RATE LAWS AND COLLISION THEORY

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Abstract.

We begin with the reaction rate formula, given in [5];

$\xi'(0) = \frac{\alpha_1 Q(T,P)(grad(Q)(T,P)\centerdot\gamma'_{12}(0))}{grad(Q)(T,P)\centerdot\gamma'_{12}(0)-cQ(T,P)}$ $(*)$

where;

$\alpha_1 = \frac{\beta^{c+1}}{\prod_{i=1}^{c} n_i}$, $\beta = \sum_{i=1}^{c} n_i$

c is the number of substances, n_i, for $1 \leq i \leq c$ are the molar amounts, Q is the equilibrium coefficient and γ is the reaction path, with $\gamma_{12}(0) = (T,P)$. As we do in the paper [5], we can write $\gamma'_{12}(0) = \lambda(cos(\theta), sin(\theta))$, and we noted that $\xi'(0)$ is monotonic in λ, so we can assume that λ is large. Then;

$\xi'(0) = \frac{\alpha_1 \lambda Q(T,P)(grad(Q)(T,P)\centerdot(cos(\theta),sin(\theta)))}{\lambda grad(Q)(T,P)\centerdot(cos(\theta),sin\theta)-cQ(T,P)}$

$= \frac{\alpha_1 Q(T,P)(grad(Q)(T,P)\centerdot(cos(\theta),sin(\theta)))}{grad(Q)(T,P)\centerdot(cos(\theta),sin\theta)-\frac{cQ(T,P)}{\lambda}}$

$\simeq \frac{\alpha_1 Q(T,P)(grad(Q)(T,P)\centerdot(cos(\theta),sin(\theta)))}{grad(Q)(T,P)\centerdot(cos(\theta),sin\theta)}$

$= \alpha_1 Q(T,P)$ $(**)$

We can expand $(**)$ as;

$\xi'(0) = Q(T,P)\frac{(\sum_{i=1}^{c} n_i)^{c+1}}{\prod_{i=1}^{c} n_i}$

$= \frac{Q(T,P)}{\prod_{i=1}^{c} n_i}(\sum_{i_1+\ldots i_j+\ldots i_c=c+1} C_{i_1}^{c+1} C_{i_2}^{c+1-i_1} \ldots C_{i_{j+1}}^{c+1-i_1-\ldots i_j} \ldots C_{i_c}^{c+1-i_1-\ldots-i_{c-1}} \prod_{j=1}^{c} n_j^{i_j})$

$= \frac{Q(T,P)}{\prod_{i=1}^{c} n_i}(\sum_{i_1+\ldots i_j+\ldots i_c=c+1} \frac{(c+1)!}{i_1!\ldots i_j!\ldots i_c!} \prod_{j=1}^{c} n_j^{i_j})$

$= Q(T,P)(\sum_{i_1+\ldots i_j+\ldots i_c=c+1} \frac{(c+1)!}{i_1!\ldots i_j!\ldots i_c!} \prod_{j=1}^{c} n_j^{i_j-1})$

$= Q(T,P)(\sum_{\mu_1+\ldots\mu_j+\ldots\mu_c=1,\mu_j\geq -1} \frac{(c+1)!}{(\mu_1+1)!\ldots(\mu_j+1)!\ldots(\mu_c+1)!} \prod_{j=1}^{c} n_j^{\mu_j})$

$= \sum_{\mu_1+\ldots\mu_j+\ldots\mu_c=1,\mu_j\geq -1} k_{\mu_1,\ldots,\mu_j,\ldots\mu_c}(T,P) \prod_{j=1}^{c} n_j^{\mu_j}\ (***)$

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where $Q(T,P)$ is the equilibrium constant and;

$k_{\mu_1,\ldots,\mu_j,\ldots\mu_c}(T,P) = \frac{(c+1)!Q(T,P)}{(\mu_1+1)!\ldots(\mu_j+1)!\ldots(\mu_c+1)!}$

For ideal and dilute solutions, we obtained in [5], the explicit formula for $Q(T,P)$;

$Q(T,P) = e^{\frac{\epsilon ln(\frac{P}{P^{\circ}})-\epsilon(T,P)}{RT}}$

where ϵ is a constant and $\epsilon(T,P)$ is an error term. If we denote the molar activation energy by $E_a = \epsilon(T,P) - \epsilon ln(\frac{P}{P^{\circ}})$, so that;

$Q(T,P) = e^{\frac{-E_a}{RT}}$

Then $(***)$ includes the Arrhenius relation in the rate constant and provides a general rate law. We want to recover a version of this formula using collision theory, based on probability, rather than thermodynamics. We consider an elementary reaction involving two substances, which we model as ideal gases, by allowing the motion of molecules to be random. We use the work in [6] as a basis for the definitions. We start with a 1-dimensional model, generalising to 3-dimensions later.

Definition 0.1. *Let* $\eta \in {}^*\mathcal{N} \setminus \mathcal{N}$, *be infinite and odd, and let* $\nu = \frac{\eta^2}{2}$, $\nu \in {}^*\mathcal{Q}_{\geq 0} \setminus \mathcal{Q}$. *We let;*

$\overline{\Omega_\eta} = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$

with the nonstandard measure μ_η, *defined by* $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$, *for* $0 \leq i \leq \eta - 1$. *We let* $L(\mu_\eta)$ *be the corresponding Loeb measure.*

Let $\overline{\Omega}_{\eta_{even}} = \{\frac{i}{\eta} : 0 \leq i \leq \eta - 1, i\ even\}$

with the corresponding counting measure μ_η, *defined by* $\mu_\eta(\frac{i}{\eta}) = \frac{1}{\eta}$, *for* $0 \leq i \leq \eta - 1$, *i even, nd Loeb measure* $L(\mu_\eta)$.

$\overline{\Omega_{\eta_{odd}}} = \{\frac{i}{\eta} : 0 \leq i \leq \eta - 1, i\ odd\}$

with the corresponding counting measure μ_η*, defined by* $\mu_\eta(\frac{i}{\eta}) = \frac{1}{\eta}$*, for* $0 \leq i \leq \eta - 1$*,* i *odd, and Loeb measure* $L(\mu_\eta)$*. We let;*

$\overline{\mathcal{T}_\nu} = \{t \in {}^*\mathcal{R}_{\geq 0}\}$

with counting measure μ_ν *and corresponding Loeb measure* $L(\mu_\nu)$*.*

$\overline{\Omega}_\kappa = \{(s_i) : 1 \leq i \leq \kappa, s_i = 1\ or\ -1\}$

so that ${}^*Card(\overline{\Omega}_\kappa) = 2^\kappa$*, with corresponding counting measure* μ_κ*,* $\mu_\kappa(s) = \frac{1}{2^\kappa}$*, and Loeb measure* $L(\mu_\kappa)$*, We let;*

$\omega_i : \overline{\Omega}_\kappa \rightarrow \{1, -1\}$*, for* $1 \leq i \leq \kappa$*, be defined by;*

$\omega_i(s) = s_i$

We let;

$\overline{\mathcal{T}_{\nu,\kappa}} = \{t \in \overline{\mathcal{T}_\nu} : 0 \leq [\nu t] \leq \kappa\}$

We let $\chi : \overline{\Omega}_\kappa \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega}_\eta$*, be defined by;*

$\chi(s,t) = \frac{1}{\eta}({}^*\sum_{j=1}^{[\nu t]} \omega_j(s))\ mod[0,1)$, $1 \leq [\nu t] \leq \kappa$

$\chi(s,0) = 0$

with corresponding ${}^\circ\chi(s,t) = (\frac{1}{\eta}({}^*\sum_{j=1}^{[\nu t]} \omega_j(s))\ mod[0,1])^\circ$

We let $\overline{\chi}_{even} : \overline{\Omega_{\eta_{even}}} \times \overline{\Omega_\kappa} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega_\eta}$ *be defined by;*

$\overline{\chi}_{even}(x,s,t) = x + 2\chi(s,t)\ mod[0,1)$

with corresponding ${}^\circ\overline{\chi}_{even} = (x + 2\chi(s,t)\ mod[0,1))^\circ$

We let $\overline{\chi}_{odd} : \overline{\Omega_{\eta_{odd}}} \times \overline{\Omega_\kappa} \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega_\eta}$ *be defined by;*

$\overline{\chi}_{odd}(x,s,t) = x + 2\chi(s,t)\ mod[0,1)$

with corresponding ${}^\circ\overline{\chi}_{even} = (x + 2\chi(s,t)\ mod[0,1))^\circ$

We define the hitting pairing time $T : \overline{\Omega_{\eta_{even}}} \times \overline{\Omega_\kappa} \times \overline{\Omega_{\eta_{odd}}} \times \overline{\Omega_\kappa} \rightarrow \overline{\mathcal{T}_{\nu,\kappa}}$ *by;*

$T(x, s_1, y, s_2) = \mu^{\circ t}({}^\circ\overline{\chi}_{even}(x, s_1, t) = {}^\circ\overline{\chi}_{odd}(y, s_2, t))$, [1]

We extend the measure μ_κ *to* $\overline{\Omega}^2_\kappa$, *by letting* $\mu_\kappa(s_1, s_2) = \frac{1}{2^{2\kappa}}$. *We denote by* $L(\mu_\kappa)$ *again the corresponding Loeb measure.*

We let $\chi_{ext,1} : \overline{\Omega}^2_\kappa \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega}_\eta$, *be defined by;*

$\chi_{ext,1}(s_1, s_2, t) = \frac{1}{\eta}({}^*\sum_{j=1}^{[\nu t]} \omega_j(s_1))$, $1 \leq [\nu t] \leq \kappa$

$\chi_{ext,1}(s_1, s_2, 0) = 0$

We let $\chi_{ext,2} : \overline{\Omega}^2_\kappa \times \overline{\mathcal{T}_{\nu,\kappa}} \rightarrow \overline{\Omega}_\eta$, *be defined by;*

$\chi_{ext,2}(s_1, s_2, t) = \frac{1}{\eta}({}^*\sum_{j=1}^{[\nu t]} \omega_j(s_2))$, $1 \leq [\nu t] \leq \kappa$

$\chi_{ext,2}(s_1, s_2, 0) = 0$

with corresponding ${}^\circ\chi_{ext,1}$ *and* ${}^\circ\chi_{ext,2}$.

