# SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 5: NO RADIATION OF LIGHT

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Abstract.

**Definition 0.1.** *We call* $(\overline{E}_0, \overline{B}_0)$*, a solution to Maxwell's equation in vacuum, good, if* $(\overline{E} + \overline{E}_0) \times \overline{B}_0 = 0$*, for some fundamental solution* $(\overline{E}, \overline{0})$ *corresponding to* $\{\rho, \overline{J}\}$ *satisfying the conditions from Lemma 4.1 in* [9]*, with* $\{\rho, \overline{J}\}$ *not vacuum and* $\{\rho, \overline{J}\} \subset S(\mathcal{R}^3 \times \mathcal{R}_{>0})$*. We call* $(\overline{E}_0, \overline{B}_0)$ *static if* $\frac{\partial \overline{E}_0}{\partial t} = \frac{\partial \overline{B}_1}{\partial t} = \overline{0}$*.*

**Definition 0.2.** *We say that a field* $\overline{C}(\overline{x}, t)$ *is simple if all the components* $c_i$*,* $1 \leq i \leq 3$ *are continuously fourth differentiable in the coordinates* $(x_1, x_2, x_3)$ *and continuously twice differentiable in the coordinate* $t$*, such that the partial derivatives all belong to* $L^1(\mathcal{R}^3)$ *for fixed* $t \geq 0$*, and, the* $L^1$*-norm of the partial derivatives is uniformly bounded for* $0 \leq t < 1$*.*

**Definition 0.3.** *We say that a real pair* $(\overline{E}, \overline{B})$*, satisfying Maxwell's equations for some* $\{\rho, \overline{J}\}$*, satisfies the strong no radiation condition if;*

$$P(r,t) = \int_{S(\overline{0},r)} (\overline{E} \times \overline{B}) \cdot d\overline{S} = 0$$

*for all* $r > 0$ *and* $t \in \mathcal{R}$*. We say that it satisfies the no radiation condition if;*

$$lim_{r \to \infty} P(r,t) = 0$$

*for all* $t \in \mathcal{R}$

**Lemma 0.4.** *For any* $\{\rho, \overline{J}\}$ *satisfying the conditions from Lemma 4.1 in* [9]*, if* $(\overline{E}, \overline{0})$ *denotes a fundamental solution, then a solution* $\{\overline{E} + \overline{E}_0, \overline{B}_0\}$*, with* $(\rho, \overline{J}, \overline{E} + \overline{E}_0, \overline{B}_0)$ *satisfying Maxwell's equations, satisfies the no radiating condition, if* $\overline{E}$*,*$\overline{E}_0$ *and* $\overline{B}_0$ *are simple and* $\{(\overline{E} + \overline{E}_0)_0, \frac{\partial (\overline{E} + \overline{E}_0)}{\partial t}|_0, (\overline{B}_0)_0, \frac{\partial \overline{B}_0}{\partial t}|_0\} \subset S(\mathcal{R}^3)$*,* $(*)$*. Moreover, we have* *that explicit representation;*

$(\overline{E}+\overline{E}_0)(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\bar{b}(\bar{k})e^{ikct}+\overline{d}(\bar{k})e^{-ikct})e^{i\bar{k}.\overline{x}}d\bar{k}$

$\overline{B}_0(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b'}(\bar{l})e^{ilct}+\overline{d'}(\bar{l})e^{-ilct})e^{i\bar{l}.\overline{x}}d\bar{l}$

*where* $\{\bar{b},\overline{d},\overline{b}'\overline{d}'\}\subset S(\mathcal{R}^3)$.

*Proof.* By Lemma 4.1 in [9], and the argument in [1], we have that;

$\square^2\overline{E}=\overline{0}$, $\overline{B}=\overline{0}$

$\square^2\overline{E}_0=\overline{0}$, $\square^2\overline{B}_0=\overline{0}$ $(*)$

Then;

$lim_{r\rightarrow\infty}P(r)=lim_{r\rightarrow\infty}\int_{S(r)}((\overline{E}+\overline{E}_0)\times(\overline{B}+\overline{B}_0))d\overline{S}(r)$

$=lim_{r\rightarrow\infty}\int_{S(r)}(\overline{E}\times\overline{B})d\overline{S}(r)+lim_{r\rightarrow\infty}\int_{S(r)}((\overline{E}+\overline{E}_0)\times\overline{B}_0)d\overline{S}(r)$

$+lim_{r\rightarrow\infty}\int_{S(r)}(\overline{E}_0\times\overline{B})d\overline{S}(r)$

$=lim_{r\rightarrow\infty}\int_{S(r)}((\overline{E}+\overline{E}_0)\times\overline{B}_0)d\overline{S}(r)$

and, by $(*)$, we have that $\square^2(\overline{E}+\overline{E}_0)=\overline{0}$ as well, $(\dagger)$.

Assume that $\overline{E}$,$\overline{E}_0$ and $\overline{B}_0$ are simple, then, $\overline{E}+\overline{E}_0$ and $\overline{B}_0$ are simple, and we have that;

$\bigtriangledown^2(\overline{E}-\overline{E}_0)-\frac{1}{c^2}\frac{\partial^2(\overline{E}-\overline{E}_0)}{\partial t^2}=\overline{0}$

so that, applying the three dimensional Fourier transform $\mathcal{F}$ to the components, and using integration by parts, we have that;

$\mathcal{F}(\bigtriangledown^2(\overline{E}-\overline{E}_0))(\bar{k},t))-\frac{1}{c^2}\frac{\partial^2(\mathcal{F}(\overline{E}-\overline{E}_0))(\bar{k},t)}{\partial t^2}$

$=-k^2\mathcal{F}(\overline{E}-\overline{E}_0)(\bar{k},t)-\frac{1}{c^2}\frac{\partial^2(\mathcal{F}(\overline{E}-\overline{E}_0))(\bar{k},t)}{\partial t^2}$

$=-k^2\overline{(}f)(\bar{k},t)-\frac{1}{c^2}\frac{\partial^2\overline{f}(\bar{k},t)}{\partial t^2}$

$= \overline{0}$

where $k^2 = k_1^2 + k_2^2 + k_3^2$, $\overline{a} = \mathcal{F}(\overline{E} - \overline{E}_0)$. For fixed $\overline{k}$, we obtain the ordinary differential equation;

$\frac{d^2\overline{a}_{\overline{k}}}{dt^2} = -c^2k^2\overline{a}_{\overline{k}}$

so that;

$\overline{a}_{\overline{k}}(t) = \overline{C}_0(\overline{k})e^{ikct} + \overline{D}_0(\overline{k})e^{-ikct}$

with;

$\overline{a}_{\overline{k}}(0) = \overline{C}_0(\overline{k}) + \overline{D}_0(\overline{k})$

$\overline{a}'_{\overline{k}}(0) = ikc\overline{C}_0(\overline{k}) - ikc\overline{D}_0(\overline{k})$ (††)

and, solving the simultaneous equations (††), we obtain that;

$\overline{C}_0(\overline{k}) = \frac{1}{2}(\overline{a}_{\overline{k}}(0) + \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))$

$\overline{D}_0(\overline{k}) = \frac{1}{2}(\overline{a}_{\overline{k}}(0) - \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))$

and;

$\mathcal{F}(\overline{E} - \overline{E}_0)(\overline{k}, t) = \overline{a}(\overline{k}, t)$

$= \frac{1}{2}(\overline{a}_{\overline{k}}(0) + \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))e^{ikct} + \frac{1}{2}(\overline{a}_{\overline{k}}(0) + \frac{1}{ikc}\overline{a}'_{\overline{k}}(0))e^{-ikct}$

$= \overline{b}(\overline{k})e^{ikct} + \overline{d}(\overline{k})e^{-ikct}$

where;

$\overline{b}(\overline{k}) = \frac{1}{2}(\mathcal{F}((\overline{E} + \overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)} + \frac{1}{ikc}\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$\overline{d}(\overline{k}) = \frac{1}{2}(\mathcal{F}((\overline{E} + \overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)} - \frac{1}{ikc}\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

Similarly;

$\mathcal{F}(\overline{B}_0)(\overline{l}, t) = \overline{a}'(\overline{l}, t) = \overline{b'}(\overline{l})e^{ilct} + \overline{d'}(\overline{l})e^{-ilct}$

where;

$\overline{b'}(\bar{l})=\frac{1}{2}(\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\bar{l},0)}+\frac{1}{ilc}\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\overline{x},0)})|_{(\bar{l},0)})$

$\overline{d'}(\bar{l})=\frac{1}{2}(\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\bar{l},0)}-\frac{1}{ilc}\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\overline{x},0)})|_{(\bar{l},0)})$

and $l^2=l_1^2+l_2^2+l_3^2$. Using the fact that $\{\overline{b}(\overline{k})e^{ikct}+\overline{d}(\overline{k})e^{-ikct},\overline{b'}(\bar{l})e^{ilct}+\overline{d'}(\bar{l})e^{-ilct}\}\subset S(\mathcal{R}^3$ for $t\in\mathcal{R}$, we can apply the inversion theorem, to obtain;

