown \times (c^2 \bigtriangledown \times \overline{b}_1) + c^2 \bigtriangledown^2 \overline{b}_1$

$= c^2 grad(div(\overline{b}_1)) - c^2 \bigtriangledown^2 \overline{b}_1 + c^2 \bigtriangledown^2 \overline{b}_1$

$= \overline{0}$

so that, by uniqueness, $\bigtriangledown \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$. Finally, we have that, $\bigtriangledown \times \overline{B} - \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t}$ satisfies the wave equation, with initial conditions $\bigtriangledown \times \overline{b}_1 - \frac{1}{c^2}\overline{e}_2 = \overline{0}$ and;

$\bigtriangledown \times \overline{b}_2 - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial^2 t}|_0$

$= \bigtriangledown \times \overline{b}_2 - \bigtriangledown^2 \overline{e}_1$

$- \bigtriangledown \times \bigtriangledown \times \overline{e}_1 - \bigtriangledown^2 \overline{e}_1$

$= -grad(div(\overline{e}_1)) + \bigtriangledown^2 \overline{e}_1 - \bigtriangledown^2 \overline{e}_1$

$= \overline{0}$

so that, by uniqueness, $\bigtriangledown \times \overline{B} = \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t}$. It follows $(\overline{E}, \overline{B})$ satisfies Maxwell's equations in vacuum, as required.

For the second claim, we can construct the potentials $\{\overline{e}_0, \overline{b}_0\}$ for $\{\overline{E}', \overline{B}'\}$ by;

$\frac{\int_{-\infty}^{t} \overline{g}(x,y,z,s)ds}{r}$ $\frac{\int_{-\infty}^{t} \overline{h}(x,y,z,s)ds}{r}$ $(r > 1)$

Choose $(\overline{E}, \overline{B})$ a solution to Maxwell's equations in vacuum with compact supports, as above. By the method in [6], we can choose $\{\overline{g}, \overline{h}\}$ such that $\bigtriangledown \times \overline{g} = \overline{E}$, $\bigtriangledown \times \overline{h} = \overline{B}$, for all times $t \in \mathcal{R}$, and, clearly, for a given $t \in \mathcal{R}$, $\overline{g}_t$ and $\overline{h}_t$ have compact supports. Then, we have that, by Maxwell's equations;

$\bigtriangledown \times \bigtriangledown \times \overline{g} = \bigtriangledown \times \overline{E}$

$= -\frac{\partial \overline{B}}{\partial t}$

$\bigtriangledown \times \bigtriangledown \times \overline{h} = \bigtriangledown \times \overline{B}$

$= \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t}$

It follows, by the fundamental theorem of calculus, and the facts;

$lim_{t\rightarrow -\infty}\overline{B}(x,y,z,t) = \overline{0}$

$lim_{t\rightarrow -\infty}\overline{E}(x,y,z,t) = \overline{0}$

that we must have;

$\bigtriangledown \times \bigtriangledown \times (\int_{-\infty}^{t} \overline{g}(x,y,z,s)ds)$

$= \int_{-\infty}^{t}(\bigtriangledown \times \bigtriangledown \times \overline{g})ds$

$= -\int_{-\infty}^{t} \frac{\partial \overline{B}}{\partial s}ds$

$= -\frac{\partial \overline{B}}{\partial t}$

$\bigtriangledown \times \bigtriangledown \times (\int_{-\infty}^{t} \overline{h}(x,y,z,s)ds)$

$= \int_{-\infty}^{t}(\bigtriangledown \times \bigtriangledown \times \overline{h})ds$

$= \frac{1}{c^2}\int_{\mathcal{R}} \frac{\partial \overline{E}}{\partial s}ds$

$= \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t}$

so that, as $\frac{\partial \overline{E}}{\partial t}$ and $\frac{\partial \overline{B}}{\partial t}$ have compact support, by the product rule $\{\bigtriangledown \times \bigtriangledown \times \overline{e}_0, \bigtriangledown \times \bigtriangledown \times \overline{b}_0\}$ are of moderate decrease. We can obtain sufficient differentiablity using a polynomial $p(r)$ for $r \leq 1$, with the property that;

$p^{(n)}(r)|_{r=1} = \frac{1}{r}^{(n)}|_{r=1}$

and defining the potentials $\{\overline{e}_0, \overline{b}_0\}$ by;

$p(r)\int_{-\infty}^{t} \overline{g}(x,y,z,s)ds$, $p(r)\int_{-\infty}^{t} \overline{h}(x,y,z,s)ds$ $(r \leq 1)$

The components of $\{\overline{E}', \overline{B}'\}$ are of very moderate decrease and analytic at infinity, as are the potentials. The constructed $(\overline{E}', \overline{B}')$ is non-oscillatory, but we need a local notion of analytic at infinity.