Lemma 0.2. *For* $\{t_1, t_2\} \subset {}^*\mathcal{T}_{\nu,\kappa}$, *the random variables* $\chi_{ext,1,t_1}$ *and* $\chi_{ext,2,t_2}$ *are* $*$*-independent, and the random variables* ${}^\circ\chi_{ext,1,t_1}$ *and* ${}^\circ\chi_{ext,2,t_2}$ *are independent. The processes* ${}^\circ\chi_{ext,1,t}$ *and* ${}^\circ\chi_{ext,2,t}$ *are rescaled Brownian motion by a factor of* $\frac{1}{\sqrt{2}}$. *The process* $B_t = {}^\circ\chi_{ext,1,t} - {}^\circ\chi_{ext,2,t}$ *is Brownian motion.*

Proof. Choose $\lambda_1, \lambda_2 \subset {}^*\mathcal{R}$, then;

$\mu_\kappa(\{(s_1, s_2) : \chi_{ext,1,t_1}(s_1, s_2) \leq \lambda_1, \chi_{ext,2,t_2}(s_1, s_2) \leq \lambda_2\})$

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[1] The set ${}^\circ\overline{\chi}_{even}(x, s_1, t) = {}^\circ\overline{\chi}_{odd}(y, s_2, t)$ is $L(\mu_\nu)$ measurable in $\mathcal{T}_{\nu,\kappa}$, as the intersection of internal sets $\bigcap_{n\in\mathcal{N}} |\overline{\chi}_{even}(x, s_1, t) - \overline{\chi}_{odd}(y, s_2, t)| < \frac{1}{n}$. Each set in the intersection has an infimum t_n, and we obtain an increasing bounded sequence $\{t_n : n \in \mathcal{N}\}$. The set $\{{}^\circ t_n : n \in \mathcal{N}\}$ is increasing and bounded, so has a limit, which we denote by $\mu^{\circ t}$.

$= \mu_\kappa(\{(s_1, s_2) : \chi_{ext,1,t_1}(s_1) \leq \lambda_1, \chi_{ext,1,t_1}(s_2) \leq \lambda_2\})$

$= \mu_\kappa(\{s_1 : \chi_{ext,1,t_1}(s_1) \leq \lambda_1\})\mu_\kappa(\{s_2 : \chi_{ext,1,t_1}(s_2) \leq \lambda_2\})$

$= \mu_\kappa(\{(s_1, s_2) : \chi_{ext,1,t_1}(s_1, s_2) \leq \lambda_1\})\mu_\kappa(\{(s_1 s_2) : \chi_{ext,1,t_1}(s_2) \leq \lambda_2\})$

For the second claim, choose $\lambda_1, \lambda_2 \subset \mathcal{R}$, then;

$L(\mu_\kappa)(\{(s_1, s_2) : {}^\circ\chi_{ext,1,t_1}(s_1, s_2) \leq \lambda_1, {}^\circ\chi_{ext,2,t_2}(s_1, s_2) \leq \lambda_2\})$

$= L(\mu_\kappa)(\{(s_1, s_2) : {}^\circ\chi_{ext,1,t_1}(s_1) \leq \lambda_1 {}^\circ\chi_{ext,1,t_1}(s_2) \leq \lambda_2\})$

$= L(\mu_\kappa)(\{s_1 : {}^\circ\chi_{ext,1,t_1}(s_1) \leq \lambda_1\})L(\mu_\kappa)(\{s_2 : {}^\circ\chi_{ext,1,t_1}(s_2) \leq \lambda_2\})$

$= L(\mu_\kappa)(\{(s_1, s_2) : {}^\circ\chi_{ext,1,t_1}(s_1, s_2) \leq \lambda_1\})L(\mu_\kappa)(\{(s_1 s_2) : {}^\circ\chi_{ext,1,t_1}(s_2) \leq \lambda_2\})$

The next claim follows from the steps in Chapter 8 of [7], or using [1], noting that the additional factor is not required in the calculation, and using the fact that $\nu = \frac{\eta^2}{2}$. It follows that, for $t_1 < t_2$, the increments ${}^\circ\chi_{ext,1,t_2} - {}^\circ\chi_{ext,1,t_1}$ and ${}^\circ\chi_{ext,2,t_2} - {}^\circ\chi_{ext,2,t_1}$ follow the normal distribution $N(0, \frac{t_2 - t_1}{2})$, with variance $\frac{t_2 - t_1}{2}$. It also follows that, for $t_1 < t_2 \leq t_3 < t_4$, the increments;

${}^\circ\chi_{ext,1,t_2} - {}^\circ\chi_{ext,1,t_1}$ and ${}^\circ\chi_{ext,1,t_4} - {}^\circ\chi_{ext,1,t_3}$ are independent

${}^\circ\chi_{ext,2,t_2} - {}^\circ\chi_{ext,2,t_1}$ and ${}^\circ\chi_{ext,2,t_4} - {}^\circ\chi_{ext,2,t_3}$ are independent, (A)

For the last claim, follow the steps in Theorem 8.8 of [7]. (i) is clear. For (ii), we have, by the above, that the increments ${}^\circ\chi_{ext,1,t_2} - {}^\circ\chi_{ext,1,t_1}$ and ${}^\circ\chi_{ext,2,t_2} - {}^\circ\chi_{ext,2,t_1}$ are independent. In particular the difference of the increments $({}^\circ\chi_{ext,1,t_2} - {}^\circ\chi_{ext,1,t_1}) - ({}^\circ\chi_{ext,2,t_2} - {}^\circ\chi_{ext,2,t_1})$ follows the normal distribution $N(0, t_2 - t_1)$, with variance $t_2 - t_1$, and so do the increments $B_{t_2} - B_{t_1}$. For (iii), we can combine (A) with the argument in the second claim. Letting;

$A = {}^\circ\chi_{ext,1,t_2} - {}^\circ\chi_{ext,1,t_1}$, $B = {}^\circ\chi_{ext,2,t_2} - {}^\circ\chi_{ext,2,t_1}$

$C = {}^\circ\chi_{ext,1,t_4} - {}^\circ\chi_{ext,1,t_3}$, $D = {}^\circ\chi_{ext,2,t_4} - {}^\circ\chi_{ext,2,t_3}$

we have that;

$P(A-B\leq x, C-D\leq y)$

$=\int_{z_1}\int_{z_2}P(B=z_1,D=z_2,A\leq x+z_1,C\leq y+z_2)dz_1dz_2$

$=\int_{z_1}\int_{z_2}P(B=z_1,D=z_2)P(A\leq x+z_1,C\leq y+z_2)dz_1dz_2$

$=\int_{z_1}\int_{z_2}P(B=z_1)P(D=z_2)P(A\leq x+z_1)P(C\leq y+z_2)dz_1dz_2$

$=\int_{z_1}P(B=z_1)P(A\leq x+z_1)dz_1\int_{z_2}P(D=z_2)P(C\leq y+z_2)dz_2$

$=\int_{z_1}P(B=z_1,A\leq x+z_1)dz_1\int_{z_2}P(D=z_2,C\leq y+z_2)dz_2$

$=P(A-B\leq x)P(C-D\leq y)$, ([2])

so that the increments $A-B$ and $C-D$ are independent.

□

Definition 0.3. *For Brownian motion $\{B_t : t\in\mathcal{R}_{\geq 0}\}$, we let τ be a stopping time with two barriers $0<x<1$ and $x-1<0$, so that;*

$\tau = min\{t : B_t = x \ or B_t = x-1\}$

We let τ_1 be the stopping time for the barrier x;

$\tau_1 = min\{t : B_t = x\}$

τ_2 the stopping time for the barrier $1-x$;

$\tau_2 = min\{t : B_t = 1-x\}$

τ_3 the stopping time for the barrier -1;

$\tau_3 = min\{t : B_t = -1\}$

τ_4 the stopping time for the barrier 1;

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[2]For a cumulative density function $F(x,y)=P(X\leq x, Y\leq y)$, by $P(X=x, Y\leq y)$, we mean $\frac{\partial F}{\partial x}(x,y)$

$\tau_4 = min\{t : B_t = 1\}$

Lemma 0.4. *We have that the probability distribution of* τ *is given by;*

$$f_\tau(t) = [-\frac{x}{\sqrt{2\pi t^3}}exp(-\frac{x^2}{2t}) - \frac{1-x}{\sqrt{2\pi t^3}}exp(-\frac{(1-x)^2}{2t})]\int_{v=t}^{\infty}\frac{1}{\sqrt{2\pi(v-t)^3}}exp(\frac{-1}{2(v-t)})dv$$

...Use to calculate expected hitting time on probability space $\overline{\Omega}_\eta^2 \times \overline{\Omega}_\kappa^2$ *and mean free path from velocity distributions, applications to fusion.*

Proof. The distributions of τ_1, τ_2, τ_3 and τ_4 are well known, see [9];

$$f_{\tau_1}(t) = \frac{x}{\sqrt{2\pi t^3}}exp(-\frac{x^2}{2t})$$

$$f_{\tau_2}(t) = \frac{1-x}{\sqrt{2\pi t^3}}exp(-\frac{(1-x)^2}{2t})$$

$$f_{\tau_3}(t) = f_{\tau_4}(t) = \frac{1}{\sqrt{2\pi t^3}}exp(-\frac{1}{2t})$$

We have that, for $t_1 < t_2$;

$$P(\tau_1 = t_1, \tau_2 = t_2) = P(\tau_2 = t_2|\tau_1 = t_1)P(\tau_1 = t_1)$$

$$= P(\tau_3 = t_2 - t_1)P(\tau_1 = t_1)$$

$$= \frac{x}{\sqrt{2\pi t_1^3}}exp(-\frac{x^2}{2t_1})\frac{1}{\sqrt{2\pi(t_2-t_1)^3}}exp(\frac{-1}{2(t_2-t_1)})$$

and for $t_1 > t_2$;

$$P(\tau_1 = t_1, \tau_2 = t_2) = P(\tau_1 = t_1|\tau_2 = t_2)P(\tau_2 = t_2)$$

$$= P(\tau_4 = t_1 - t_2)P(\tau_2 = t_2)$$

$$= \frac{1-x}{\sqrt{2\pi t_2^3}}exp(-\frac{(1-x)^2}{2t_2})\frac{1}{\sqrt{2\pi(t_1-t_2)^3}}exp(\frac{-1}{2(t_1-t_2)})$$

as the increments B_{t_1} and B_{t-t_1} are independent.

