A NOTE ON POLAR AND CARTESIAN DERIVATIVES

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Abstract. We give a formula for the n'th partial radial derivative from polar coordinates, in terms of weight n cartesian partial derivatives.

Lemma 0.1. *Let $f \in C^\infty(\mathcal{R}^3)$, with the conversion from cartesian to polar coordinates given by;*

$$x = rsin(\theta)cos(\phi),\ y = rsin(\theta)sin(\phi),\ z = rcos(\theta)$$

for $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$, see [1]. Then, for $n \in \mathcal{N}$, with;

$$R_n = \{(n_1, n_2, n_3) \in \mathcal{N}^3, n_1 + n_2 + n_3 = n\}$$

we have that;

$$r^n \frac{\partial^n f(r,\theta,\phi)}{\partial r^n} = \sum_{(n_1,n_2,n_3)\in R_n} \frac{n!}{n_1!n_2!n_3!} \frac{\partial^n f(x,y,z)}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} x^{n_1} y^{n_2} z^{n_3}$$

Proof. The proof is by induction. The case $n = 1$ follows from the fact that;

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$= \frac{\partial f}{\partial x} sin(\theta)cos(\phi) + \frac{\partial f}{\partial y} sin(\theta)sin(\phi) + \frac{\partial f}{\partial z} cos(\theta)$$

$$= \frac{\partial f}{\partial x}\frac{x}{r} + \frac{\partial f}{\partial y}\frac{y}{r} + \frac{\partial f}{\partial z}\frac{z}{r}$$

so that;

$$r\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z$$

Suppose, inductively, that;

$$\frac{\partial^n f}{\partial r^n} = \frac{1}{r^n} \sum_{(n_1,n_2,n_3)\in R_n} \frac{n!}{n_1!n_2!n_3!} \frac{\partial^n f}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} x^{n_1} y^{n_2} z^{n_3}$$

Then, using the product rule;

$$\frac{\partial^{n+1}f}{\partial r^{n+1}}=\left(\frac{x}{r}\frac{\partial f}{\partial x}+\frac{y}{r}\frac{\partial f}{\partial y}+\frac{z}{r}\frac{\partial f}{\partial z}\right)\left(\frac{1}{r^n}\sum_{(n_1,n_2,n_3)\in R_n}\frac{n!}{n_1!n_2!n_3!}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}x^{n_1}y^{n_2}z^{n_3}\right)$$

$$=\sum_{(n_1,n_2,n_3)\in R_n}\frac{n!}{n_1!n_2!n_3!}\Big[\Big(\frac{(r^2n_1x^{n_1}-nx^{n_1+2})y^{n_2}z^{n_3}}{r^{n+3}}+\frac{(r^2n_2y^{n_2}-ny^{n_2+2})x^{n_1}z^{n_3}}{r^{n+3}}$$

$$+\frac{(r^2n_3z^{n_3}-nz^{n_3+2})x^{n_1}y^{n_2}}{r^{n+3}}\Big)\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}+\frac{x^{n_1+1}y^{n_2}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1+1}\partial y^{n_2}\partial z^{n_3}}+\frac{x^{n_1}y^{n_2+1}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2+1}\partial z^{n_3}}$$

$$+\frac{x^{n_1}y^{n_2}z^{n_3+1}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3+1}}\Big]$$

$$=\sum_{(n_1,n_2,n_3)\in R_n}\frac{n!}{n_1!n_2!n_3!}\Big[\Big(\frac{((x^2+y^2+z^2)n_1x^{n_1}-nx^{n_1+2})y^{n_2}z^{n_3}}{r^{n+3}}+\frac{((x^2+y^2+z^2)n_2y^{n_2}-ny^{n_2+2})x^{n_1}z^{n_3}}{r^{n+3}}$$

$$+\frac{((x^2+y^2+z^2)n_3z^{n_3}-nz^{n_3+2})x^{n_1}y^{n_2}}{r^{n+3}}\Big)\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}+\frac{x^{n_1+1}y^{n_2}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1+1}\partial y^{n_2}\partial z^{n_3}}$$

$$+\frac{x^{n_1}y^{n_2+1}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2+1}\partial z^{n_3}}+\frac{x^{n_1}y^{n_2}z^{n_3+1}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3+1}}\Big]$$