$(\overline{E}+\overline{E}_0)(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b}(\overline{k})e^{ikct}+\overline{d}(\overline{k})e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$\overline{B}_0(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b'}(\bar{l})e^{ilct}+\overline{d'}(\bar{l})e^{-ilct})e^{i\bar{l}\centerdot\overline{x}}d\bar{l}$

As we noted above, $\{\overline{b}e^{ikct}+\overline{d}e^{-ikct},\overline{b'}e^{ilct}+\overline{d'}e^{-ilct}\}\subset S(\mathcal{R}^3$ for $t\in\mathcal{R}$, so that, by the fact that the Fourier transform preserves the Schwartz class, see [14], we must have that $\{(\overline{E}+\overline{E}_0)_t,(\overline{B}_0)_t\}\subset S(R^3)$ for $t\in\mathcal{R}$. Then, for $n\geq 3$ and the definition of the Schwartz class;

$|P(r,t)|=|\int_{S(r)}((\overline{E}+\overline{E}_0)_t\times(\overline{B}_0)_t)d\overline{S}|$

$\leq\int_{S(r)}|((\overline{E}+\overline{E}_0)_t\times(\overline{B}_0)_t)\centerdot\hat{\overline{n}}|dS(r)|$

$\leq\int_{S(r)}|(\overline{E}+\overline{E}_0)_t||(\overline{B}_0)_t|dS(r)$

$\leq 4\pi r^2\frac{C_{1,n}}{r^n}\frac{D_{1,n}}{r^n}$

$=\frac{4\pi C_{1,n}D_{1,n}}{r^{2n-2}}$

so clearly;

$lim_{r\rightarrow\infty}P(r,t)=0$

□

**Definition 0.5.** *Fix a real propagation vector* $\overline{k}_0$ *and a real vector* $\overline{d}_0$ *with* $\overline{k}_0\centerdot\overline{d}_0=0$. *Let;*

$\overline{E}_0(\overline{x},t)=\overline{d}_0e^{-ik_0ct}e^{i\overline{k}_0\centerdot\overline{x}}$

$\overline{B}_0(\overline{x},t)=\overline{d}_0'e^{-ik_0ct}e^{i\overline{k}_0\centerdot\overline{x}}$

*where* $\overline{d}_0' = \frac{1}{c}(\overline{k}_0 \times \overline{d}_0)$. *Then, see* [1], *the pair* $(\overline{E}_0, \overline{B}_0)$ *solves Maxwell's equation in vacuum, and so does* $(Re(\overline{E}_0), Re(\overline{B}_0))$. *We call* $(Re(\overline{E}_0), Re(\overline{B}_0))$ *a monochromatic solution.*

**Lemma 0.6.** *For a monochromatic solution* $(Re(\overline{E}_0), Re(\overline{B}_0))$ *to Maxwell's equation in vacuum, we have that* $P(r,t) = O(r)$. *In particularly,* $(Re(\overline{E}_0), Re(\overline{B}_0))$ *doesn't satisfy satisfy the no radiation condition unless* $\overline{E}_0 = \overline{d}_0$ *and* $\overline{B}_0 = 0$, *or* $\overline{E}_0 = \overline{B}_0 = \overline{0}$, *in which cases* $(Re(\overline{E}_0), Re(\overline{B}_0))$ *is constant. Any constant real solution* $(\overline{E}_1, \overline{B}_1)$ *satisfies the strong no radiation and no radiation conditions*

*Proof.* We have, for a monochromatic solution, that;

$$Re(\overline{E}_0)(\overline{x},t) = \frac{\overline{d}_0}{4}(e^{ik_0ct}e^{i\overline{k}_0\cdot\overline{x}}+e^{ik_0ct}e^{-i\overline{k}_0\cdot\overline{x}}+e^{-ik_0ct}e^{i\overline{k}_0\cdot\overline{x}}+e^{-ik_0ct}e^{-i\overline{k}_0\cdot\overline{x}})$$

$$Re(\overline{B}_0)(\overline{x},t) = \frac{\overline{d}_0'}{4}(e^{ik_0ct}e^{i\overline{k}_0\cdot\overline{x}}+e^{ik_0ct}e^{-i\overline{k}_0\cdot\overline{x}}+e^{-ik_0ct}e^{i\overline{k}_0\cdot\overline{x}}+e^{-ik_0ct}e^{-i\overline{k}_0\cdot\overline{x}})$$

so that $Re(\overline{E}_0) \times Re(\overline{B}_0)$

$$= \frac{(\overline{d}_0 \times \overline{d}_0')}{16}(e^{2ik_0ct}e^{2i\overline{k}_0\cdot\overline{x}} + e^{2ik_0ct}e^{-2i\overline{k}_0\cdot\overline{x}} + e^{-2ik_0ct}e^{2i\overline{k}_0\cdot\overline{x}} + e^{-2ik_0ct}e^{-2i\overline{k}_0\cdot\overline{x}}$$

$$+2e^{2ik_0ct} + 2e^{-2ik_0ct} + 2e^{2i\overline{k}_0\cdot\overline{x}} + 2e^{-2i\overline{k}_0\cdot\overline{x}} + 4)$$

By the divergence theorem, we have that;

$$P(r,t) = \int_{S(\overline{0},r)}(Re(\overline{E}_0) \times Re(\overline{B}_0))d\overline{S}(r)$$

$$= \int_{B(\overline{0},r)} \nabla\cdot(\frac{(\overline{d}_0 \times \overline{d}_0')}{16}(e^{2ik_0ct}e^{2i\overline{k}_0\cdot\overline{x}}+e^{2ik_0ct}e^{-2i\overline{k}_0\cdot\overline{x}}+e^{-2ik_0ct}e^{2i\overline{k}_0\cdot\overline{x}}+e^{-2ik_0ct}e^{-2i\overline{k}_0\cdot\overline{x}}$$

$$+2e^{2ik_0ct} + 2e^{-2ik_0ct} + 2e^{2i\overline{k}_0\cdot\overline{x}} + 2e^{-2i\overline{k}_0\cdot\overline{x}} + 4))dB(r)$$

$$= \int_{B(\overline{0},r)} \frac{(\overline{d}_0 \times \overline{d}_0')}{16} \cdot 2i\overline{k}_0(e^{2i\overline{k}_0\cdot\overline{x}}(e^{2ik_0ct} + e^{-2ik_0ct} + 2) - e^{-2i\overline{k}_0\cdot\overline{x}}(e^{2ik_0ct}$$

$$+e^{-2ik_0ct} + 2))dB(r)$$

$$= \frac{(\overline{d}_0 \times \overline{d}_0')}{16} \cdot 2i\overline{k}_0(e^{2ik_0ct} + e^{-2ik_0ct} + 2)(2(\frac{2\pi r}{|2\overline{k}_0|})^{\frac{3}{2}} J_{\frac{3}{2}}(r|2\overline{k}_0|))$$

$$= \frac{(\overline{d}_0 \times \overline{d}_0')}{4} \cdot i\overline{k}_0(e^{2ik_0ct} + e^{-2ik_0ct} + 2)((\frac{\pi r}{|\overline{k}_0|})^{\frac{3}{2}} J_{\frac{3}{2}}(2r|\overline{k}_0|))$$

$$= \frac{(\overline{d}_0 \times \overline{d}_0')}{4} \cdot i\overline{k}_0(e^{2ik_0ct} + e^{-2ik_0ct} + 2)(\frac{\pi r}{|\overline{k}_0|})^{\frac{3}{2}}(\frac{1}{\pi r|\overline{k}_0|})^{\frac{1}{2}}(P_1(\frac{1}{2r|\overline{k}_0|})sin(2r|\overline{k}_0|)$$

$-Q_0(\frac{1}{2r|\overline{k}_0|})cos(2r|\overline{k}_0|))$

$= \frac{(\overline{d}_0 \times \overline{d}_0')}{4} \centerdot i\overline{k}_0(e^{2ik_0ct} + e^{-2ik_0ct} + 2)(\frac{\pi r}{|\overline{k}_0|})^{\frac{3}{2}}(\frac{1}{\pi r|\overline{k}_0|})^{\frac{1}{2}}((\frac{P_{1,1}}{2r|\overline{k}_0|})sin(2r|\overline{k}_0|)$

$-Q_{0,0}cos(2r|\overline{k}_0|))$

$= \frac{(\overline{d}_0 \times \overline{d}_0')}{4} \centerdot i\overline{k}_0(e^{2ik_0ct} + e^{-2ik_0ct} + 2)(\frac{\pi}{|\overline{k}_0|})^{\frac{3}{2}}(\frac{1}{\pi|\overline{k}_0|})^{\frac{1}{2}}((\frac{P_{1,1}}{2|\overline{k}_0|})sin(2r|\overline{k}_0|)$