It follows that;

$$P(\tau > t) = P(\tau_1 > t, \tau_2 > t)$$

$$= \int_{u=t}^{\infty}\int_{v=u}^{\infty}\frac{x}{\sqrt{2\pi u^3}}exp(-\frac{x^2}{2u})\frac{1}{\sqrt{2\pi(v-u)^3}}exp(\frac{-1}{2(v-u)})dvdu$$

$+\int_{v=t}^{\infty}\int_{u=v}^{\infty}\frac{1-x}{\sqrt{2\pi v^3}}exp(-\frac{(1-x)^2}{2v})\frac{1}{\sqrt{2\pi(u-v)^3}}exp(\frac{-1}{2(u-v)})dudv$

and, using the FTC;

$f_\tau(t)=-\frac{d}{dt}P(\tau>t)$

$=-\int_{v=t}^{\infty}\frac{x}{\sqrt{2\pi t^3}}exp(-\frac{x^2}{2t})\frac{1}{\sqrt{2\pi(v-t)^3}}exp(\frac{-1}{2(v-t)})dv$

$-\int_{u=t}^{\infty}\frac{1-x}{\sqrt{2\pi t^3}}exp(-\frac{(1-x)^2}{2t})\frac{1}{\sqrt{2\pi(u-t)^3}}exp(\frac{-1}{2(u-t)})du$

$=-\frac{x}{\sqrt{2\pi t^3}}exp(-\frac{x^2}{2t})\int_{v=t}^{\infty}\frac{1}{\sqrt{2\pi(v-t)^3}}exp(\frac{-1}{2(v-t)})dv$

$-\frac{1-x}{\sqrt{2\pi t^3}}exp(-\frac{(1-x)^2}{2t})\int_{v=t}^{\infty}\frac{1}{\sqrt{2\pi(v-t)^3}}exp(\frac{-1}{2(v-t)})dv$

$=[-\frac{x}{\sqrt{2\pi t^3}}exp(-\frac{x^2}{2t})-\frac{1-x}{\sqrt{2\pi t^3}}exp(-\frac{(1-x)^2}{2t})]\int_{v=t}^{\infty}\frac{1}{\sqrt{2\pi(v-t)^3}}exp(\frac{-1}{2(v-t)})dv$

□

Lemma 0.5. *Let $\nu>0$ be infinite, $\{a,b\}\subset\mathcal{R}_{>0}$, $B:\overline{\Omega}_\kappa\times\overline{\mathcal{T}}_{\nu,\kappa}\rightarrow{}^*\mathcal{R}$ be nonstandard Brownian motion;*

$B(t,\omega)=\frac{1}{\sqrt{\nu}}{}^*\sum_{i=1}^{[\nu t]}\omega_i$

with stopping times;

$\tau_1=min_{t\in\overline{\mathcal{T}}_{\nu,\kappa}}\{B_t=\frac{[a\sqrt{\nu}]}{\sqrt{\nu}}\}$

$\tau_2=min_{t\in\overline{\mathcal{T}}_{\nu,\kappa}}\{B_t=\frac{-[b\sqrt{\nu}]}{\sqrt{\nu}}\}$

$\tau_3=min_{t\in\overline{\mathcal{T}}_{\nu,\kappa}}\{B_t=-\frac{[b\sqrt{\nu}]}{\sqrt{\nu}}-\frac{[a\sqrt{\nu}]}{\sqrt{\nu}}\}$

then, if $\{t_1,t_2\}\subset\overline{\mathcal{T}}_{\nu,\kappa}$, with $0<t_1<t_2$;

$\mu_\kappa(\tau_1=t_1,\tau_2=t_2)=\mu_\kappa(\tau_1=t_1)\mu_\kappa(\tau_3=t_2-t_1)$

Proof. We have that;

$(\tau_1=t_1,\tau_2=t_2)=\{\omega:B_{t_1}(\omega)=\frac{[a\sqrt{\nu}]}{\sqrt{\nu}},B_t(\omega)\cap\{\frac{[a\sqrt{\nu}]}{\sqrt{\nu}},-\frac{[b\sqrt{\nu}]}{\sqrt{\nu}}\}=$

$\emptyset,0\leq t<t_1,B_t(\omega)\neq-\frac{[b\sqrt{\nu}]}{\sqrt{\nu}},t_1<t<t_2,B_{t_2}(\omega)=-\frac{[b\sqrt{\nu}]}{\sqrt{\nu}}\}$ $(*)$

Let $pr_1 : \overline{\Omega}_\kappa \rightarrow \overline{\Omega}_{[t_1 v]}$ be the projection onto the first $[t_1\nu]$ coordinates, and define $X_{t_1} \subset \overline{\Omega}_{[t_1 v]}$ by $pr_1((\tau_1 = t_1))$. Clearly, we have that $\mu_{[t_1\nu]}(X_{t_1}) = \mu_\kappa(\tau_1 = t_1)$. Let $pr_2 : \overline{\Omega}_\kappa \rightarrow \overline{\Omega}_{[t_2\nu]-[t_1\nu]}$ be the projection onto the first $[t_2\nu] - [t_1\nu]$ coordinates , and define $X_{t_1,t_2} \subset \overline{\Omega}_{[t_2\nu]-[t_1\nu]}$ by $pr_2((\tau_3 = t_2 - t_1))$. Clearly, we have that $\mu_{[t_2\nu]-[t_1\nu]}(X_{t_1,t_2}) = \mu_\kappa(\tau_3 = t_2 - t_1)$. Let $pr_3 : \overline{\Omega}_\kappa \rightarrow \overline{\Omega}_{[t_2\nu]-[t_1\nu]}$ be the projection onto coordinates $[t_1\nu]+1$ to $[t_2\nu]$, then we have that, by $(*)$;

$$\omega \in (\tau_1 = t_1, \tau_2 = t_2) \text{ iff } pr_1(\omega) \in X_{t_1} \text{ and } pr_3(\omega) \in X_{t_1,t_2}$$

Let $pr_4 : \overline{\Omega}_\kappa \rightarrow \overline{\Omega}_{[t_2\nu]}$ be the projection onto the first $[t_2\nu]$ coordinates, and let $X_{t_2} = pr_4(\tau_1 = t_1, \tau_2 = t_2)$, then;

$$\mu_\kappa(\tau_1 = t_1, \tau_2 = t_2) = \mu_{[t_2\nu]}(X_{t_2})$$

$$= \frac{{}^*Card(X_{t_2})}{2^{[t_2\nu]}}$$

$$= \frac{{}^*Card(X_{t_1}){}^*Card(X_{t_1,t_2})}{2^{[t_1\nu]}2^{[t_2\nu]-[t_1\nu]}}$$

$$= \mu_{[t_1\nu]}(X_{t_1})\mu_{[t_2\nu]-[t_1\nu]}(X_{t_1,t_2})$$

$$= \mu_\kappa(\tau_1 = t_1)\mu_\kappa(\tau_3 = t_2 - t_1)$$

□

Definition 0.6. *Let $f : \mathcal{R}^2 \rightarrow \mathcal{R}$, in the variables (t, x) be analytic, such that, on a bounded region $V \subset \mathcal{R}^2$, all the partial derivatives $\frac{\partial^{i+j} f}{\partial x^i \partial t^j} \leq E_V i!j!$, for some $E_V \in \mathcal{R}$, with transfer $f^* : {}^*\mathcal{R}^2 \rightarrow {}^*\mathcal{R}$, let $B : \overline{\Omega}_\kappa \times \overline{\mathcal{T}}_{\nu,\kappa} \rightarrow {}^*\mathcal{R}$ be nonstandard Brownian motion, and let $g : \overline{\Omega}_\kappa \times \overline{\mathcal{T}}_{\nu,\kappa} \rightarrow {}^*\mathcal{R}$ be defined by;*

$$g(t, \omega) = f^*(\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))$$

We define;

$$dg_{\frac{[t\nu]}{\nu}}(\omega) = g(\omega, \tfrac{[t\nu]+1}{\nu}) - g(\omega, \tfrac{[t\nu]}{\nu})$$

$$dt = \tfrac{1}{\nu}$$

$$dB_{\frac{[t\nu]}{\nu}}(\omega) = \frac{\omega_{\frac{[t\nu]+1}{\nu}}}{\sqrt{\nu}}$$

We define the nonstandard derivatives;

$$\frac{\partial f^*}{\partial t}|_{\frac{[t\nu]}{\nu},\omega}=\frac{\partial f^*}{\partial t}|_{\frac{[t\nu]}{\nu},B_{\frac{[t\nu]}{\nu}}(\omega)}=\nu(f^*(\frac{[t\nu]+1}{\nu},B_{\frac{[t\nu]}{\nu}}(\omega))-f^*(\frac{[t\nu]}{\nu},B_{\frac{[t\nu]}{\nu}}(\omega)))$$

$$\frac{\partial f^*}{\partial B_t}|_{\frac{[t\nu]}{\nu},\omega}=\frac{\partial f^*}{\partial B_t}|_{\frac{[t\nu]}{\nu},B_{\frac{[t\nu]}{\nu}}(\omega)}=\nu(f^*([t\nu],B_{\frac{[t\nu]+1}{\nu}}(\omega))-f^*(\frac{[t\nu]}{\nu},B_{\frac{[t\nu]}{\nu},\omega}))$$

$$(\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega}=(\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},B_{\frac{[t\nu]}{\nu}}(\omega)}=\frac{\partial f}{\partial x}^*|_{(\omega,\frac{[t\nu]}{\nu})}$$

$$(\frac{\partial^2 f}{\partial B_t^2})^*|_{\frac{[t\nu]}{\nu},\omega}=(\frac{\partial^2 f}{\partial B_t^2})^*|_{\frac{[t\nu]}{\nu},B_{\frac{[t\nu]}{\nu}}(\omega)}=\frac{\partial^2 f}{\partial x^2}^*|_{\omega,\frac{[t\nu]}{\nu}}$$

We define the filtration $\{\mathcal{F}_{\frac{i}{\nu}}:0\leq i\leq\kappa\}$ *on* $\overline{\Omega}_\kappa$ *by letting* $\mathcal{F}_{\frac{i}{\nu}}$ *be generated as a* $*$-σ *algebra by the basic sets;*

$$U_{\overline{k}_i}=\{\overline{\omega}\in\overline{\Omega}_\kappa:(\overline{\omega}(j))_{1\leq j\leq i}=\overline{k}_i\}$$

where $\overline{k}_i$ *is a sequence of* 1*'s and* -1*'s of length* i*.*

We say that a process $M:\overline{\Omega}_\kappa\times\overline{\mathcal{T}}_{\nu,\kappa}\rightarrow{}^*\mathcal{R}$ *is adapted to the filtrartion if* M_t *is* $*$*-measurable with respect to* $\mathcal{F}_{\frac{[t\nu]}{\nu}}$*. We define internal integrals by;*

For $t_1<t_2$*;*

$$\int_{t_1}^{t_2}M(t,\omega)dt=\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]}{\nu}}M(t,\omega)dt=\frac{1}{\nu}{}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]}{\nu}}M(\frac{i}{\nu},\omega)$$

For $t_1<t_2$*;*

$$\int_{t_1}^{t_2}M(t,\omega)dB_t=\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]}{\nu}}M(t,\omega)dB_t=\frac{1}{\sqrt{\nu}}{}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]}{\nu}}M(\frac{i}{\nu},\omega)\omega_{i+1}$$

If M_t *is adapted to the filtration, we define;*

$$E(M_t|\mathcal{F}_s)=E(M_{\frac{[t\nu]}{\nu}}|\mathcal{F}_{\frac{[s\nu]}{\nu}})$$

to be the orthogonal projection of $M_{\frac{[t\nu]}{\nu}}$ *onto the* $*$*-subspace of* $*$*-measurable random variables with respect to* $\mathcal{F}_{\frac{[s\nu]}{\nu}}$*, see* [8] *for more details, so that;*

$$E(M_t|\mathcal{F}_0)=E(M_t)=\int_{\overline{\Omega_\kappa}}M_t(\omega)d\mu_\kappa(\omega)$$

We define M_t to be a nonstandard martingale if $E(M_t|\mathcal{F}_s)=M_s$

We define M_t to be a quasi-nonstandard martingale, on $[0,\lambda]$ if for $0\leq\frac{[s\nu]}{\nu}\leq\frac{[t\nu]}{\nu}\leq\frac{[\lambda\nu]}{\nu}$;

$E(M_t|\mathcal{F}_s)\simeq M_s$ *and* $|E(M_t|\mathcal{F}_s)-M_s|\leq\frac{C}{\nu^{\frac{1}{12}}}$

for some $C\in\mathcal{R}$.