□

Remarks 0.76. *By considering the following family of examples, we can support the theory of this paper. As we are considering the radiation condition at $t=0$, the data of $\{\frac{\partial \overline{E}}{\partial t}|_{t=0}, \frac{\partial \overline{B}}{\partial t}|_{t=0}\}$ is not involved in the Fourier transform calculation, and there is no need for these initial conditions to be the second derivative of a potential which is $O(\frac{1}{r})$. For $a(r)$ analytic at infinity, with $a(r)$ $O(1)$, we let $\beta_a(r)=\frac{1}{(1+a(r)r)^5}$, so that β_a is $O(\frac{1}{r^5})$. As $a(r)$ is analytic at infinity, we have that;*

$a(\frac{1}{r})=g(r)$

where g is analytic at 0, then, by the chain rule;

$a'(\frac{1}{r})\frac{-1}{r^2}=g'(r)$

so that;

$\frac{1}{r}a'(\frac{1}{r})=-rg'(r)$

and $lim_{r\rightarrow\infty}ra'(r)=lim_{r\rightarrow 0}\frac{1}{r}a'(\frac{1}{r})$

$=lim_{r\rightarrow 0}-rg'(r)$

$=0$ (A)

We let the potentials be defined by;

$\overline{e}_0=\beta_a(r)x^5(0,1,0)$, $(x>0)$

$\overline{e}_0=(0,0,0)$, $(x\leq 0)$

$\overline{b}_0=\beta_a(r)x^5(0,0,1)$, $(x>0)$

$\overline{b}_0=(0,0,0)$, $(x\leq 0)$

which are $O(1)$ but belong to $C^4(\mathcal{R}^3)$. We have that;

$\overline{E}_0=\bigtriangledown\times\overline{e}_0=(-\frac{x^5\beta'_a z}{r},0,5x^4\beta_a+\frac{x^6\beta'_a}{r})$,

$(x>0)$

$\overline{E}_0 = \overline{0}$, $(x \leq 0)$

$\overline{B}_0 = \bigtriangledown \times \overline{b}_0 = (\frac{x^5\beta'_a y}{r}, -5x^4\beta_a - \frac{x^6\beta'_a}{r}, 0)$,

$(x > 0)$

$\overline{B}_0 = \overline{0}$, $(x \leq 0)$

which are $O(\frac{1}{r})$. *We have that;*

$\overline{E}_0 \times \overline{B}_0 = (25x^8\beta_a^2 + \frac{10x^{10}\beta_a\beta'_a}{r} + \frac{x^{12}\beta'^2_a}{r^2}, \frac{5x^9\beta_a\beta'_a y}{r} + \frac{x^{11}\beta'^2_a y}{r^2}, \frac{5x^9\beta_a\beta'_a z}{r} + \frac{x^{11}\beta'^2_a z}{r^2})$

and;

$(\overline{E}_0 \times \overline{B}_0) \centerdot \hat{n}$

$= (\overline{E}_0 \times \overline{B}_0) \centerdot \frac{(x,y,z)}{r}$

$= \frac{25x^9\beta_a^2}{r} + \frac{10x^{11}\beta_a\beta'_a}{r^2} + \frac{x^{13}\beta'^2_a}{r^3} + \frac{5x^9\beta'_a\beta_a y^2}{r^2} + \frac{x^{11}\beta'^2_a y^2}{r^3} + \frac{5x^9\beta'_a\beta_a z^2}{r^2} + \frac{x^{11}\beta'^2_a z^2}{r^3}$

$= \frac{25x^9\beta_a^2}{r} + \frac{10x^{11}\beta'_a\beta_a}{r^2} + 5x^9\beta'_a\beta_a - \frac{5x^9\beta'_a\beta_a x^2}{r^2} + \frac{x^{11}\beta'^2_a}{r}$

where $x^2 + y^2 + z^2 = r^2$ *and* $\hat{n}$ *is the unit normal to the sphere* $S(r)$. *It follows that, using a polar coordinate change* $x = rsin(\theta)cos(\phi)$, $y = rsin(\theta)sin(\phi)$, $z = cos(\theta)$, $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$;
;