$-Q_{0,0}rcos(2r|\overline{k}_0|))$

Clearly, $P(r,t) = O(r)$ unless $\overline{d}_0 \times \overline{d}_0' \centerdot \overline{k}_0 = 0$, in which case either $\overline{k}_0 = \overline{0}$ or $\overline{d}_0 = \overline{0}$. In the first case, we obtain that $\overline{E}_0 = \overline{d}_0$ and $\overline{B}_0 = \overline{0}$, in the second case, we obtain that $\overline{E}_0 = \overline{B}_0 = \overline{0}$. The last claim is clear by the divergence theorem and the fact that $\bigtriangledown \centerdot (\overline{E}_1 \times \overline{B}_1) = 0$. □

**Lemma 0.7.** *For any $\{\rho, \overline{J}\}$ satisfying the conditions from Lemma 4.1 in [9], if $(\overline{E}, \overline{0})$ denotes a fundamental solution, then a solution $\{\overline{E} + \overline{E}_0, \overline{B}_0\}$, with $(\rho, \overline{J}, \overline{E} + \overline{E}_0, \overline{B}_0)$ satisfying Maxwell's equations such that $\{\overline{E}, \overline{E}_0, \overline{B}_0\}$ are simple and $\{(\overline{E} + \overline{E}_0)_0, \frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_0, (\overline{B}_0)_0, \frac{\partial \overline{B}_0}{\partial t}|_0\} \subset S(\mathcal{R}^3)$, satisfies the strong no-radiation condition, using the integral representation in Lemma 0.4, when;*

$\overline{a}(\overline{k}, t) \times \overline{a'}(\overline{l}, t) = \overline{0}$ (†)

*or when $\overline{B}_0$ is parallel to $\overline{E} + \overline{E}_0$. In either of these cases, the no radiation condition holds as well.*

*If $\{\overline{E}, \overline{E}_0, \overline{B}_0\}$ are simple, then $\{\overline{E}+\overline{E}_0, \overline{B}_0\}$ satisfies the no-radiation condition when...?*

*Proof.* Using the result of Lemma 0.4, we can use the integral representations of $\overline{E} + \overline{E}_0$ and $\overline{B}_0$ to compute;

$((\overline{E} + \overline{E}_0) \times \overline{B}_0)(\overline{x}, t)$

$= \frac{1}{(2\pi)^3} \int_{\mathcal{R}^6} (\overline{b}(\overline{k}) \times \overline{b'}(\overline{l}))e^{i(\overline{k}+\overline{l}) \centerdot \overline{x}} e^{i(k+l)ct} d\overline{k} d\overline{l}$

$+\frac{1}{(2\pi)^3} \int_{\mathcal{R}^6} (\overline{b}(\overline{k}) \times \overline{d'}(\overline{l}))e^{i(\overline{k}+\overline{l}) \centerdot \overline{x}} e^{i(k-l)ct} d\overline{k} d\overline{l}$

$+\frac{1}{(2\pi)^3} \int_{\mathcal{R}^6} (\overline{d}(\overline{k}) \times \overline{b'}(\overline{l}))e^{i(\overline{k}+\overline{l}) \centerdot \overline{x}} e^{i(l-k)ct} d\overline{k} d\overline{l}$

$+\frac{1}{(2\pi)^3}\int_{\mathcal{R}^6}(\overline{d}(\overline{k})\times\overline{d'}(\overline{l}))e^{i(\overline{k}+\overline{l})\centerdot\overline{x}}e^{-i(k+l)ct}d\overline{k}d\overline{l}$, $(\dagger\dagger)$

Clearly, if $(\dagger)$ is satisfied, then we obtain that $(\overline{E}+\overline{E}_0)\times\overline{B}_0=\overline{0}$, so that $\bigtriangledown\centerdot((\overline{E}+\overline{E}_0)\times\overline{B}_0)=0$, and using the divergence theorem, $P(r,t)=0$ for all $r>0$ and $t\in\mathcal{R}_{\geq 0}$, and $lim_{r\rightarrow\infty}P(r,t)=0$, for all $t\in\mathcal{R}_{\geq 0}$, so that the strong no radiation and no radiation conditions hold. Similarly, if $\overline{B}_0$ is parallel to $\overline{E}+\overline{E}_0$, then $(\overline{E}+\overline{E}_0)\times\overline{B}_0=\overline{0}$, so that $((\overline{E}+\overline{E}_0),\overline{B}_0)$ satisfies the strong no radiation and the no radiation conditions again.

If $\{\overline{E},\overline{E}_0,\overline{B}_0\}$ are simple, then, we have that;

$\mathcal{F}((\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2})^2(\overline{E}+\overline{E}_0))(\overline{k},t)=(k_1^2+k_2^2+k_3^2)^2\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)$

so that, for $|\overline{k}|\geq 1$, $\leq i\leq 3$;

$|\mathcal{F}(\overline{E}+\overline{E}_0)_i(\overline{k},t)|\leq\frac{1}{|\overline{k}|^4}\int_{\mathcal{R}^3}|(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2})(\overline{E}+\overline{E}_0)_i|d\overline{x}$

$\leq\frac{C_{i,t}}{|\overline{k}|^4}$

and, similarly;

$|\mathcal{F}(\overline{B}_0)_i(\overline{k},t)|\leq\frac{D_{i,t}}{|\overline{k}|^4}$

where $\{C_{i,t},D_{i,t}\}\subset\mathcal{R}_{\geq 0}$

Similarly;

$|\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)|$

$\leq\sum_{i=1}^3|\mathcal{F}(\overline{E}+\overline{E}_0)_i(\overline{k},t)|$

$\leq\frac{C_t}{|\overline{k}|^4}$

where $C_t=\sum_{i=1}^3 C_{i,t}$

and $|\mathcal{F}(\overline{B}_0)(\overline{k},t)|$

$\leq\frac{D_t}{|\overline{k}|^4}$ $(\sharp)$

Clearly, we have that $\mathcal{F}(\overline{E}+\overline{E}_0)(\overline{k},t)$ and $\mathcal{F}(\overline{B}_0)(\overline{k},t)$ are differentiable and therefore bounded on $B(\overline{0},1)$, so that, using polar coordinates, with $k_1 = Rsin(\theta)cos(\phi)$, $k_2 = Rsin(\theta)sin(\phi)$, $k_3 = Rcos(\theta)$
;

$|\int_{\mathcal{R}^3}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}d\overline{k}|$

$= |\int_{B(\overline{0},1)}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}d\overline{k} + \int_{\mathcal{R}^3\setminus B(\overline{0},1)}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}d\overline{k}|$

$\leq C_{i,t,1} + |\int_{R>1}\int_0^{\pi}\int_{-\pi}^{\pi}\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t}(R,\theta,\phi)R^2 sin(\theta)dRd\theta d\phi$

$\leq C_{i,t,1} + \int_{R>1}\int_0^{\pi}\int_{-\pi}^{\pi}R^2\frac{C_{i,t}}{R^4}dR$

$\leq C_{i,t,1} + 2\pi^2 C_{i,t}\int_1^{\infty}\frac{1}{R^2}dR$

$= C_{i,t,1} + 2\pi^2 C_{i,t}$

so that, for $1 \leq i \leq 3$, $\mathcal{F}(\overline{E}+\overline{E}_0)_{i,t} \in L^1(\mathcal{R}^3)$, and, similarly, $\mathcal{F}(\overline{B}_0)_{i,t} \in L^1(\mathcal{R}^3)$. Following the proof of Lemma 0.4, we can still use the inversion theorem integral and the integral representations for $((\overline{E}+\overline{E}_0),\overline{B}_0)$, and the computation $(\dagger\dagger)$ holds again. We have, using polar coordinates, that;

$|\int_{B(\overline{0},1)}\frac{1}{ikc}\mathcal{F}(\frac{\partial\overline{E}+\overline{E}_{0,i}}{\partial t}|_{\overline{x},0})(\overline{k})d\overline{k}|$

$\leq \int_0^1\int_0^{\pi}\int_{-\pi}^{\pi}|\mathcal{F}(\frac{\partial\overline{E}+\overline{E}_{0,i}}{\partial t}|_{\overline{x},0})(R,\theta,\phi)|\frac{1}{R}R^2 dRd\theta \text{p}\dot{h}i$

$= \frac{2\pi^2}{2} = \pi^2$

so that the components, $\frac{1}{ikc}\mathcal{F}(\frac{\partial\overline{E}+\overline{E}_{0,i}}{\partial t}|_{\overline{x},0})(\overline{k})$ for $1 \leq i \leq 3$, are integrable on $B(\overline{0},1)$, and, therefore, so are the components of $\{\overline{b},\overline{b}',\overline{d},\overline{d}'\}$.
Applying the result $(\sharp)$, we obtain that, for $\overline{k}| > 1$;