Lemma 0.7. *We have that;*

$$dg_{\frac{[t\nu]}{\nu}}(\omega)=((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt+((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}+C_{\frac{[t\nu]}{\nu}}(\omega)$$

where $|C_{\frac{[t\nu]}{\nu}}(\omega)|\leq\frac{C_{\frac{[t\nu]}{\nu},\omega}}{\nu^{\frac{3}{2}}}$ *and* $C_{\frac{[t\nu]}{\nu},\omega}\in\mathcal{R}_{>0}$ *if* $\frac{[t\nu]}{\nu}$ *and* $B_{\frac{[t\nu]}{\nu}}(\omega)$ *are finite.*

There exist $\{\lambda_1,\lambda_2\}\subset{}^*\mathcal{N}$ *infinite, and* $V_{\lambda_1,\lambda_2}\subset\overline{\Omega}_\kappa$, *such that for* $0\leq\frac{[t_1\nu]}{\nu}<\frac{[t_2\nu]}{\nu}\leq\frac{[\lambda_2\nu]}{\nu}$, *with t_1 and t_2 finite,* $\omega\in V_{\lambda_1,\lambda_2}$, *we have that;*

$$g(\tfrac{[t_2\nu]}{\nu},\omega)-g(\tfrac{[t_1\nu]}{\nu},\omega)\simeq\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt+\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}$$

and, moreover;

$$|g(\tfrac{[t_2\nu]}{\nu},\omega)-g(\tfrac{[t_1\nu]}{\nu},\omega)-(\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt+\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}})|$$

$$\leq\nu^{-\frac{1}{4}}$$

with $\mu_\kappa(V_{\lambda_1,\lambda_2})\simeq 1$ *and* $\mu_\kappa(\overline{\Omega}_\kappa\setminus V_{\lambda_1,\lambda_2})\leq\frac{1}{\lambda_1}$;

For g_t constant on $\overline{\Omega}_\kappa\setminus V_{\lambda_1,\lambda_2}$, for t finite, if $((\frac{\partial f}{\partial t})^*|_{\omega,t}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,t})=0$, *for* $0\leq\frac{[t_1\nu]}{\nu}\leq t\leq\frac{[t_2\nu]}{\nu}\leq\lambda_2$, *then;*

$$|E(g_{\frac{[t_2\nu]}{\nu}}-g_{\frac{[t_1\nu]}{\nu}}|\mathcal{F}_{\frac{[t_1\nu]}{\nu}})|\leq\frac{C}{\nu^{\frac{1}{12}}}$$

$$\simeq 0$$

and g_t is a quasi-nonstandard martingale on $[0,T]$, for T finite.

Proof. We have that;

$dg_{\frac{[t\nu]}{\nu}}(\omega) = g(\omega, \frac{[t\nu]+1}{\nu}) - g(\omega, \frac{[t\nu]}{\nu})$

$= f^*(\frac{[t\nu]+1}{\nu}, B_{\frac{[t\nu]+1}{\nu}}(\omega)) - f^*(\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))$

As f is analytic, for $\{t,x\} \subset \mathcal{R}$, $\{h_1, h_2\} \subset \mathcal{R}$, with $max(|h_1|, |h_2|) < \frac{1}{2}$, we have that;

$f(t+h_1, x+h_2) = f(t,x) + h_1 \frac{\partial f}{\partial t}|_{t,x} + h_2 \frac{\partial f}{\partial x}|_{t,x} + \frac{h_2^2}{2}\frac{\partial^2 f}{\partial x^2}|_{t,x}$

$+ \sum_{(i,j):i\geq 1, j\geq 1} \frac{\partial^{i+j} f}{\partial t^i \partial x^j}|_{t,x} \frac{h_1^i h_2^j}{i!j!} + \sum_{i\geq 2} \frac{\partial^i f}{\partial t^i}|_{t,x} \frac{h_1^i}{i!} + \sum_{j\geq 3} \frac{\partial^j f}{\partial x^j}|_{t,x} \frac{h_2^j}{j!}$

so that;

$|f(t+h_1, x+h_2) - f(t,x) - h_1 \frac{\partial f}{\partial t}|_{t,x} - h_2 \frac{\partial f}{\partial x}|_{t,x} - \frac{h_2^2}{2}\frac{\partial^2 f}{\partial x^2}|_{t,x}|$

$\leq M_{t,x}|h_1||h_2| \sum_{(i,j):i\geq 1, j\geq 1} |h_1|^{i-1}|h_2|^{j-1} + M_{t,x}|h_1|^2 \sum_{i\geq 2} |h_1|^{i-2} + M_{t,x}|h_2|^3 \sum_{i\geq 3} |h_1|^{i-3}$

$= M_{t,x}|h_1||h_2| \sum_{(i,j):i\geq 0, j\geq 0} |h_1|^i |h_2|^j + \frac{M_{t,x}|h_1|^2}{1-|h_1|} + \frac{M_{t,x}|h_2|^3}{1-|h_2|}$

$\leq M_{t,x}|h_1||h_2| \sum_{i\geq 0} \frac{|h_1|^i}{1-|h_2|} + 2M_{t,x}|h_1|^2 + 2M_{t,x}|h_2|^3$

$\leq \frac{M_{t,x}|h_1||h_2|}{(1-|h_1|)(1-|h_2|)} + 2M_{t,x}|h_1|^2 + 2M_{t,x}|h_2|^3$

$\leq 4M_{t,x}|h_1||h_2| + 2M_{t,x}|h_1|^2 + 2M_{t,x}|h_2|^3$

By transfer, we obtain that, for $\{t,x\} \subset {}^*\mathcal{R}$, $D \in {}^*\mathcal{R}$, $|(t,x)| \leq D$, $\{h_1, h_2\} \subset {}^*\mathcal{R}_{>0}$, with $max(|h_1|, |h_2|) < \frac{1}{2}$;

$|f^*(t+h_1, x+h_2) - f^*(t,x) - h_1(\frac{\partial f}{\partial t})^*|_{t,x} - h_2(\frac{\partial f}{\partial x})^*|_{t,x} - \frac{h_2^2}{2}(\frac{\partial^2 f}{\partial x^2})^*|_{t,x}|$

$\leq 4M_{t,x}|h_1||h_2| + 2M_{t,x}|h_1|^2 + 2M_{t,x}|h_2|^3$

with $M_{t,x} \leq M_D$, and $M_D \in \mathcal{R}$ if (t,x) is finite, so that, with;

$h_1 = \frac{1}{\nu} < \frac{1}{2}$

$h_2 = B_{\frac{[t\nu]+1}{\nu}} - B_{\frac{[t\nu]}{\nu}} = \frac{\omega_{\frac{[t\nu]+1}{\nu}}}{\sqrt{\nu}}$

$|h_2| \leq \frac{1}{\sqrt{\nu}} < \frac{1}{2}$

$h_2^2 = \frac{1}{\nu}$

$h_1^2 = \frac{1}{\nu^2}$

$|h_2^3 \leq \frac{1}{\nu^{\frac{3}{2}}}$

we have that;

$$|f^*(\tfrac{[t\nu]+1}{\nu}, B_{\frac{[t\nu]+1}{\nu}}(\omega)) - f^*(\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)) - h_1(\tfrac{\partial f}{\partial t})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)) - h_2(\tfrac{\partial f}{\partial x})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))$$

$$-\tfrac{h_2^2}{2}(\tfrac{\partial^2 f}{\partial x^2})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))|$$

$$= |f^*(\tfrac{[t\nu]+1}{\nu}, B_{\frac{[t\nu]+1}{\nu}}(\omega)) - f^*(\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)) - \tfrac{1}{\nu}(\tfrac{\partial f}{\partial t})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))$$

$$-\tfrac{\omega_{\frac{[t\nu]+1}{\nu}}}{\sqrt{\nu}}(\tfrac{\partial f}{\partial x})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)) - \tfrac{1}{2\nu}(\tfrac{\partial^2 f}{\partial x^2})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))|$$

$$= |f^*(\tfrac{[t\nu]+1}{\nu}, B_{\frac{[t\nu]+1}{\nu}}(\omega)) - f^*(\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)) - [(\tfrac{\partial f}{\partial t})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))$$

$$+\tfrac{1}{2}(\tfrac{\partial^2 f}{\partial x^2})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))]dt - (\tfrac{\partial f}{\partial x})^*|_{(}\tfrac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))dB_{\frac{[t\nu]}{\nu}}(\omega)|$$

$$\leq 4M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}|h_1||h_2| + 2M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}|h_1|^2 + 2M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}|h_2|^3$$

$$\leq \frac{4M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}}{\nu^{\frac{3}{2}}} + \frac{2M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}}{\nu^2} + \frac{2M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}}{\nu^{\frac{3}{2}}}$$

$$\leq \frac{6M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}}{\nu^{\frac{3}{2}}}$$

with $M_{\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)}$ finite, if $(\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega))$ is finite.