$$=\sum_{(n_1,n_2,n_3)\in R_n}\frac{n!}{n_1!n_2!n_3!}\Big[\frac{1}{r^{n+3}}\big[(n_1x^{n_1+2}y^{n_2}z^{n_3}+n_2x^{n_1}y^{n_2+2}z^{n_3}+n_3x^{n_1}y^{n_2}z^{n_3+2}$$

$$-nx_1^{n+2}y^{n_2}z^{n_3}-nx^{n_1}y^{n_2+2}z^{n_3}-nx^{n_1}y^{n_2}z^{n_3+2})+(n_1x^{n_1}y^{n_2+2}z^{n_3}$$

$$+n_1x^{n_1}y^{n_2}z^{n_3+2}+n_2x^{n_1+2}y^{n_2}z^{n_3}+n_2x^{n_1}y^{n_2}z^{n_3+2}+n_3x^{n_1+2}y^{n_2}z^{n_3}$$

$$+n_3x^{n_1}y^{n_2+2}z^{n_3})\big]\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}+\frac{x^{n_1+1}y^{n_2}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1+1}\partial y^{n_2}\partial z^{n_3}}$$

$$+\frac{x^{n_1}y^{n_2+1}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2+1}\partial z^{n_3}}+\frac{x^{n_1}y^{n_2}z^{n_3+1}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3+1}}\Big]$$

$$=\sum_{(n_1,n_2,n_3)\in R_n}\frac{n!}{n_1!n_2!n_3!}\Big[\frac{1}{r^{n+3}}\big[-((n_2+n_3)x^{n_1+2}y^{n_2}z^{n_3}+(n_1+n_3)x^{n_1}y^{n_2+2}z^{n_3}$$

$$+(n_1+n_2)x^{n_1}y^{n_2}z^{n_3+2})+(n_1x^{n_1}y^{n_2+2}z^{n_3}+n_1x^{n_1}y^{n_2}z^{n_3+2}+n_2x^{n_1+2}y^{n_2}z^{n_3}$$

$$+n_2x^{n_1}y^{n_2}z^{n_3+2}+n_3x^{n_1+2}y^{n_2}z^{n_3}+n_3x^{n_1}y^{n_2+2}z^{n_3})\big]\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}+\frac{x^{n_1+1}y^{n_2}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1+1}\partial y^{n_2}\partial z^{n_3}}$$

$$+\frac{x^{n_1}y^{n_2+1}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2+1}\partial z^{n_3}}+\frac{x^{n_1}y^{n_2}z^{n_3+1}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3+1}}\Big]$$

$$=\sum_{(n_1,n_2,n_3)\in R_n}\frac{n!}{n_1!n_2!n_3!}\Big[\frac{x^{n_1+1}y^{n_2}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1+1}\partial y^{n_2}\partial z^{n_3}}$$

$$+\frac{x^{n_1}y^{n_2+1}z^{n_3}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2+1}\partial z^{n_3}}+\frac{x^{n_1}y^{n_2}z^{n_3+1}}{r^{n+1}}\frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3+1}}\Big]$$

$$=\sum_{(m_1,m_2,m_3)\in R_{n+1}}\left(\frac{n!}{(m_1-1)!m_2!m_3!}+\frac{n!}{m_1!(m_2-1)!m_3!}+\frac{n!}{m_1!m_2!(m_3-1)!}\right)\frac{x^{m_1}y^{m_2}z^{m_3}}{r^{n+1}}\frac{\partial^{n+1}f}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}}$$

$$=\sum_{(m_1,m_2,m_3)\in R_{n+1}}\frac{n!(m_1+m_2+m_3)}{m_1!m_2!m_3!}\frac{x^{m_1}y^{m_2}z^{m_3}}{r^{n+1}}\frac{\partial^{n+1}f}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}}$$

$= \sum_{(n_1,n_2,n_3)\in R_{n+1}} \frac{(n+1)!}{n_1!n_2!n_3!} \frac{x^{n_1}y^{n_2}z^{n_3}}{r^{n+1}} \frac{\partial^n f}{\partial x^{n_1}\partial y^{n_2}\partial z^{n_3}}$

$\square$

References

[1] Vector Analysis, D. Bourne and P. Kendall, Oldbourne Mathematical Series, (1967).

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