$|\overline{b}(\overline{k}) + \overline{d}(\overline{k})| \leq \frac{C_0}{|\overline{k}|^4}$

$|e^{ikct}\overline{b}(\overline{k}) + e^{ikct}\overline{d}(\overline{k})| \leq \frac{C_0}{|\overline{k}|^4}$

$|e^{ikct}\overline{b}(\overline{k}) + e^{-ikct}\overline{d}(\overline{k})| \leq \frac{C_t}{|\overline{k}|^4}$

$|(e^{ikct} - e^{-ikct})\overline{d}(\overline{k})|$

$= 2|sin(kct)\overline{d}(\overline{k})|$

$\leq \frac{C_0+C_t}{|k|^4}$

so that at time $t = \frac{\pi}{2kc}$, we have that;

$|\overline{d}(\overline{k})| \leq \frac{C_0+C_{\frac{\pi}{2kc}}}{|k|^4}$

$\leq \frac{C_0+E}{|k|^4}$

where $E \in \mathcal{R}_{>0}$ is the uniform bound for $t \in [0,1]$, and, similarly, for $|\overline{k}| > 1$;

$max(|\overline{b}|, |\overline{b}'|, |\overline{d}|, |\overline{d}'|)(\overline{k}) \leq \frac{F}{|k|^4}$

for some $F \in \mathcal{R}_{>0}$. In particularly, we have that the components $\{\overline{b}, \overline{b}', \overline{d}, \overline{d}'\}$ belong to $L^1(\overline{R}^3)$ and we can apply the calculation in (††).

By the divergence theorem, we have that;

$\int_{S(\overline{0},r)}(\overline{b}(\overline{k}) \times \overline{d'}(\overline{l}))e^{i(\overline{k}+\overline{l})\cdot\overline{x}}e^{i(k-l)ct}d\overline{S}(r)$

$= \int_{B(\overline{0},r)} \bigtriangledown \cdot ((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l}))e^{i(\overline{k}+\overline{l})\cdot\overline{x}}e^{i(k-l)ct})dB(r)$

$= \int_{B(\overline{0},r)}((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l})) \cdot i(\overline{k}+\overline{l}))e^{i(\overline{k}+\overline{l})\cdot\overline{x}}e^{i(k-l)ct}dB(r)$

$= ((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l})) \cdot i(\overline{k}+\overline{l}))(\frac{2\pi r}{|\overline{k}+\overline{l}|})^{\frac{3}{2}}J_{\frac{3}{2}}(r|\overline{k}+\overline{l}|)e^{i(k-l)ct}$

$= ((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l})) \cdot i(\overline{k}+\overline{l}))(\frac{2\pi r}{|\overline{k}+\overline{l}|})^{\frac{3}{2}}(\frac{2}{\pi(r|\overline{k}+\overline{l}|)})^{\frac{1}{2}}(P_1(\frac{1}{r|\overline{k}+\overline{l}|})sin(r|\overline{k}+\overline{l}|)$

$-Q_0(\frac{1}{r|\overline{k}+\overline{l}|})cos(r|\overline{k}+\overline{l}|))e^{i(k-l)ct}$

$= ((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l})) \cdot i(\overline{k}+\overline{l}))(\frac{2\pi r}{|\overline{k}+\overline{l}|})^{\frac{3}{2}}(\frac{2}{\pi(r|\overline{k}+\overline{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{r|\overline{k}+\overline{l}|}sin(r|\overline{k}+\overline{l}|)$

$-Q_{0,0}cos(r|\overline{k}+\overline{l}|))e^{i(k-l)ct}$

$= ((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l})) \cdot i(\overline{k}+\overline{l}))(\frac{2\pi}{|\overline{k}+\overline{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\overline{k}+\overline{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{|\overline{k}+\overline{l}|}sin(r|\overline{k}+\overline{l}|)e^{i(k-l)ct}$

$-((\overline{b}(\overline{k}) \times \overline{d'}(\overline{l})) \cdot i(\overline{k}+\overline{l}))(\frac{2\pi}{|\overline{k}+\overline{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\overline{k}+\overline{l}|)})^{\frac{1}{2}}Q_{0,0}rcos(r|\overline{k}+\overline{l}|))e^{i(k-l)ct}$ (∗)

By (∗), we have that;

$lim_{r\rightarrow\infty}P(r)=\frac{1}{(2\pi)^3}lim_{r\rightarrow\infty}\int_{\mathcal{R}^6}((\overline{b}(\overline{k})\times\overline{d'}(\bar{l})).i(\overline{k}+\bar{l}))(\frac{2\pi}{|\overline{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\overline{k}+\bar{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{|\overline{k}+\bar{l}|}$

$sin(r|\overline{k}+\bar{l}|)e^{i(k-l)ct}d\overline{k}d\bar{l}$

$-\frac{1}{(2\pi)^3}lim_{r\rightarrow\infty}\int_{\mathcal{R}^6}((\overline{b}(\overline{k})\times\overline{d'}(\bar{l})).i(\overline{k}+\bar{l}))(\frac{2\pi}{|\overline{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\overline{k}+\bar{l}|)})^{\frac{1}{2}}Q_{0,0}$

$rcos(r|\overline{k}+\bar{l}|))e^{i(k-l)ct}d\overline{k}d\bar{l}$

Let $g(\overline{k},\bar{l},t)=\frac{1}{(2\pi)^3}(\overline{b}(\overline{k})\times\overline{d'}(\bar{l})).i(\overline{k}+\bar{l}))(\frac{2\pi}{|\overline{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\overline{k}+\bar{l}|)})^{\frac{1}{2}}\frac{P_{1,1}}{|\overline{k}+\bar{l}|}e^{i(k-l)ct}$

and $h(\overline{k},\bar{l},t)=-\frac{1}{(2\pi)^3}(\overline{b}(\overline{k})\times\overline{d'}(\bar{l})).i(\overline{k}+\bar{l})(\frac{2\pi}{|\overline{k}+\bar{l}|})^{\frac{3}{2}}(\frac{2}{\pi(|\overline{k}+\bar{l}|)})^{\frac{1}{2}}Q_{0,0}e^{i(k-l)ct}$
$(***)$

Then $\{g,h\}\subset S(\mathcal{R}^3\times\mathcal{R}_{>0})$ and, we have that;

$lim_{r\rightarrow\infty}P(r,t)$

$=lim_{r\rightarrow\infty}\int_{\mathcal{R}^6}g(\overline{k},\bar{l},t)d\overline{k}sin(r|\overline{k}+\bar{l}|)d\bar{l}$

$+lim_{r\rightarrow\infty}r\int_{\mathcal{R}^6}h(\overline{k},\bar{l},t)d\overline{k}cos(r|\overline{k}+\bar{l}|)d\bar{l}$

From $(***)$, we have that;

$g(\overline{k},\bar{l},t)=\frac{iP_{1,1}}{2\pi^2}(\overline{b}(\overline{k})\times\overline{d}'(\bar{l})).\frac{\overline{u}(\overline{k},\bar{l})}{|\overline{k}+\bar{l}|^2}e^{i(k-l)ct}$

where $\overline{u}(\overline{k},\bar{l})$ is a unit vector, so that, using Fubini's Theorem, and a change of variables $\overline{k}'=\overline{k}+\bar{l}$, we have;

$\int_{\mathcal{R}^6}(g(\overline{k},\bar{l},t)e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$

$=\int_{\mathcal{R}^6}\frac{iP_{1,1}}{2\pi^2}(\overline{b}(\overline{k})\times\overline{d}'(\bar{l})).\frac{\overline{u}(\overline{k},\bar{l})}{|\overline{k}+\bar{l}|^2}e^{i(k-l)ct}e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$

$=\int_{\mathcal{R}^6}\frac{\overline{\phi}(\overline{k},\bar{l},t)}{|\overline{k}+\bar{l}|^2}e^{i(r|\overline{k}+\bar{l}|)}d\overline{k}d\bar{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}^3}\frac{\phi(\overline{k},\bar{l},t)}{|\overline{k}+\bar{l}|^2}e^{i(r|\overline{k}+\bar{l}|)}d\overline{k})d\bar{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}^3}\frac{\phi(\overline{k}'-\bar{l},\bar{l},t)}{|\overline{k}'|^2}e^{i(r|\overline{k}'|)}d\overline{k}')d\bar{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}^3}\frac{\phi(\overline{k}-\bar{l},\bar{l},t)}{|\overline{k}|^2}e^{i(r|\overline{k}|)}d\overline{k})d\bar{l}$

where $\phi(\overline{k},\overline{l},t)=\frac{iP_{1,1}}{2\pi^2}(\overline{b}(\overline{k})\times\overline{d}'(\overline{l}))\centerdot\overline{u}(\overline{k},\overline{l})e^{i(k-l)ct}$

It follows, switching to polars coordinates;

$k_1=Rsin(\theta)cos(\phi)$, $k_2=Rsin(\theta)sin(\phi)$, $k_3=Rcos(\theta)$

that;

$\int_{\mathcal{R}^6}(g(\overline{k},\overline{l},t)e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}d\overline{k}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{q(R,\theta,\phi,t,\overline{l})}{R^2}e^{irR}R^2sin(\theta)dRd\theta)d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}q(R,\theta,\phi,t,\overline{l})e^{irR}sin(\theta)dRd\theta)d\overline{l}$ (2)

where $q(R,\theta,\phi,t,\overline{l})=\phi(\overline{k}-\overline{l},\overline{l},t)$.