For the second claim, we can use the result in [1], see also [7], that a.e (V) $L(\mu_\kappa)$, for $\frac{[t_2\nu]}{\nu}$ finite, $0 \leq t \leq \frac{[t_2\nu]}{\nu}$, the map $(\frac{[t\nu]}{\nu}, \omega) \mapsto B_{\frac{[t\nu]}{\nu}}(\omega)$, $(\dagger)$ is near standard and finite. We can approximate V by V_n, $n \in \mathcal{N}$, such that V_n is μ_κ measurable, $V_n \subset V_{n+1} \subset V$, and $\mu_\kappa(\overline{\Omega}_\kappa \setminus V_n) \leq \frac{1}{n}$, then, as the map $(\dagger)$ is internal, $|\frac{[t\nu]}{\nu}, B_{\frac{[t\nu]}{\nu}}(\omega)| \leq M_n$, with $M_n \in \mathcal{R}_{>0}$. By assumption, we can then assume that, for $(i,j) \in \mathcal{Z}^2_{\geq 0}$, $\omega \in V_n$, $0 \leq t \leq \frac{[t_2\nu]}{\nu}$, with t_2 finite, $\frac{|^*\partial^{i+j} f}{\partial t^i \partial x^j} \leq R_n i! j!$, for $|(t,x)| \leq M_n$, with $R_n \in \mathcal{R}_{>0}$. Then, using the previous result, for $\omega \in V_n$, $0 \leq \frac{[t_1\nu]}{\nu} \leq \frac{[t_2\nu]}{\nu}$;

$g(\frac{[t_2\nu]}{\nu},\omega)-g(\frac{[t_1\nu]}{\nu},\omega)={}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}dg_{\frac{i}{\nu}}$

$={}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}[((\frac{\partial f}{\partial t})^*|_{\omega,\frac{i}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{i}{\nu}})dt+((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{i}{\nu}})dB_{\frac{i}{\nu}}+C_{\frac{i}{\nu}}(\omega)]$

$\simeq{}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}[((\frac{\partial f}{\partial t})^*|_{\omega,\frac{i}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{i}{\nu}})dt+((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{i}{\nu}})dB_{\frac{i}{\nu}}]$

$=\frac{1}{\nu}{}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{i}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{i}{\nu}})+\frac{1}{\sqrt{\nu}}{}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{i}{\nu}})\omega_{\frac{i+1}{\nu}}$

$=\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt+\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}$

as for $t_1<t_2$ finite;

$|{}^*\sum_{i=\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}C_{\frac{i}{\nu}}(\omega)|$

$\leq[t_2\nu]max_{0\leq i\leq[t_2\nu]-1}|C_{\frac{i}{\nu}}(\omega)|$

$\leq\frac{R_n[t_2\nu]}{\nu^{\frac{3}{2}}}$

$\leq\frac{\nu^{\frac{5}{4}}}{\nu^{\frac{3}{2}}}$

$=\nu^{-\frac{1}{4}}$

$\simeq 0$

where R_n is the uniform bound in $M_{t,x}$ given above. Fixing $\frac{[t_2\nu]}{\nu}$ finite, letting n vary with $\mu_\kappa(\overline{\Omega}_\kappa\setminus V_{n,\frac{[t_2\nu]}{\nu}})<\frac{1}{n}$, $\overline{\Omega}_\kappa\setminus V_{n,\frac{[t_2\nu]}{\nu}}$ decreasing, we have that;

$\{n\in\mathcal{N}:|g(\frac{[t_2\nu]}{\nu},\omega)-g(\frac{[t_1\nu]}{\nu},\omega)-(\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt$

$+\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}})|\leq\nu^{-\frac{1}{4}},for\ \omega\in V_{n,\frac{[t_2\nu]}{\nu}},0\leq\frac{[t_1\nu]}{\nu}\leq$

$\frac{[t_2\nu]}{\nu}\}$

contains $\mathcal{N}$, so by overflow, contains $\lambda_1\in{}^*\mathcal{N}$ infinite, and we find $V_{\lambda_1,\frac{[t_2\nu]}{\nu}}$ with $\mu_\kappa(\overline{\Omega}_\kappa\setminus V_{\lambda_1,\frac{[t_2\nu]}{\nu}})<\frac{1}{\lambda_1}$, such that, for $\omega\in V_{\lambda_1,\frac{[t_2\nu]}{\nu}}$;

$|g(\frac{[t_2\nu]}{\nu},\omega)-g(\frac{[t_1\nu]}{\nu},\omega)-(\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt$

$+\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}})| \leq \nu^{-\frac{1}{4}}$ (X)

for $0 \leq \frac{[t_1\nu]}{\nu} \leq \frac{[t_2\nu]}{\nu}$. We then have that, for $m \in \mathcal{N}$, the statement (X) holds for $V_{\lambda_1,m}$, so that by overflow again, we can find $\lambda_2 \in {}^*\mathcal{N}$, such that (X) holds for V_{λ_1,λ_2}. In particular, $\mu_\kappa(\overline{\Omega}_\kappa \setminus V_{\lambda_1,\lambda_2}) < \frac{1}{\lambda_1} \simeq 0$

For the final claim, if $((\frac{\partial f}{\partial t})^*|_{\omega,t} + \frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,t}) = 0$, for $\frac{[t_1\nu]}{\nu} \leq t \leq \frac{[t_2\nu]}{\nu} \leq \frac{[\lambda_2\nu]}{\nu}$, then, by the second claim, for $\omega \in V_{\lambda_1,\frac{[\lambda_2\nu]}{\nu}}$;

$g(\frac{[t_2\nu]}{\nu},\omega)-g(\frac{[t_1\nu]}{\nu},\omega) \simeq \int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial t})^*|_{\omega,\frac{[t\nu]}{\nu}}+\frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\omega,\frac{[t\nu]}{\nu}})dt+\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}$

$= \int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}$ (D)

whereas, if $\frac{[t_1\nu]}{\nu} \leq t \leq \frac{[t_2\nu]}{\nu}$, with t_1, t_2 finite, as g_t is constant, for $\omega \in \overline{\Omega}_\kappa \setminus V_{\lambda_1,\lambda_2}$;

$g(\frac{[t_2\nu]}{\nu},\omega) - g(\frac{[t_1\nu]}{\nu},\omega) = 0$ (C)

It follows, using the method of [8], Lemma 0.13, and $(C),(D)$;

$E(g_{\frac{[t_2\nu]}{\nu}} - g_{\frac{[t_1\nu]}{\nu}}|\mathcal{F}_{\frac{[t_1\nu]}{\nu}})$

$\simeq E(\int_{\frac{[t_1\nu]}{\nu}}^{\frac{[t_2\nu]-1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}|\mathcal{F}_{\frac{[t_1\nu]}{\nu}})$

$= 0$

with $|E(g_{\frac{[t_2\nu]}{\nu}} - g_{\frac{[t_1\nu]}{\nu}}|\mathcal{F}_{\frac{[t_1\nu]}{\nu}})| \leq \frac{1}{\nu^{\frac{1}{4}}} \leq \frac{1}{\nu^{\frac{1}{12}}}$

and, for $0 < s < t$ finite;

$E(g_t|\mathcal{F}_s) = E(g_t - g_s + g_s|\mathcal{F}_s)$

$\simeq E(g_s|\mathcal{F}_s)$

$= g_s$

with $|E(g_t|\mathcal{F}_s) - g_s| \leq \frac{1}{\nu^{\frac{1}{14}}}$

so that g_t is a quasi-nonstandard martingale on $[0,T]$, for T finite.

$\square$

Lemma 0.8. *Let B_t be nonstandard Brownian motion, then if $x = k\sqrt{\nu}$, where $0 \leq k \leq [t\nu]$;*

$Pr(|B_t| \geq \frac{k}{\nu}) \leq 2^* exp(\frac{-k^2}{2[t\nu]})$

In particularly, $Pr(|B_t| \geq x) \leq 2^* exp(-\frac{x^2}{2t})$.

Proof. For n finite, with $X_{n,t} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[tn]} \omega_i$, we have that, for $0 \leq k \leq [tn]$;

$Pr(X_{n,t} \geq \frac{k}{\sqrt{n}}) = Pr(X'_{n,t} \geq k)$

$= Pr(\frac{X'+1}{2} \geq \frac{k+1}{2})$

$Pr(X_{n,t} \leq \frac{-k}{\sqrt{n}}) = Pr(X'_{n,t} \leq -k)$

$= Pr(\frac{X'+1}{2} \leq \frac{-k+1}{2})$

where $X' = \sum_{i=1}^{[tn]} \omega_i$ and $\frac{X'+1}{2}$ follows the Binomial distribution with probability $\frac{1}{2}$ and $[tn]$ trials. We have tht $E(\frac{X'+1}{2}) = \frac{1}{2}$, so, by Hoeffding's inequality;

$Pr(\frac{X'+1}{2}) \geq \frac{k+1}{2} \leq e^{\frac{-k^2}{2[tn]}}$

$Pr(\frac{X'+1}{2}) \leq \frac{-k+1}{2} \leq e^{\frac{-k^2}{2[tn]}}$

so that $Pr(|X_{n,t}| \geq \frac{k}{\sqrt{n}}) = Pr(X'_{n,t} \geq k) + Pr(X'_{n,t} \leq -k) \leq 2e^{\frac{-k^2}{2[tn]}}$

The result is uniform in $n \in \mathcal{R}_{>0}$, so transfers to the case where $\nu \in {}^*\mathcal{R}_{>0}$, and gives the first result. Then substituting, we have that ${}^*exp(-\frac{k^2}{2[t\nu]}) = {}^*exp(-\frac{x^2\nu}{2[t\nu]}) \leq {}^*exp(-\frac{x^2}{2t})$, which gives the second result.

□

Lemma 0.9. *Let $f_\lambda(x,t) = e^{\alpha x - \frac{\alpha^2 t}{2}}$, where $\alpha = \sqrt{2i\lambda}$, for the principal root, $\lambda \in \mathcal{R}$, then;*

$|f_\lambda(x,t)| \leq e^{\sqrt{|\lambda|}|x|}$, *for the positive square root.*

and, similarly, for $\lambda \neq 0$;

$\frac{|\frac{\partial^{i+j}f_\lambda}{\partial x^i \partial t^j}|}{i!j!} \leq max(1, e^{6|\lambda|ln(|2\lambda|)}e^{\sqrt{|\lambda|}|x|})$, *uniformly in* $(i,j) \in \mathcal{Z}^2_{\geq 0}$

for $\lambda = 0$*;*

$\frac{|\frac{\partial^{i+j}f_\lambda}{\partial x^i \partial t^j}|}{i!j!} \leq 1$, *uniformly in* $(i,j) \in \mathcal{Z}^2_{\geq 0}$

Proof. For the first claim, we have that;

$|f_\lambda(x,t)| = |e^{\sqrt{2i\lambda}x - i\lambda t}|$

$= |e^{\sqrt{2i\lambda}x}|$

$= |e^{\sqrt{2\lambda}x(cos(\frac{\pi}{4})+isin(\frac{\pi}{4}))}|$, $(\lambda \geq 0)$

$= |e^{\sqrt{2\lambda}x\frac{1}{\sqrt{2}}}|$

$= e^{\lambda x}$

$\leq e^{\lambda|x|}$

$|f_\lambda(x,t)| = |e^{\sqrt{-2\lambda}x(cos(\frac{3\pi}{4})+isin(\frac{3\pi}{4}))}|$, $(\lambda \leq 0)$

$= |e^{\sqrt{-2\lambda}xcos(\frac{3\pi}{4})}|$

$= |e^{-\frac{1}{\sqrt{2}}\sqrt{-2\lambda}x}|$

$= |e^{-\sqrt{-\lambda}x}|$

$\leq e^{\sqrt{-\lambda}|x|}$

For the second claim, using the first part;