$\int_{S(\overline{0},r)}(\overline{E}_0 \times \overline{B}_0) \centerdot d\overline{S}(r)$

$= \frac{25\beta_a^2}{r}\int_{S(\overline{0},r),x\geq 0} x^9 dS(r) + \frac{\beta'_a\beta_a}{r^2}\int_{S(\overline{0},r),x\geq 0}(10x^{11} - 5x^{11})dS(r) + \beta'_a\beta_a\int_{S(\overline{0},r),x\geq 0} 5x^9 dS(r)$

$+ \frac{\beta'^2_a}{r}\int_{S(\overline{0},r),x\geq 0} x^{11} dS(r)$

$= \frac{25\beta_a^2 r^2}{r}\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^9 sin(\theta)d\theta d\phi + \frac{5\beta'_a\beta_a r^2}{r^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^{11} sin(\theta)d\theta d\phi + 5\beta'_a\beta_a r^2\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^9 sin(\theta)d\theta d\phi$

$+ \frac{\beta'^2_a r^2}{r}\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^{11} sin(\theta)d\theta d\phi$

$= \frac{25\beta_a^2 r^2 r^9}{r}\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^9(\theta)cos^9(\phi)sin(\theta)d\theta d\phi$

$+\frac{5\beta_a'\beta_a r^2 r^{11}}{r^2}\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^{11}(\theta)cos^{11}(\phi)sin(\theta)d\theta d\phi$

$+5\beta_a'\beta_a r^2 r^9\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^9(\theta)cos^9(\phi)sin(\theta)d\theta d\phi$

$+\frac{\beta_a'^2 r^2 r^{11}}{r}\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^{11}(\theta)cos^{11}(\phi)sin(\theta)d\theta d\phi$

$=25\beta_a^2 r^{10}(\frac{9.7.5.3.1}{10.8.6.4.2})\pi(\frac{8.6.4.2}{9.7.5.3.1})2$

$+5\beta_a'\beta_a r^{11}(\frac{11.9.7.5.3.1}{12.10.8.6.4.2})\pi(\frac{10.8.6.4.2}{11.9.7.5.3.1})2$

$+5\beta_a'\beta_a r^{11}(\frac{9.7.5.3.1}{10.8.6.4.2})\pi(\frac{8.6.4.2}{9.7.5.3.1})2$

$+\beta_a'^2 r^{12}(\frac{11.9.7.5.3.1}{12.10.8.6.4.2})\pi(\frac{10.8.6.4.2}{11.9.7.5.3.1})2$

$=\beta_a^2 r^{10}(5\pi)+\beta_a'\beta_a r^{11}(\frac{11\pi}{6})+\beta_a'^2 r^{12}(\frac{\pi}{6})$

We have that, by the chain rule;

$\beta_a'(r)=-\frac{5}{(1+ra(r))^6}(ra(r))'$

$=-\frac{5a(r)}{(1+ra(r))^6}-\frac{5ra'(r)}{(1+ra(r))^6}$

and;

$lim_{r\rightarrow\infty}r^5\beta_a=lim_{r\rightarrow\infty}\frac{r^5}{(1+ra(r))^5}$

$=lim_{r\rightarrow\infty}\frac{1}{a(r)^5}=\frac{1}{f^5}$

where $f=lim_{r\rightarrow\infty}a(r)$.

and, using the result (A);

$lim_{r\rightarrow\infty}r^6\beta_a'=lim_{r\rightarrow\infty}-\frac{5a(r)r^6}{(1+ra(r))^6}-\frac{5r^7a'(r)}{(1+ra(r))^6}$

$=lim_{r\rightarrow\infty}-\frac{5a(r)}{a(r)^6}-\frac{5ra'(r)}{a(r)^6}$

$=-\frac{5}{f^5}$

It follows that, noting that $lim_{r\rightarrow\infty}r^5\beta_a=\frac{1}{f^5}$, $lim_{r\rightarrow\infty}r^6\beta_a'=-\frac{5}{f^5}$ *;*

$lim_{r\rightarrow\infty}P(r) = lim_{r\rightarrow\infty}(\beta^2 r^{10}(5\pi) + \beta'\beta r^{11}(\frac{11\pi}{6}) + \beta'^2 r^{12}(\frac{\pi}{6}))$