From $(***)$ again, we have that;

$h(\overline{k},\overline{l},t)=\frac{-iQ_{0,0}}{2\pi^2}(\overline{b}(\overline{k})\times\overline{d}'(\overline{l}))\centerdot\frac{\overline{u}(\overline{k},\overline{l})}{|\overline{k}+\overline{l}|}e^{i(k-l)ct}$

where $\overline{u}(\overline{k},\overline{l})$ is a unit vector, so that, using Fubini's Theorem, and a change of variables $\overline{k}'=\overline{k}+\overline{l}$, we have;

$\int_{\mathcal{R}^6}(h(\overline{k},\overline{l},t)e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}$

$=\int_{\mathcal{R}^6}\frac{-iQ_{0,0}}{2\pi^2}(\overline{b}(\overline{k})\times\overline{d}'(\overline{l}))\centerdot\frac{\overline{u}(\overline{k},\overline{l})}{|\overline{k}+\overline{l}|}e^{i(k-l)ct}e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}$

$=\int_{\mathcal{R}^6}\frac{\overline{\theta}(\overline{k},\overline{l},t)}{|\overline{k}+\overline{l}|}e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}^3}\frac{\theta(\overline{k},\overline{l},t)}{|\overline{k}+\overline{l}|}e^{i(r|\overline{k}+\overline{l}|)}d\overline{k})d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}^3}\frac{\theta(\overline{k}'-\overline{l},\overline{l},t)}{|\overline{k}'|}e^{i(r|\overline{k}'|)}d\overline{k}')d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}^3}\frac{\theta(\overline{k}-\overline{l},\overline{l},t)}{|\overline{k}|}e^{i(r|\overline{k}|)}d\overline{k})d\overline{l}$

where $\theta(\overline{k},\overline{l},t)=\frac{-iQ_{0,0}}{2\pi^2}(\overline{b}(\overline{k})\times\overline{d}'(\overline{l}))\centerdot\overline{u}(\overline{k},\overline{l})e^{i(k-l)ct}$

It follows, switching to polars coordinates;

$k_1=Rsin(\theta)cos(\phi)$, $k_2=Rsin(\theta)sin(\phi)$, $k_3=Rcos(\theta)$

that;

$\int_{\mathcal{R}^6}(h(\overline{k},\overline{l},t)e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}d\overline{k}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{p(R,\theta,\phi,t,\overline{l})}{R}e^{irR}R^2 sin(\theta)dRd\theta)d\overline{l}$

$= \int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}p(R,\theta,\phi,t,\overline{l})e^{irR}Rsin(\theta)dRd\theta)d\overline{l}$ (3)

where $p(R,\theta,\phi,t,\overline{l}) = \theta(\overline{k}-\overline{l},\overline{l},t)$.

Write $\overline{b}(\overline{k}) = \overline{b}_1(\overline{k}) + i\overline{b}_2(\overline{k})$, $\overline{d}'(\overline{l}) = \overline{d}_1'(\overline{l}) + i\overline{d}_2'(\overline{l})$

where;

$\overline{b}_1(\overline{k}) = \frac{1}{2}Re(\mathcal{F}((\overline{E}+\overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)}) + \frac{1}{2kc}Im(\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$\overline{b}_2(\overline{k}) = \frac{1}{2}Im(\mathcal{F}((\overline{E}+\overline{E}_0)|_{(\overline{x},0)})|_{(\overline{k},0)}) - \frac{1}{2kc}Re(\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$\overline{d}_1'(\overline{l}) = \frac{1}{2}Re(\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\overline{l},0)}) - \frac{1}{2lc}Im(\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{l},0)})$

$\overline{d}_2'(\overline{l}) = \frac{1}{2}Im(\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\overline{l},0)}) + \frac{1}{2lc}Re(\mathcal{F}(\frac{\partial(\overline{B}_0)}{\partial t}|_{(\overline{x},0)})|_{(\overline{l},0)})$

We have that;

$q(R,\theta,\phi,t,\overline{l})$

$= \frac{iP_{1,1}}{2\pi^2}[(\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l}) - \overline{b}_{2,\overline{l}}(R,\theta,\phi)\times\overline{d}_2'(\overline{l}))$

$\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)$

$-\frac{P_{1,1}}{2\pi^2}[(\overline{b}_{2,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l}) + \overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}_2'(\overline{l}))$

$\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)$ (1)

and, similarly;

$p(R,\theta,\phi,t,\overline{l})$

$= \frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l}) - \overline{b}_{2,\overline{l}}(R,\theta,\phi)\times\overline{d}_2'(\overline{l}))$

$\cdot \overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)$

$+\frac{Q_{0,0}}{2\pi^2}[(\overline{b}_{2,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l})+\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}_2'(\overline{l}))$

$\cdot \overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)$ (4)

where $\overline{b}_{1,\overline{l}}(\overline{k})=\overline{b}_1(\overline{k}-\overline{l})$, $\overline{b}_{2,\overline{l}}(\overline{k})=\overline{b}_2(\overline{k}-\overline{l})$, $\overline{u}_{\overline{l}}(\overline{k},\overline{l})=\overline{u}(\overline{k}-\overline{l},\overline{l})$, $\mu(\overline{k},\overline{l},t)=e^{i(|\overline{k}-\overline{l}|-|\overline{l}|)ct}$

and, from (1), (2), we have that;

$\int_{\mathcal{R}^6} g(\overline{k},\overline{l},t)e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[(\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l})-\overline{b}_{2,\overline{l}}(R,\theta,\phi)$

$\times\overline{d}_2'(\overline{l}))\cdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)-\frac{P_{1,1}}{2\pi^2}[(\overline{b}_{2,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l})+\overline{b}_{1,\overline{l}}(R,\theta,\phi)$

$\times\overline{d}_2'(\overline{l}))\cdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)dRd\theta)d\overline{l}$

and, from (4), (3);

$\int_{\mathcal{R}^6} h(\overline{k},\overline{l},t)e^{i(r|\overline{k}+\overline{l}|)}d\overline{k}d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l})-\overline{b}_{2,\overline{l}}(R,\theta,\phi)$

$\times\overline{d}_2'(\overline{l}))\cdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)+\frac{Q_{0,0}}{2\pi^2}[(\overline{b}_{2,\overline{l}}(R,\theta,\phi)\times\overline{d}_1'(\overline{l})+\overline{b}_{1,\overline{l}}(R,\theta,\phi)$

$\times\overline{d}_2'(\overline{l}))\cdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}Rsin(\theta)dRd\theta d\phi)d\overline{l}$

Write $\overline{b}_1(\overline{k})=\overline{b}_{11}(\overline{k})+\frac{\overline{b}_{12}(\overline{k})}{k}$, $\overline{d}_1'(\overline{l})=\overline{d}_{11}'(\overline{l})+\frac{\overline{d}_{12}'(\overline{l})}{l}$

Then;

$\overline{b}_{1,\overline{l}}(\overline{k})=\overline{b}_1(\overline{k}-\overline{l})=\overline{b}_{11}(\overline{k}-\overline{l})+\frac{\overline{b}_{12}(\overline{k}-\overline{l})}{|\overline{k}-\overline{l}|}$

and;

$\overline{b}_{1,\overline{l}}(R,\theta,\phi)=\overline{b}_{11,\overline{l}}(R,\theta,\phi)+\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}$

where $\overline{b}_{11,\overline{l}}(\overline{k})=\overline{b}_{11}(\overline{k}-\overline{l})$ and $\overline{b}_{12,\overline{l}}(\overline{k})=\overline{b}_{12}(\overline{k}-\overline{l})$

Then, we have that;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}'_1(\overline{l}))\centerdot\overline{u_{\overline{l}}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[(\overline{b}_{11,\overline{l}}(R,\theta,\phi)+\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})$

$\times(\overline{d}'_{11}(\overline{l})+\frac{\overline{d}'_{12}(\overline{l})}{l})]\centerdot\overline{u_{\overline{l}}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\overline{l}$

and, we have that;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{1,\overline{l}}(R,\theta,\phi)\times\overline{d}'_1(\overline{l}))\centerdot\overline{u_{\overline{l}}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}Rsin(\theta)$

$dRd\theta d\phi)d\overline{l}$

$=\int_{\mathcal{R}^3}(\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{-iQ_{0,0}}{2\pi^2}[(\overline{b}_{11,\overline{l}}(R,\theta,\phi)+\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})$