$|\frac{\partial^{i+j}f_\lambda}{i!j!\partial x_i \partial t_j}| = \frac{|\alpha^i(-1)^j\alpha^j||f_\lambda(x,t)|}{i!j!}$

$\leq \frac{|\alpha|^{i+j}e^{\sqrt{|\lambda|}|x|}}{i!j!}$

$\leq \frac{|2\lambda|^{\frac{i+j}{2}}}{i!j!}e^{\sqrt{|\lambda|}|x|}$

We have that, for $i \geq 6|\lambda|$, $j \geq 6|\lambda|$, $i! \geq |2\lambda|^{\frac{i}{2}}$, $j! \geq |2\lambda|^{\frac{j}{2}}$, so that;

$|\frac{\partial^{i+j}f_\lambda}{i!j!\partial x_i\partial t_j}| \leq max(1, max_{1\leq i,j\leq 6|\lambda|}\frac{|2\lambda|^{\frac{i+j}{2}}}{i!j!}e^{\sqrt{|\lambda|}|x|}$

$\leq max(1, |2\lambda|^{\frac{6|\lambda|+6|\lambda|}{2}})e^{\sqrt{|\lambda|}|x|}$

$\leq max(1, |2\lambda|^{6|\lambda|})e^{\sqrt{|\lambda|}|x|}$

$= max(1, e^{6|\lambda|ln(|2\lambda|)})e^{\sqrt{|\lambda|}|x|}$ $(\lambda \neq 0)$

$|\frac{\partial^{i+j}f_\lambda}{i!j!\partial x_i\partial t_j}| \leq 1$, $(\lambda = 0)$

□

Lemma 0.10. *For $\lambda \in \mathcal{R}_{\neq 0}$ fixed, we can obtain infinite x_0 and t_0, such that for $|x| \leq x_0$, $0 \leq t \leq t_0$;*

(i). $\frac{e^{6|\lambda|ln(|2\lambda|)}{}^*exp(\sqrt{|\lambda|}|x|)[t\nu]}{\nu^{\frac{3}{2}}} \simeq 0$

(ii). ${}^*exp(-\frac{x_0^2}{2t_0}) \simeq 0$

Proof. Let $t_0 = log^*(\nu)$, $x_0 = \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$, then, for $|x| \leq x_0$;

$|{}^*exp(\sqrt{|\lambda|}|x|)| \leq |{}^*exp(\sqrt{|\lambda|}x_0)|$

$= |{}^*exp(\frac{log^*(\nu)}{3})|$

$= \nu^{\frac{1}{3}}$

so that;

$\frac{{}^*exp(\sqrt{|\lambda|}|x|)[t\nu]}{\nu^{\frac{3}{2}}} \leq \frac{\nu^{\frac{1}{3}}[t\nu]}{\nu^{\frac{3}{2}}}$

$\leq \frac{\nu^{\frac{1}{3}}[log^*(\nu)\nu]}{\nu^{\frac{3}{2}}}$

$\leq \frac{\nu^{\frac{1}{3}}(log^*(\nu)\nu+1)}{\nu^{\frac{3}{2}}}$

$= \frac{log^*(\nu)}{\nu^{\frac{1}{6}}} + \frac{1}{\nu^{\frac{1}{6}}}$

$\simeq 0$

and, as $e^{6|\lambda|ln(|2\lambda|)}$ is finite, we have that;

$$\frac{e^{6|\lambda|ln(|2\lambda|)} {}^*exp(\sqrt{|\lambda|}|x|)[t\nu]}{\nu^{\frac{3}{2}}} \simeq 0$$

which gives (i). For (ii), we have that;

$${}^*exp(-\frac{x_0^2}{2t_0}) = {}^*exp(-\frac{\frac{log^*(\nu)^2}{9|\lambda|}}{2log^*(\nu)})$$

$$= {}^*exp(-\frac{log^*(\nu)}{18|\lambda|})$$

$$= \nu^{-\frac{1}{18|\lambda|}}$$

$$\simeq 0$$

□

Definition 0.11. *For $\lambda \in \mathcal{R}_{\neq 0}$, we define stopped nonstandard Brownian motion $\overline{B_{t,\lambda}} : \overline{\Omega}_\kappa \times \mathcal{T}_{\nu,\kappa} \rightarrow {}^*\mathcal{R}$ by:*

$$\overline{B_{t,\lambda}}(\omega) = B_t(\omega),\ if\ max_{0\leq t'\leq t}|B_{t'}(\omega)| \leq \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$$

$$\overline{B_{t,\lambda}}(\omega) = \frac{log^*(\nu)}{3\sqrt{|\lambda|}},\ if\ max_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$$

$$and\ for\ min_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}},\ B_{t'}(\omega) > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$$

$$\overline{B_{t,\lambda}}(\omega) = -\frac{log^*(\nu)}{3\sqrt{|\lambda|}},\ if\ max_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$$

$$and\ for\ min_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}},\ B_{t'}(\omega) < -\frac{log^*(\nu)}{3\sqrt{|\lambda|}}$$

Lemma 0.12. *For $0 \leq t \leq {}^*log(v)$, we have that;*

$$\mu_\kappa(max_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{{}^*log(\nu)}{3\sqrt{|\lambda|}}) \simeq 0$$

Proof. We have, using Lemma 0.8 and the reflection principle for random walks, see [9], that;

$$\mu_\kappa(max_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{{}^*log(\nu)}{3\sqrt{|\lambda|}})$$

$$\leq \mu_\kappa(max_{0\leq t'\leq t}B_{t'}(\omega) > \frac{{}^*log(\nu)}{3\sqrt{|\lambda|}}) + \mu_\kappa(min_{0\leq t'\leq t}B_{t'}(\omega) < -\frac{{}^*log(\nu)}{3\sqrt{|\lambda|}})$$

$< 2\mu_\kappa(B_t(\omega) > \frac{{}^*log(\nu)}{3\sqrt{|\lambda|}}) + 2\mu_\kappa(B_t(\omega) < -\frac{{}^*log(\nu)}{3\sqrt{|\lambda|}})$

$= 2\mu_\kappa(|B_t(\omega)| > \frac{{}^*log(\nu)}{3\sqrt{|\lambda|}})$

$\leq 2\mu_\kappa(|B_t(\omega)| > \frac{[{}^*log(\nu)\sqrt{\nu}]}{3\sqrt{|\lambda|\nu}})$

$\leq 4^*exp(-\frac{(\frac{[{}^*log(\nu)\sqrt{\nu}]}{3\sqrt{|\lambda|\nu}})^2}{2t})$

$\leq 4^*exp(-\frac{(\frac{{}^*log(\nu)\sqrt{\nu}-1}{3\sqrt{|\lambda|\nu}})^2}{2t})$

$= 4^*exp(-\frac{(\frac{{}^*log(\nu)-2}{3\sqrt{|\lambda|}})^2}{2t})$

$\leq 4^*exp(-\frac{(\frac{{}^*log(\nu)-2}{3\sqrt{|\lambda|}})^2}{2^*log(\nu)})$

$= 4^*exp(-\frac{{}^*log(\nu)^2-4^*log(\nu)+4}{18|\lambda|^*log(\nu)})$

$= 4^*exp(-\frac{{}^*log(\nu)+4-\frac{4}{{}^*log(\nu)}}{18|\lambda|})$

$\leq 8\nu^{\frac{-1}{18|\lambda|}}{}^*exp(\frac{4}{18|\lambda|})$

$\simeq 0$

□

Lemma 0.13. *If* $X_t : \overline{\Omega}_\kappa \rightarrow {}^*\mathcal{R}$ *is a* $\mathcal{F}_t$*-measurable random variable, with* $X_t \simeq 0$*, then, for* $0 \leq s \leq t$*,* $E(X_t|\mathcal{F}_s) \simeq 0$ *as well.*

Proof. For $n \in \mathcal{N}$, we have that $|X_t| < \frac{1}{n}$, so that by Jensen's inequality and monotonicity, we have;

$|E(X_t|\mathcal{F}_s)| \leq E(|X_t||\mathcal{F}_s)$

$< E(\frac{1}{n}|\mathcal{F}_s)$

$= \frac{1}{n}E(1|\mathcal{F}_s)$

$= \frac{1}{n}$

As $n \in \mathcal{N}$ was arbitrary, we obtain that $E(X_t|\mathcal{F}_s) \simeq 0$.

□

Definition 0.14. *For* $\alpha \in \mathcal{C}$, *we define* $M_{\alpha,t} = {}^*exp(\alpha B_{\frac{[t\nu]}{\nu}} - \frac{\alpha^2[t\nu]}{2\nu})$. *For* $\alpha = \sqrt{2i\lambda}$, *we define the stopped process* $\overline{M}_{\alpha,t}$ *by:*

$\overline{M}_{\alpha,t}(\omega) = M_{\alpha,t}(\omega)$, *if* $max_{0\leq t'\leq t}|B_{t'}(\omega)| \leq \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$

$\overline{M}_{\alpha,t}(\omega) = {}^*exp(\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}} - \frac{\alpha^2[t'\nu]}{2\nu})$, *if* $max_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$

and for $min_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$, $B_{t'}(\omega) > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$

$\overline{M}_{\alpha,t}(\omega) = {}^*exp(-\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}} - \frac{\alpha^2[t'\nu]}{2\nu})$, *if* $max_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$

and for $min_{0\leq t'\leq t}|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$, $B_{t'}(\omega) < -\frac{log^*(\nu)}{3\sqrt{|\lambda|}}$

Lemma 0.15. *For* $\alpha \in \mathcal{C}$, $\alpha = \sqrt{2i\lambda}$, $\overline{M}_{\alpha,t}$ *is a quasi-nonstandard martingale.*

Proof. Let $U_t \subset \overline{\Omega}_\kappa$ be defined by;

$U_t = \{\omega : max_{0\leq t'\leq t}B_{t'} \leq \frac{log^*(\nu)}{3\sqrt{|\lambda|}}\}$

$V \subset \overline{\Omega}_\kappa \times \mathcal{T}_{\nu,\kappa}$ be defined by;

$V = \{(t,\omega) : 0 \leq t \leq {}^*log(\nu), \omega \in U_t\}$

$V^c = \{(t,\omega) : 0 \leq t \leq {}^*log(\nu), \omega \notin U_t\}$

For $\omega \in \overline{\Omega}_\kappa$, let;

$t_\omega = min_{t',0\leq t'\leq {}^*log(\nu)}(|B_{t'}(\omega)| > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}) - \frac{1}{\nu}$

be the partial function, so that $\omega \in U_{t_\omega}$ but $\omega \notin U_{t_\omega+\frac{1}{\nu}}$. Let $V^* \subset V$ be defined by;

$V^* = \{(t_\omega,\omega) : \omega \in \overline{\Omega}_\kappa,\ t_\omega\ defined\}$

Then, for $(t,\omega) \in V^c$, we have that $d\overline{M}_{\alpha,t}|_{\frac{[t\nu]}{\nu},\omega} = 0$. For $(t,\omega) \in V \setminus V^*$, , by the definition of V and V^*, the process $\overline{M}_{\alpha,t}$ agrees with $M_{\alpha,t}$ at (t,ω) and $(t+\frac{1}{\nu},\omega)$. We have that, letting $f(t,x) = exp(\alpha x - \frac{\alpha^2 t}{2})$;

$((\frac{\partial f}{\partial t})^*|_{(t,\omega)} + \frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{(t,\omega)} = 0$

so, following the proof of Lemma 0.7 and using Lemma 0.9;

$d\overline{M}_{\alpha,t}|_{\omega,\frac{[t\nu]}{\nu}} = dM_{\alpha,t}|_{\omega,\frac{[t\nu]}{\nu}}$

$= ((\frac{\partial f}{\partial t})^*|_{\frac{[t\nu]}{\nu},\omega} + \frac{1}{2}(\frac{\partial^2 f}{\partial B_t^2})^*|_{\frac{[t\nu]}{\nu},\omega})dt + ((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}} + C_{\frac{[t\nu]}{\nu}}(\omega)$

$= ((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}} + C_{\frac{[t\nu]}{\nu}}(\omega)$

where;

$|C_{\frac{[t\nu]}{\nu}}(\omega)| \leq \frac{e^{6|\lambda|ln(|2\lambda|)} {}^*exp(\sqrt{|\lambda|}|x|)}{\nu^{\frac{3}{2}}}$, $|x| \leq \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$.