$= \frac{1}{f^{10}}(5\pi - \frac{55\pi}{6} + \frac{25\pi}{6}) = 0$

Once we have $\{\overline{E}_0, \overline{B}_0\}$, *we construct* $\{\frac{\partial\overline{E}}{\partial t}|_{t=0}, \frac{\partial\overline{B}}{\partial\partial t}|_{t=0}\}$ *by;*

$\frac{\partial\overline{E}}{\partial t}|_{t=0} = c^2(\bigtriangledown \times \overline{B}_0)$

$\frac{\partial\overline{B}}{\partial t}|_{t=0} = -(\bigtriangledown \times \overline{E}_0)$

so that $\{\frac{\partial\overline{E}}{\partial t}|_{t=0}, \frac{\partial\overline{B}}{\partial t}|_{t=0}\} \subset C^3(\mathcal{R}^3)$. *We then construct* $\{\overline{E}, \overline{B}\}$ *satisfying the wave equations* $\square^2\overline{E} = \overline{0}$, $\square^2\overline{B} = \overline{0}$, *with initial conditions* $\{\overline{E}_0, \overline{B}_0, \frac{\partial\overline{E}}{\partial t}|_{t=0}, \frac{\partial\overline{E}}{\partial t}|_{t=0}\}$ *using Kirchoff's formula, see* [4], *(need initial conditions* C^3, C^2*). As proved in Lemma 0.75,* $(\overline{E}, \overline{B})$ *is a solution to Maxwell's equations in vacuum.*

An interesting consequence of the radiation question concerning light is Planck's heuristic formula $E = hf$. *If the spectrum of the electromagnetic field of light is localised to avoid radiation losses, then we can use Plancherel's formula to conclude that;*

$\int_{\mathcal{R}^3} |\overline{E}|^2 + |\overline{B}^2| d\overline{x} \simeq g|\overline{k}|^2$

so that over a cycle, the energy;

$E \simeq g|\overline{k}|^2 \frac{2\pi}{c|\overline{k}|} = \frac{2\pi g}{c|\overline{k}|} = 4\pi^2 gf$

with the frequency $f = \frac{c|\overline{k}|}{2\pi}$ *and* g *the small bandwidth of the spectrum. During an excitation of a charge and current configuration, with the total charge and current conserved, we can pass through a phase of zero current and charge, in which case the difference in the energies* $E_1 - E_2$ *stored in the electromagnetic fields is interchangeable with the electromagnetic energy of light at a particular frequency. The Balmer series for the difference in electromagnetic energy of current and charge confined to a sphere was predicted in* [14].

Lemma 0.77. *There exists a unique fundamental solution* $(\overline{E}, \overline{0})$, *with* $\overline{E}$ *decaying in the sense of* [11], *for given* $(\rho, \overline{J})$, *not vacuum. Without any decay condition, the difference* $\overline{E} - \overline{E}'$ *of two such solutions* $\{\overline{E}, \overline{E}'\}$, *is either* $\overline{0}$ *or static and unbounded with* $\bigtriangledown \cdot \overline{E} = 0$ *and* $\bigtriangledown \times \overline{E} = \overline{0}$,

$(*)$*, with the possibility* $(*)$ *being satisfiable. If* $(\overline{E}_0, \overline{B}_0)$ *is a solution to Maxwell's equation in vacuum, then we cannot have that* $\overline{E} + \overline{E}_0 = \overline{0}$.

Proof. Suppose there exist two fundamental solutions $(\overline{E}, \overline{0})$ and $(\overline{E}', \overline{0})$, then $(0, \overline{0}, \overline{E} - \overline{E}', \overline{0})$ is a solution to Maxwell's equations in vacuum. It follows from Maxwell's fourth equation, that;

$$\frac{\partial(\overline{E}-\overline{E}')}{\partial t} = \overline{0}$$

and, from the relations in Lemma 4.1 of [12], that;

$$\square^2(\overline{E} - \overline{E}') = \bigtriangledown^2(\overline{E} - \overline{E}') = 0$$