$\times(\overline{d}'_{11}(\overline{l})+\frac{\overline{d}'_{12}(\overline{l})}{l})]\centerdot\overline{u_{\overline{l}}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}Rsin(\theta)dRd\theta d\phi)d\overline{l}$

From ($\sharp$), we have that the real and imaginary components of;

$\{\mathcal{F}((\overline{B}_0)|_{(\overline{x},0)})|_{(\overline{l},0)},\mathcal{F}((\overline{E}+\overline{E}_0)|_{(\overline{x},0)})|_{(\overline{l},0)},\mathcal{F}((\frac{\partial\overline{B}_0}{\partial t}|_{(\overline{x},0)})|_{(\overline{l},0)},\mathcal{F}(\frac{\partial(\overline{E}+\overline{E}_0)}{\partial t})|_{(\overline{x},0)})|_{(\overline{l},0)}\}$

decay faster than $\frac{1}{|l|^4}$ (need $\frac{1}{|l|^6}$?). It follows that the components of;

$\{\overline{b}_{11,\overline{l}}(\overline{k})\times\overline{d}'_{11}(\overline{l}),\frac{\overline{b}_{11,\overline{l}}(\overline{k})\times\overline{d}'_{12}(\overline{l})}{l},\frac{\overline{b}_{12,\overline{l}}(\overline{k})\times\overline{d}'_{11}(\overline{l})}{|\overline{k}-\overline{l}|},\frac{\overline{b}_{12,\overline{l}}(\overline{k})\times\overline{d}'_{12}(\overline{l})}{|\overline{k}-\overline{l}|l}\}$

decay faster than $\frac{1}{|\overline{k}|^4|\overline{l}|^4|\overline{k}-\overline{l}|}$, and, as $\overline{u_{\overline{l}}}(\overline{k},\overline{l})$ is a unit vector, $|\nu(\overline{k},\overline{l},t)|=1$, $|sin(\theta(\overline{k}))|\leq 1$, so do the components of;

$\{[(\overline{b}_{11,\overline{l}}(\overline{k})\times\overline{d}'_{11}(\overline{l}))\centerdot\overline{u_{\overline{l}}}(\overline{k},\overline{l})]\nu(\overline{k},\overline{l},t)sin(\theta(\overline{k})),[(\frac{\overline{b}_{11,\overline{l}}(\overline{k})\times\overline{d}'_{12}(\overline{l})}{l})\centerdot\overline{u_{\overline{l}}}(\overline{k},\overline{l}))]\nu(\overline{k},\overline{l},t)sin(\theta(\overline{k})),$

$[(\frac{\overline{b}_{12,\overline{l}}(\overline{k})\times\overline{d}'_{11}(\overline{l})}{|\overline{k}-\overline{l}|})\centerdot\overline{u_{\overline{l}}}(\overline{k},\overline{l})]\nu(\overline{k},\overline{l},t)sin(\theta(\overline{k})),[(\frac{\overline{b}_{12,\overline{l}}(\overline{k})\times\overline{d}'_{12}(\overline{l})}{|\overline{k}-\overline{l}|l})\centerdot\overline{u_{\overline{l}}}(\overline{k},\overline{l})]\nu(\overline{k},\overline{l},t)sin(\theta(\overline{k}))\}$

Noting that, for $C\in\mathcal{R}_{>0}$, $D\in\mathcal{R}_{>0}$ and fixed $\overline{l}\in\mathcal{R}^3$, $\overline{l}\neq\overline{0}$, without loss of generality, assuming that $D<|\overline{l}|$?;

$|\int_{|\overline{k}|>D}\frac{C}{|\overline{k}|^4||\overline{l}|^4|\overline{k}-\overline{l}|}|d\overline{k}$

$=|\int_{D<|\overline{k}|<|\overline{l}|+1}\frac{C}{|\overline{k}|^4||\overline{l}|^4|\overline{k}-\overline{l}|}|d\overline{k}+\int_{D>|\overline{l}|+1}\frac{C}{|\overline{k}|^4||\overline{l}|^4|\overline{k}-\overline{l}|}|d\overline{k}|$

$\leq |\int_{D<|\bar{k}|<|\bar{l}|+1} \frac{C}{|\bar{k}|^4||\bar{l}|^4|\bar{k}-\bar{l}|} d\bar{k}| + |\int_{|\bar{k}|>|\bar{l}|+1>D} \frac{C}{|\bar{k}|^4||\bar{l}|^4|\bar{k}-\bar{l}|} d\bar{k}|$

$\leq \frac{C}{D^4|\bar{l}|^4} \int_{Ann(D,|\bar{l}|+1)} \frac{1}{|\bar{k}-\bar{l}|} d\bar{k} + \frac{1}{|\bar{l}|^4} \int_{|\bar{k}|>|\bar{l}|+1} \frac{C}{|\bar{k}|^4} d\bar{k}$

$= \frac{C}{D^4|\bar{l}|^4} \int_{Ann_{\bar{l}}(D,|\bar{l}|+1)} \frac{1}{|\bar{k}|} d\bar{k} + \frac{1}{|\bar{l}|^4} \int_0^\pi \int_{-\pi}^\pi \int_{|\bar{l}|+1}^\infty \frac{CR^2 sin(\theta)}{R^4} dR d\theta d\theta d\phi$

$\leq \frac{C}{D^4|\bar{l}|^4} \int_{B(\bar{0},2|\bar{l}|+2D+1)} \frac{1}{|\bar{k}|} d\bar{k} + \frac{1}{|\bar{l}|^4} \int_0^\pi \int_{-\pi}^\pi \int_{|\bar{l}|+1}^\infty \frac{C}{R^2} dR d\theta d\theta d\phi$

$\leq \frac{2\pi^2 C}{D^4|\bar{l}|^4} \int_0^{2|\bar{l}|+2D+1} \frac{R^2}{R} dR + \frac{2\pi^2 C}{(|\bar{l}|+1)|\bar{l}|^4}$

$\leq \frac{\pi^2 C(2|\bar{l}|+2D+1)^2}{D^4|\bar{l}|^4} + \frac{2\pi^2 C}{D|\bar{l}|^4}$

It follows, that for fixed $r \in \mathcal{R}_{>0}$, we can choose $D_r, E_r$ such that, for fixed $r \in \mathcal{R}_{>0}$;

$\int_{|\bar{k}|>D_r} \int_{|\bar{l}>E_r} |\alpha(\bar{k},\bar{l},t)| d\bar{k} d\bar{l}$

$\leq \int_{|\bar{l}|>E_r} \frac{1}{|\bar{l}|^4 r^2}$

(see note above for faster decay)

$\leq \frac{2\pi^2}{E_r r^2}$

where;

$\alpha(\bar{k},\bar{l},t) = \alpha(R,\theta,\phi,\bar{l},t) = \frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi) + \frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}) \times (\bar{d}'_{11}(\bar{l}) + \frac{\bar{d}'_{12}(\bar{l})}{l})] \centerdot \overline{u_{\bar{l}}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t) sin(\theta)$

$\beta(\bar{k},\bar{l},t) = \beta(R,\theta,\phi,\bar{l},t) = \frac{-iQ_{0,0}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi) + \frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}) \times (\bar{d}'_{11}(\bar{l}) + \frac{\bar{d}'_{12}(\bar{l})}{l})] \centerdot \overline{u_{\bar{l}}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t) sin(\theta)$

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}} \int_{0\leq\theta<\pi} \int_{0\leq\phi\leq 2\pi} \frac{-iQ_{0,0}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi) + \frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})$

$\times(\bar{d}'_{11}(\bar{l}) + \frac{\bar{d}'_{12}(\bar{l})}{l})] \centerdot \overline{u_{\bar{l}}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t) e^{irR} R sin(\theta) dR d\theta d\phi) d\bar{l}$

$= \int_{\mathcal{R}^3} \int_{\mathcal{R}_{>0}} \int_{0\leq\theta<\pi} \int_{0\leq\phi\leq 2\pi} \beta(R,\theta,\phi,\bar{l},t) e^{irR} R d\theta d\phi) d\bar{l}$

Splits as four terms, the worst of which is;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}} \int_{0\leq\theta<\pi} \int_{0\leq\phi\leq 2\pi} \frac{-iQ_{0,0}}{2\pi^2}[\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$

$\times\frac{\overline{d}'_{12}(\overline{l})}{l}]\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)RdRd\theta d\phi)d\overline{l}$

$=\int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\beta_4(R,\theta,\phi,\overline{l},t)e^{irR}RdRd\theta d\phi)d\overline{l}$