For $(t,\omega) \in V^*$, we have that, using Lemma 0.9 again;

$|d\overline{M}_{\alpha,t}|_{\omega,\frac{[t\nu]}{\nu}}| = |\overline{M}_{\alpha,t}|_{\frac{[t\nu]+1}{\nu},\omega} - \overline{M}_{\alpha,t}|_{\frac{[t\nu]}{\nu},\omega}|$

$= |{}^*exp(\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}} - \frac{\alpha^2[t\nu]+1}{2\nu}) - {}^*exp(\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}} - \frac{c}{\sqrt{\nu}} - \frac{\alpha^2[t\nu]}{2\nu})|$

$= |{}^*exp(\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}}){}^*exp(-\frac{\alpha^2[t\nu]}{2\nu})||{}^*exp(-\frac{\alpha^2}{2\nu}) - {}^*exp(-\frac{c}{\sqrt{\nu}})|$

$= |{}^*exp(\sqrt{2i\lambda}\frac{log^*(\nu)}{3\sqrt{|\lambda|}}){}^*exp(-\frac{i\lambda[t\nu]}{\nu})||{}^*exp(-\frac{i\lambda}{\nu}) - {}^*exp(-\frac{c}{\sqrt{\nu}})|$

$= |{}^*exp(\sqrt{2i\lambda}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})||1 - \frac{i\lambda}{\nu} + O(\frac{1}{\nu^2}) - 1 + \frac{c}{\sqrt{\nu}} - O(\frac{1}{\nu})|$

$\leq^* exp(\sqrt{|\lambda|}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})\frac{G}{\sqrt{\nu}}$

$= \frac{G\nu^{\frac{1}{3}}}{\sqrt{\nu}}$

$\simeq 0.$

with $B_{\frac{[t\nu]+1}{\nu}}(\omega) > \frac{log^*(\nu)}{3\sqrt{|\lambda|}}$, $0 < c \leq 1$, $G \in \mathcal{R}_{>0}$

and;

$|d\overline{M}_{\alpha,t}|_{\frac{[t\nu]}{\nu},\omega}| = |\overline{M}_{\alpha,t}|_{\frac{[t\nu]+1}{\nu},\omega} - \overline{M}_{\alpha,t}|_{\frac{[t\nu]}{\nu},\omega}|$

$= |{}^*exp(-\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}} - \frac{\alpha^2[t\nu]+1}{2\nu}) - {}^*exp(-\alpha\frac{log^*(\nu)}{3\sqrt{|\lambda|}} + \frac{c}{\sqrt{\nu}} - \frac{\alpha^2[t\nu]}{2\nu})|$

$= |^*exp(-\alpha \frac{log^*(\nu)}{3\sqrt{|\lambda|}})^*exp(-\frac{\alpha^2[t\nu]}{2\nu})||^*exp(-\frac{\alpha^2}{2\nu}) - {}^*exp(\frac{c}{\sqrt{\nu}})|$

$= |^*exp(-\sqrt{2i\lambda}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})^*exp(-\frac{i\lambda[t\nu]}{\nu})||^*exp(-\frac{i\lambda}{\nu}) - {}^*exp(\frac{c}{\sqrt{\nu}})|$

$= |^*exp(-\sqrt{2i\lambda}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})||1 - \frac{i\lambda}{\nu} + O(\frac{1}{\nu^2}) - 1 - \frac{c}{\sqrt{\nu}} - O(\frac{1}{\nu})|$

$\leq^* exp(\sqrt{|\lambda|}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})\frac{G}{\sqrt{\nu}}$

$= \frac{G\nu^{\frac{1}{3}}}{\sqrt{\nu}}$

$\simeq 0 \ (A)$

with $B_{\frac{[t\nu]+1}{\nu}}(\omega) < -\frac{log^*(\nu)}{3\sqrt{|\lambda|}}$, $0 < c \leq 1$, $G \in \mathcal{R}_{>0}$

It follows, using the proof of Lemma 0.7 again, that, for $0 \leq t \leq^* log(\nu)$;

$\overline{M}_{\alpha,t} - \overline{M}_{\alpha,0} = \int_0^{(t\wedge t_\omega)-\frac{1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}} + \epsilon(\omega,t) + \delta(\omega,t) \ (C)$

where, using Lemma 0.10 (i), (A), and the fact that $t \leq^* log(\nu)$;

$|\epsilon(\omega,t)| \leq \frac{e^{6|\lambda|ln(|2\lambda|)*}exp(\sqrt{|\lambda|}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})[t\nu]}{\nu^{\frac{3}{2}}}$

$\simeq 0$

$|\delta(\omega,t)| \leq \frac{G\nu^{\frac{1}{3}}}{\sqrt{\nu}}$

$\simeq 0$

We have that $t_\omega = \tau - \frac{1}{\nu}$, where τ is the stopping time for the barrier $\frac{log^*(\nu)}{3\sqrt{|\lambda|}}$, so that;

$|\int_0^{(t\wedge t_\omega)-\frac{1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}} - \int_0^{(t\wedge\tau)}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}}|$

$\leq |\int_0^{(t\wedge t_\omega)-\frac{1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}} - \int_0^{(t\wedge t_\omega)}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}}|$

$+|\int_0^{(t\wedge t_\omega)}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}} - \int_0^{(t\wedge\tau)}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}}|$

$= |(\frac{\partial f}{\partial B_t})^*|_{\frac{[(t\wedge t_\omega)\nu]}{\nu},\omega}dB_{\frac{[(t\wedge t_\omega)\nu]}{\nu}}| + |(\frac{\partial f}{\partial B_t})^*|_{\frac{[\tau\nu]}{\nu},\omega}dB_{\frac{[\tau\nu]}{\nu}}|$

$= |(\frac{\partial f}{\partial B_t})^*|_{\frac{[(t\wedge t_\omega)\nu]}{\nu},\omega} dB_{\frac{[(t\wedge t_\omega)\nu]}{\nu}}|$

$\leq \frac{1}{\sqrt{\nu}}|(\frac{\partial f}{\partial B_t})^*|_{\frac{[(t\wedge t_\omega)\nu]}{\nu},\omega}|$

$\leq \frac{\alpha\nu^{\frac{1}{3}}}{\sqrt{\nu}}$

$\simeq 0\ (B)$

By proofs in [8], we have that $\int_0^t((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}$ is a nonstandard martingale, and by Lemma 0.16, $\int_0^{(t\wedge\tau)}((\frac{\partial f}{\partial B_t})^*|_{\omega,\frac{[t\nu]}{\nu}})dB_{\frac{[t\nu]}{\nu}}$ is a nonstandard martingale as well. It follows from (B) and Lemma 0.13, that;

$E(\int_0^{(t\wedge t_\omega)-\frac{1}{\nu}}((\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega})dB_{\frac{[t\nu]}{\nu}}|\mathcal{F}_s)$

$\simeq E(\int_0^{t\wedge\tau}(\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega}dB_{\frac{[t\nu]}{\nu}}|\mathcal{F}_s)$

$= \int_0^{s\wedge\tau}(\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega}dB_{\frac{[t\nu]}{\nu}}$

$\simeq \int_0^{(s\wedge t_\omega)-\frac{1}{\nu}}(\frac{\partial f}{\partial B_t})^*|_{\frac{[t\nu]}{\nu},\omega}dB_{\frac{[t\nu]}{\nu}}$

and from (C) and Lemma 0.13 again, that;

$E(\overline{M}_{\alpha,t} - \overline{M}_{\alpha,0}|\mathcal{F}_s) \simeq \overline{M}_{\alpha,s} - \overline{M}_{\alpha,0}$

As $\overline{M}_{\alpha,0} = 1$, we obtain that;

$E(\overline{M}_{\alpha,t}|\mathcal{F}_s) \simeq M_{\alpha,s}$

as well. By the proof, using the explicit inequality in Lemma 0.13, we have that;

$|E(\overline{M}_{\alpha,t} - \overline{M}_{\alpha,0}|\mathcal{F}_s) - (\overline{M}_{\alpha,s} - \overline{M}_{\alpha,0})|$

$\leq \frac{2G\nu^{\frac{1}{3}}}{\sqrt{\nu}} + \frac{2\alpha\nu^{\frac{1}{3}}}{\sqrt{\nu}} + \frac{2e^{6|\lambda|ln(|2\lambda|)}*exp(\sqrt{|\lambda|}\frac{log^*(\nu)}{3\sqrt{|\lambda|}})[^*log(\nu)\nu]}{\nu^{\frac{3}{2}}}$

$\leq 2G\nu^{-\frac{1}{6}} + 2\alpha\nu^{-\frac{1}{6}} + 2e^{6|\lambda|ln(|2\lambda|)}\frac{\nu^{\frac{1}{3}}*log(\nu)\nu}{\nu^{\frac{3}{2}}}$

$\leq 2G\nu^{-\frac{1}{6}} + 2\alpha\nu^{-\frac{1}{6}} + 2e^{6|\lambda|ln(|2\lambda|)}\frac{^*log(\nu)}{\nu^{\frac{1}{6}}}$

$\leq \frac{H_\lambda}{\nu^{\frac{1}{12}}}$

where $H_\lambda \in \mathcal{R}_{>0}$ depends on λ. Clearly, we then obtain that;

$|E(\overline{M}_{\alpha,t}|\mathcal{F}_s) - \overline{M}_{\alpha,s}| \leq \frac{H_\lambda}{\nu^{\frac{1}{12}}}$

as well, so that $\overline{M}_{\alpha,t}$ is a quasi nonstandard martingale, for $0 \leq t \leq {}^*log(\nu)$.