By the decaying condition and properties of harmonic functions, we have that $\overline{E} - \overline{E}' = \overline{0}$, so that $\overline{E} = \overline{E}'$. Without the decay condition, we must have that $\overline{E} - \overline{E}'$ is unbounded or $\overline{E} - \overline{E}' = \overline{0}$, and from Maxwell's first and second equations, we must have that $\bigtriangledown \cdot \overline{E} = 0$ and $\bigtriangledown \times \overline{E} = \overline{0}$ as well. The satisfiable claim follows from the fact that we can construct a solution $(0, \overline{0}, \overline{E}_0, \overline{0})$ to Maxwell's equations in free space, by the requirements that;

(i). $\bigtriangledown \cdot \overline{E}_0 = 0$

(ii). $\frac{\partial \overline{E}_0}{\partial t} = \overline{0}$

(iii). $\bigtriangledown \times \overline{E}_0 = \overline{0}$

It is possible to satisfy the requirements $(i), (iii)$, for a function $\overline{f} : \mathcal{R}^3 \rightarrow \mathcal{R}$, so that we can define $\overline{E}_0(\overline{x}, t) = \overline{f}(\overline{x})$ to satisfy the conditions $(i), (ii), (iii)$. For the last claim, suppose that $\overline{E} + \overline{E}_0 = \overline{0}$, then $\overline{E} = -\overline{E}_0$ and we have that, by Maxwell's equations, and $(\overline{E}_0, \overline{B}_0)$ a vacuum solution;

$$\bigtriangledown \cdot \overline{E} = -\bigtriangledown \cdot \overline{E}_0 = \frac{\rho}{\epsilon_0} = 0$$

so that $\rho = 0$. Using the fact that $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$ and $\square^2 \overline{J} = \overline{0}$, we have that $\frac{\partial \overline{J}}{\partial t} = \overline{0}$ and $\bigtriangledown^2 \overline{J} = \overline{0}$, so that, as $\overline{J} \in S(\mathcal{R}^3)$, we must have that $\overline{J} = \overline{0}$ and $(\rho, \overline{J})$ is a vacuum solution, contradicting the hypotheses. $\square$

References

[1] Algebraic Geometry for Scientists and Engineers, S. Abhyankar, AMS, Volume 35, (1990).

[2] On convergence and growth of partial sums of Fourier series, L. Carlsen, Acta Mathematica, 116 (1): 135157, (1966).

[3] Basics of Bessel Functions, Joella Deal, Portland State University Thesis, (2018).

[4] Partial Differential Equations, L. C. Evans, Graduate Studies in Mathematics, Volume 19, AMS, (1998).

[5] Fourier Analysis and its Applications, Folland, G. B., Wadsworth, (1992).

[6] An Introduction to Electrodynamics, Third Edition, D. Griffiths, Pearson Education, (2008)

[7] Fourier Analysis, T. Korner, CUP, (1988)

[8] Maths Stack Exchange; https://math.stackexchange.com/questions/3310890/integral-of-exp-over-the-unit-ball

[9] Maths Stack Exchange; https://math.stackexchange.com/questions/291877/bounding-the-solution-of-a-wave-equation-in-3-dimensions

[10] Simple Proofs of the Riemann-Lebesgue Lemma using Nonstandard Analysis, Tristram de Piro, available at http://www.curvalinea.net (51), (2019)

[11] Some Arguments for the Wave Equation in Quantum Theory 2, Tristram de Piro, Open Journal of Mathematical Sciences, available at http://www.curvalinea.net (59), (2022)

[12] Some Arguments for the Wave Equation in Quantum Theory 3, Tristram de Piro, submitted to the Open Journal of Mathematical Sciences, available at http://www.curvalinea.net (63), (2022)

[13] A Simple Proof of the Fourier Inversion Theorem Using Nonstandard Analysis, Tristram de Piro, available at http://www.curvalinea.net (19), (2013)

[14] Some Arguments for the Wave Equation in Quantum Theory, Tristram de Piro, Open Journal of Mathematical Sciences, available at http://www.curvalinea.net (58), (2021)

[15] The Continuity Equation and Particle Motion, Tristram de Piro, in progress, (2022)

[16] College Calculus with Analytic Geometry, Protter, Murray H.; Morrey, Charles B. Jr, (2nd ed.), Reading: Addison-Wesley, (1970)

[17] Fourier Analysis, An Introduction, E. Stein and R. Sakarchi, Princeton Lectures in Mathematics, (2003).

[18] Introfuction to Fourier Analysis on Euclidean Spaces, E. Stein and G. Weiss, Princeton University Press, (1971).

Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP
E-mail address: `t.depiro@curvalinea.net`