Again, fix $\overline{l}\neq\overline{0}$, with $\theta\neq cos^{-1}(\frac{l_3}{l})=\theta_{0,\overline{l}}$ and $\phi\neq tan^{-1}(\frac{l_2}{l_1})=\phi_{0,\overline{l}}$. By the result of Lemma 0.18 (change to $\beta_4$ factor), we can assume that the real and imaginary parts of $\frac{\partial R\beta_4(R,\theta,\phi,\overline{l},t)}{\partial R}$ are oscillatory, then as $lim_{R\rightarrow 0}R\beta_4(R,\theta,\phi,\overline{l},t)=0$ and $lim_{R\rightarrow 0}\frac{\partial R\beta_4(R,\theta,\phi,\overline{l},t)}{\partial R}=M\in\mathcal{R}$, we can apply the result of Lemma 0.13, and assume that;

$|\int_{\mathcal{R}_{>0}}\beta_4(R,\theta,\phi,\overline{l},t)e^{irR}RdR|\leq\frac{4\sqrt{2}||\frac{\partial R\beta_4}{\partial R}||_\infty+D_{\overline{l}}}{r^2}$ (remove $\sqrt{2}$ and include spacing $\delta_{\overline{l}}$)

for sufficiently large $r\in\mathcal{R}_{>0}$, where;

$||\frac{\partial R\beta_4}{\partial R}||_\infty=||\beta_4+R\frac{\partial\beta_4}{\partial R}||_\infty$

$\leq||\beta_4||_\infty+||R\frac{\partial\beta_4}{\partial R}||_\infty$

$=|\frac{-iQ_{0,0}}{2\pi^2}[\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}\times\frac{\overline{d}'_{12}(\overline{l})}{l}]\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})sin(\theta)|$

$+|\frac{-iQ_{0,0}}{2\pi^2}[\frac{\partial}{\partial R}(\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})\times\frac{\overline{d}'_{12}(\overline{l})}{l}]\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})sin(\theta)|$

$+|\frac{-iQ_{0,0}}{2\pi^2}[\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}\times\frac{\overline{d}'_{12}(\overline{l})}{l}]\centerdot\frac{\partial}{\partial R}(\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l}))sin(\theta)|$

$\leq\frac{Q_{0,0}}{2\pi^2}|\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}||\frac{\overline{d}'_{12}(\overline{l})}{l}|$

$+\frac{Q_{0,0}}{2\pi^2}|\frac{\partial}{\partial R}(\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})||\frac{\overline{d}'_{12}(\overline{l})}{l}|$

$+\frac{Q_{0,0}}{2\pi^2}|\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}||\frac{\overline{d}'_{12}(\overline{l})}{l}||\frac{\partial}{\partial R}(\frac{\overline{k}}{|\overline{k}|})|$

$=\frac{Q_{0,0}}{2\pi^2}|\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}||\frac{\overline{d}'_{12}(\overline{l})}{l}|$

$+\frac{Q_{0,0}}{2\pi^2}|\frac{\partial}{\partial R}(\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})||\frac{\overline{d}'_{12}(\overline{l})}{l}|$

and $D_{\overline{l}}$ is the sum of the decay rates for the real and imaginary components of $\frac{\partial R\beta_4}{\partial R}$. Fix $\kappa>0$, then, as, for fixed $\overline{l}\neq\overline{0}$, $R\beta_4(\overline{k},\overline{l})\in L^1(\mathcal{R}^3)$, we can choose $\theta_{0,\overline{l},\kappa_1}<\theta_{0,\overline{l}}<\theta_{0,\overline{l},\kappa_2}$, $\phi_{0,\overline{l},\kappa_1}<\phi_{0,\overline{l}}<\phi_{0,\overline{l},\kappa_2}$, such that;

$|\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}} R\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi| \leq \frac{\kappa}{(l+1)^4}$

Then;

$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi} R\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq |\int_{\mathcal{R}_{>0}}\int_{([0,\pi)\times[0,2\pi)\setminus[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}]\times[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}])} R\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$+|\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}} R\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq |\int_{\mathcal{R}_{>0}}\int_{V_{\bar{l},\kappa_1,\kappa_2}} R\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi| + \frac{\kappa}{(l+1)^4}$

$\leq \int_{V_{\bar{l},\kappa_1,\kappa_2}}(|\int_{\mathcal{R}_{>0}} R\beta_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|)d\theta d\phi + \frac{\kappa}{(l+1)^4}$

$\leq 2\pi^2\frac{4\sqrt{2}||\frac{\partial R\beta_4}{\partial R}|_{V_{\bar{l},\kappa_1,\kappa_2}}||_\infty + D_{\bar{l}}}{r^2} + \frac{\kappa}{(l+1)^4}$

$\leq \frac{2\pi^2}{r^2}(\frac{2\sqrt{2}Q_{0,0}}{\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{2\sqrt{2}Q_{0,0}}{\pi^2}|\frac{\partial}{\partial R}(\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})||\frac{\bar{d}'_{12}(\bar{l})}{l}| + D_{\bar{l}}) + \frac{\kappa}{(l+1)^4}$

$= \frac{2\pi^2}{r^2}(\frac{2\sqrt{2}Q_{0,0}}{\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{2\sqrt{2}Q_{0,0}}{\pi^2}|\frac{\frac{\partial}{\partial R}(\bar{b}_{12,\bar{l}}(R,\theta,\phi))}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$

$+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)<(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l},\frac{\partial}{\partial R}((Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l})>}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|^3}||\frac{\bar{d}'_{12}(\bar{l})}{l}|+$
$D_{\bar{l}}) + \frac{\kappa}{(l+1)^4}$

$\leq \frac{2\pi^2}{r^2}(\frac{2\sqrt{2}Q_{0,0}}{\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{2\sqrt{2}Q_{0,0}}{\pi^2}|\frac{\frac{\partial}{\partial R}(\bar{b}_{12,\bar{l}}(R,\theta,\phi))}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{2\sqrt{2}Q_{0,0}}{\pi^2}\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)||((sin(\theta)cos(\phi),sin(\theta)sin(\phi),cos(\theta)))|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}| + D_{\bar{l}}) + \frac{\kappa}{(l+1)^4}$

$\leq \frac{2\pi^2}{r^2}(\frac{2\sqrt{2}Q_{0,0}}{\pi^2}\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{2\sqrt{2}Q_{0,0}}{\pi^2}\frac{|\frac{\partial}{\partial R}(\bar{b}_{12,\bar{l}}(R,\theta,\phi))|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

$+\frac{2\sqrt{6}Q_{0,0}}{\pi^2}\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|+D_{\bar{l}})+\frac{\kappa}{(l+1)^4}$ $(F)$

where;

$V_{\bar{l},\kappa_1,\kappa_2}=([0,\pi)\times[0,2\pi)\setminus[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}]\times[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}])$

.................

Using the fact that $R\frac{|\bar{b}_{12,\bar{l}}(R,\theta,\phi)|}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|_{[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}]\times[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}]\times\mathcal{R}_{>0}}$ is integrable, need to split $\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}R|\beta_4(R,\theta,\phi,\bar{l},t)|dRd\theta d\phi$

into $\int_{|R|>r}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}R|\beta_4(R,\theta,\phi,\bar{l},t)|dRd\theta d\phi$ $(A)$

and $\int_{|R|<r}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}R|\beta_4(R,\theta,\phi,\bar{l},t)|dRd\theta d\phi$ $(B)$

Can control $(A)$ as $\frac{1}{r^2(l+1)^4}$ due to decay, vary $(B)$ as $\frac{1}{r^{\frac{5}{4}}(1+l)^4}$, similarly to below, then angles $\theta_{0,\bar{l},\kappa_2}-\theta_{0,\bar{l},\kappa_1}$ and $\phi_{0,\bar{l},\kappa_2}-\phi_{0,\bar{l},\kappa_1}$ can vary as $(\frac{1}{r^{\frac{5}{4}}})^{\frac{1}{3}}=\frac{1}{r^{\frac{5}{12}}}$. Then last and worst term in $(F)$ varies as $\frac{1}{\frac{1}{r^{\frac{5}{12}}}^2}=r^{\frac{5}{6}}$.

Integrating and looking at all components, for sufficiently large $r\in\mathcal{R}_{>0}$. Follows that,

$|\int_{\mathcal{R}^6}h(\bar{k},\bar{l},t)e^{ir|\bar{k}+\bar{l}}d\bar{k}d\bar{l}|\leq\frac{Fr^{\frac{5}{6}}}{r^2}+\frac{H}{r^{\frac{5}{4}}}+\frac{J}{r^2}$

where $\{F,H,J\}\subset\mathcal{R}$. Follows that?(split again $Re(h),Im(h)$)

$|\int_{\mathcal{R}^6}h(\bar{k},\bar{l},t)cos(r|\bar{k}+\bar{l}|)d\bar{k}d\bar{l}\leq\frac{F'r^{\frac{5}{6}}}{r^2}+\frac{H'}{r^{\frac{5}{4}}}+\frac{J'}{r^2}$

for sufficiently large $r'>r$, invoking uniform version of Lemma 0.12 again. In particular;

$lim_{r\rightarrow\infty}r\int_{\mathcal{R}^6}h(\bar{k},\bar{l},t)cos(r|\bar{k}+\bar{l}|)d\bar{k}d\bar{l}=lim_{r\rightarrow\infty}\frac{1}{r}=lim_{r\rightarrow\infty}\frac{F'r^{\frac{5}{6}}}{r}+\frac{H'}{r^{\frac{1}{4}}}+\frac{J'}{r}=0$

so no radiation condition holds.