□

Lemma 0.16. *If M_t is a nonstandard martingale, and τ is a stopping time for the barrier $\frac{[a\sqrt{\nu}]}{\sqrt{\nu}}$, with $a \in {}^*\mathcal{R}$, $\tau = min\{t : B_t = \frac{[a\sqrt{\nu}]}{\sqrt{\nu}}, t \in \mathcal{T}_{\nu,\kappa}\}$, then the process $M_{t\wedge\tau}$ is a nonstandard martingale. In particular the process $M_{\alpha,t\wedge\tau}$ is a nonstandard martingale. The process $\overline{M}_{\alpha,t\wedge\tau}$, for $\alpha = \sqrt{2i\lambda}$, $\lambda \in \mathcal{R}_{>0}$, τ is a stopping time for the barrier $\frac{[a\sqrt{\nu}]}{\sqrt{\nu}}$, with $a \in \mathcal{R}_{>0}$, is a quasi-nonstandard martingale. The process $\overline{M}_{\alpha,t\wedge\tau}$, for $\alpha = \sqrt{2i\lambda}$, $\lambda \in \mathcal{R}_{<0}$, τ is a stopping time for the barrier $\frac{-[a\sqrt{\nu}]}{\sqrt{\nu}}$, with $a \in \mathcal{R}_{>0}$, is a quasi-nonstandard martingale.*

Proof. For the first claim, the proof for the discrete case can be found in [9]. It is sufficient to show that the event $(\tau \leq \frac{i}{\nu}) \in \mathcal{F}_{\frac{i}{\nu}}$. This follows as;

$(\tau \leq \frac{i}{\nu})$ iff $\bigwedge_{\overline{\omega}_i} \sum_{j=1}^{i} (\overline{\omega}_i)_j = [a\sqrt{\nu}]$

where $\overline{\omega}_i$ is a sequence of 0's nd 1's of length i. The disjunction is a $*$-finite union of the basic sets $U_{\overline{k}_i}$, so belongs to the $*$-σ algebra $\mathcal{F}_{\frac{i}{\nu}}$. The last claim is a consequence of this lemma and lemma 0.15.

For the second claim..... □

Lemma 0.17. *We have that, for $\lambda \in \mathcal{R}_{>0}$, $a \in \mathcal{R}_{>0}$;*

$E(\overline{M}_{\alpha,\tau}) \simeq 1,\ E(exp^*(-\lambda\tau)) \simeq {}^*exp(-\frac{\sqrt{2\lambda}[a\nu]}{\sqrt{\nu}})$

Proof. As $\overline{M}_{\alpha,t\wedge\tau}$ is a quasi nonstandard martingale, we have that;

$E(\overline{M}_{\alpha,t\wedge\tau}) \simeq E(\overline{M}_{\alpha,0\wedge\tau}) = E(\overline{M}_{\alpha,0}) = 1\ (*)$

....

Let $\kappa_1 = \nu^{\frac{4}{3}} < \nu^{\frac{3}{2}} < \kappa$, so that $\frac{\kappa_1}{\nu^{\frac{3}{2}}} = \frac{1}{\nu^{\frac{1}{6}}} \simeq 0$, and $(*)$ goes through $\simeq$.

By Lemma 0.18, we have that;

$P(\tau \geq \frac{\kappa_1}{\nu}) \leq \frac{A[a\sqrt{\nu}]}{\sqrt{[\frac{\kappa_1}{\nu}\nu]}}$

$= \frac{A[a\sqrt{\nu}]}{\sqrt{[\kappa_1]}}$

$= \frac{A[a\sqrt{\nu}]}{\sqrt{[\nu^{\frac{4}{3}}]}}$

$\simeq 0$

We have that;

$M_{\alpha,\tau}|_{(\tau\geq\frac{\kappa_1}{\nu})^c} = M_{\alpha,\frac{\kappa_1}{\nu}\wedge\tau}|_{(\tau\geq\frac{\kappa_1}{\nu})^c}$

so that as $M_{\alpha,t\wedge\tau}$ is bounded by ${}^*exp(\alpha\frac{[a\sqrt{\nu}]}{\sqrt{\nu}})$, we have that $E(M_{\alpha,\tau}) \simeq 1$, with;

$|E(M_{\alpha,\tau}) - 1| \leq \frac{2A[a\sqrt{\nu}]}{\sqrt{[\nu^{\frac{4}{3}}]}}{}^*exp(\alpha\frac{[a\sqrt{\nu}]}{\sqrt{\nu}})$

....

□

Lemma 0.18. *We have that, for* $\kappa \geq max(2, 3a, a^2)$*;*

$P(T_a \geq \kappa) \leq \frac{C_a}{\sqrt{\kappa}}$

where $C_a = \frac{8ae\sqrt{6}}{\sqrt{\pi}}$*, for a random walk, starting at* 0*, with steps* 1 *and* -1*, and barrier* $a > 0$*, stopping time* T_a*;*

For nonstandard Brownian motion B_t*, with barrier* $\frac{[a\sqrt{\nu}]}{\sqrt{\nu}}$*,* $a \in \mathcal{R}$*, and stopping time* τ*, we have that there exists* $A \in \mathcal{R}$*, with;*

$P(\tau \geq \frac{[t\nu]}{\nu}) \leq A\frac{[a\sqrt{\nu}]}{\sqrt{[t\nu]}}$

for $[t\nu] \geq max(2, 3[a\sqrt{\nu}], [a\sqrt{\nu}]^2)$*. In particular, for* $t \geq a^2+1$*, when* $t \in \mathcal{R}$*, we have that;*

$P(\tau \geq \frac{[t\nu]}{\nu}) \leq \frac{2\sqrt{2}Aa}{\sqrt{t}}$.

Proof. We have that, see [3];

$P(T_a = n) = \frac{a}{n} C^n_{\frac{n-a}{2}} \frac{1}{2^n}$

for $n \geq a > 0$, $n - a$ even.

It follows that, using Stirling's approximation, for $\kappa > max(2, 3a, a^2)$;

$P(T_a \geq \kappa) = \sum_{n=\kappa, n-a \ even}^{\infty} \frac{a}{n} C^n_{\frac{n-a}{2}} \frac{1}{2^n}$

$= \sum_{n=\kappa, n-a \ even}^{\infty} \frac{a}{n} \frac{n!}{\frac{n-a}{2}! \frac{n+a}{2}!} \frac{1}{2^n}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{a}{n} \frac{\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}}{\sqrt{2\pi(\frac{n-a}{2})}(\frac{n-a}{2e})^{\frac{n-a}{2}} \sqrt{2\pi(\frac{n+a}{2})}(\frac{n+a}{2e})^{\frac{n+a}{2}} e^{\frac{1}{12(\frac{n-a}{2})+1}} e^{\frac{1}{12(\frac{n+a}{2})+1}}} \frac{1}{2^n}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a}{n} \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi(\frac{n-a}{2})}(\frac{n-a}{2e})^{\frac{n-a}{2}} \sqrt{2\pi(\frac{n+a}{2})}(\frac{n+a}{2e})^{\frac{n+a}{2}}} \frac{1}{2^n}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a}{n} \frac{\sqrt{3}}{\sqrt{\pi}\sqrt{n}} \frac{(\frac{n}{e})^n}{(\frac{n-a}{2e})^{\frac{n-a}{2}} (\frac{n+a}{2e})^{\frac{n+a}{2}}} \frac{1}{2^n}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} \frac{(\frac{n}{e})^n}{(\frac{n-a}{2e})^{\frac{n}{2}} (\frac{n+a}{2e})^{\frac{n}{2}}} \frac{1}{(\frac{n-a}{2e})^{\frac{-a}{2}} (\frac{n+a}{2e})^{\frac{a}{2}}} \frac{1}{2^n}$

$= \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} \frac{(\frac{n}{e})^n}{(\frac{n-a}{2e})^{\frac{n}{2}} (\frac{n+a}{2e})^{\frac{n}{2}}} (\frac{n-a}{n+a})^{\frac{a}{2}} \frac{1}{2^n}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} \frac{(\frac{n}{e})^n}{(\frac{n-a}{2e})^{\frac{n}{2}} (\frac{n+a}{2e})^{\frac{n}{2}}} \frac{1}{2^n}$

$= \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} (\frac{4n^2}{n^2-a^2})^{\frac{n}{2}} \frac{1}{2^n}$

$= \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} (\frac{n^2}{n^2-a^2})^{\frac{n}{2}}$

$= \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} (1 + \frac{a^2}{n^2-a^2})^{\frac{n}{2}}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} (1 + \frac{2a^2}{n^2})^{\frac{n}{2}}$

$= \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} ((1 + \frac{2a^2}{n^2})^{n^2})^{\frac{1}{2n}}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4a\sqrt{3}}{\sqrt{\pi}n\sqrt{n}} e^{\frac{a^2}{n}}$

$\leq \sum_{n=\kappa, n-a \ even}^{\infty} \frac{4ae\sqrt{3}}{\sqrt{\pi}n\sqrt{n}}$

$\leq \frac{4ae\sqrt{3}}{\sqrt{\pi}} \int_{\kappa-1}^{\infty} \frac{dx}{x\sqrt{x}}$

$= \frac{4ae\sqrt{3}}{\sqrt{\pi}} [\frac{-2}{x^{\frac{1}{2}}}]_{\kappa-1}^{\infty}$

$= \frac{4ae\sqrt{3}}{\sqrt{\pi}} \frac{2}{(\kappa-1)^{\frac{1}{2}}}$

$\leq \frac{4ae\sqrt{3}}{\sqrt{\pi}} \frac{2\sqrt{2}}{\kappa^{\frac{1}{2}}}$

$\leq \frac{C_a}{\kappa^{\frac{1}{2}}}$

where $C_a = \frac{8ae\sqrt{6}}{\sqrt{\pi}}$

For the next claim, just observe that the above proof is uniform in a random walk with a barrier at $[a\sqrt{n}]$ for $n \in \mathcal{N}$, so by transfer, we can obtain the result for infinite $\nu \in {}^*\mathcal{N}$, rescaling the walk by a factor of $\frac{1}{\sqrt{\nu}}$ and moving the barrier to $\frac{[a\sqrt{\nu}]}{\sqrt{\nu}}$, the constant A being $\frac{8e\sqrt{6}}{\sqrt{\pi}}$. The last claim is just a simple exercise in nonstandard arithmetic, noting that for $t \geq a^2+1$, the max condition is automatically satisfied for $[t\nu]$. $\square$

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