..........................
Similarly;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[(\overline{b}_{11,\overline{l}}(R,\theta,\phi)+\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|})$

$\times(\overline{d}'_{11}(\overline{l})+\frac{\overline{d}'_{12}(\overline{l})}{l})]\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\overline{l}$

$=\int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha(R,\theta,\phi,\overline{l},t)e^{irR}dRd\theta d\phi)d\overline{l}$

Splits as four terms, the worst of which is;

$\int_{\mathcal{R}^3}(\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\frac{iP_{1,1}}{2\pi^2}[\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}$

$\times\frac{\overline{d}'_{12}(\overline{l})}{l}]\centerdot\overline{u}_{\overline{l}}(R,\theta,\phi,\overline{l})]\mu(R,\theta,\phi,\overline{l},t)e^{irR}sin(\theta)dRd\theta d\phi)d\overline{l}$

$=\int_{\mathcal{R}^3}\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\overline{l},t)e^{irR}dRd\theta d\phi)d\overline{l}$

Again, fix $\overline{l}\neq\overline{0}$, with $\theta\neq cos^{-1}(\frac{l_3}{l})=\theta_{0,\overline{l}}$ and $\phi\neq tan^{-1}(\frac{l_2}{l_1})=\phi_{0,\overline{l}}$. By the result of Lemma 0.18, we can assume that the real and imaginary parts of $\alpha_4(R,\theta,\phi,\overline{l},t)$ are oscillatory, then as $lim_{R\rightarrow 0}\alpha_4(R,\theta,\phi,\overline{l},t)=M\in\mathcal{R}$, we can apply the result of Lemmas 0.15, 0.17 and 0.8, and assume that;

$|\int_{\mathcal{R}_{>0}}\alpha_4(R,\theta,\phi,\overline{l},t)e^{irR}dR|$

$\leq|\int_{\mathcal{R}_{>0}}Re(\alpha_4)(R,\theta,\phi,\overline{l},t)e^{irR}dR|+|\int_{\mathcal{R}_{>0}}Im(\alpha_4)(R,\theta,\phi,\overline{l},t)e^{irR}dR|$

$\leq\frac{2}{r}(\frac{n_{\overline{l},\theta,\phi,Re}||Re(\alpha_4)||_\infty}{\xi_{Re}}+\frac{D_{\overline{l},\theta,\phi,Re}}{n_{\overline{l},\theta,\phi}\xi_{Re}})$

$+\frac{2}{r}(\frac{n_{\overline{l},\theta,\phi,Im}||Im(\alpha_4)||_\infty}{\xi_{Im}}+\frac{D_{\overline{l},\theta,\phi,Im}}{n_{\overline{l},\theta,\phi}\xi_{Im}})$

so that, for $l>1$;

$|\int_{\mathcal{R}_{>0}}\alpha_4(R,\theta,\phi,\overline{l},t)e^{irR}dR|$

$\leq\frac{2}{r}(\frac{4\sqrt{3}l||Re(\alpha_4)||_\infty}{\xi_{Re}}+\frac{C2^{\frac{5}{2}}|\frac{\overline{d}'_{12}(\overline{l})}{l}|}{4\sqrt{3}l\xi_{Re}})$

$+\frac{2}{r}(\frac{4\sqrt{3}l||Im(\alpha_4)||_\infty}{\xi_{Im}}+\frac{C2^{\frac{5}{2}}|\frac{\overline{d}'_{12}(\overline{l})}{l}|}{4\sqrt{3}l\xi_{Im}})$

$\leq\frac{2}{r\xi}(4\sqrt{3}l(||Re(\alpha_4)||_\infty+||Im(\alpha_4)||_\infty)+\frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\overline{l})}{l}|}{4\sqrt{3}l})$

$\leq \frac{2}{r\xi}(4\sqrt{6}l||\alpha_4||_\infty + \frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}l})$

and, similarly, for $0 < l \leq 1$;

$|\int_{\mathcal{R}_{>0}} \alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|$

$\leq \frac{2}{r\xi}(4\sqrt{6}||\alpha_4||_\infty + \frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}})$ $(D)$

for sufficiently large $r \in \mathcal{R}_{>0}$, where $\xi_{Re} > 0, \xi_{Im} > 0$ are constants independent of $\bar{l},\theta,\phi$, $\xi = min(\xi_{Re},\xi_{Im}) > 0$, $\{D_{\bar{l},\theta,\phi,Re}, D_{\bar{l},\theta,\phi,Im}\}$ are the decay rates for the real and imaginary components of $\alpha_4(R,\theta,\phi,\bar{l},t)$. We have that;

$||\alpha_4||_\infty = |\frac{iP_{1,1}}{2\pi^2}[\frac{\overline{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|} \times \frac{\overline{d}'_{12}(\bar{l})}{l}] \centerdot \overline{u}_{\bar{l}}(R,\theta,\phi,\bar{l})sin(\theta)|$

$\leq \frac{P_{1,1}}{2\pi^2}|\frac{\overline{b}_{12,\bar{l}}(R,\theta,\phi)sin(\theta)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

$= |\frac{P_{1,1}}{2\pi^2}\frac{\overline{b}_{12,\bar{l}}(\bar{k})}{k^2|\bar{k}-\bar{l}|}||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

where;

$\frac{P_{1,1}}{2\pi^2}\frac{\overline{b}_{12,\bar{l}}(\bar{k})}{k^2|\bar{k}-\bar{l}|} = \frac{P_{1,1}}{2\pi^2}\frac{\overline{b}_{12,\bar{l}}(R,\theta,\phi)sin(\theta)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}$

Fix $\kappa > 0$, then, as, for fixed $\bar{l} \neq \bar{0}$, $\frac{|\overline{b}_{12,\bar{l}}(\bar{k})|}{k^2|\bar{k}-\bar{l}|} \in L^1(\mathcal{R}^3)$, we can choose $\theta_{0,\bar{l},\kappa_1} < \theta_{0,\bar{l}} < \theta_{0,\bar{l},\kappa_2}$, $\phi_{0,\bar{l},\kappa_1} < \phi_{0,\bar{l}} < \phi_{0,\bar{l},\kappa_2}$, such that;

$|\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}} \frac{P_{1,1}}{2\pi^2}\frac{|\overline{b}_{12,\bar{l}}(\bar{k})|}{k^2|\bar{k}-\bar{l}|}(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi| \leq \kappa'$

Then;

$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq |\int_{\mathcal{R}_{>0}}\int_{([0,\pi)\times[0,2\pi)\setminus[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}]\times[\phi_{0,\bar{l},\kappa_1},\phi_{0,\bar{l},\kappa_2}])}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$+|\int_{\mathcal{R}_{>0}}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq |\int_{\mathcal{R}_{>0}}\int_{V_{\bar{l},\kappa_1,\kappa_2}}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi| + \kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

$\leq \int_{V_{\bar{l},\kappa_1,\kappa_2}}(|\int_{\mathcal{R}>0}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dR|)d\theta d\phi+\kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

Using $(D)$, it follows that, for $l>1$;

$|\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq 2\pi^2\frac{2}{r\xi}(4\sqrt{6}l||\alpha_4|_{V_{\bar{l},\kappa_1,\kappa_2}}||_\infty+\frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}l})+\kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}l}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|_{V_{\bar{l},\kappa_1,\kappa_2}}||\frac{\overline{d}'_{12}(\bar{l})}{l}|+\frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}l})$

$+\kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

and, for $0<l\leq 1$;

$|\int_{\mathcal{R}>0}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\bar{l},t)e^{irR}dRd\theta d\phi|$

$\leq 2\pi^2\frac{2}{r\xi}(4\sqrt{6}||\alpha_4|_{V_{\bar{l},\kappa_1,\kappa_2}}||_\infty+\frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}})+\kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|_{V_{\bar{l},\kappa_1,\kappa_2}}||\frac{\overline{d}'_{12}(\bar{l})}{l}|+\frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}})$

$+\kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|$ $(H)$

Fix $\delta>0$ arbitrary, then we have that, for $l>\delta$, sufficiently small $0<\kappa<min(\frac{\delta}{2},\delta^2)$;

$\int_{\mathcal{R}>0}\int_{\theta_{0,\bar{l},\kappa_1}\leq\theta\leq\theta_{0,\bar{l},\kappa_2}}\int_{\phi_{0,\bar{l},\kappa_1}\leq\phi\leq\phi_{0,\bar{l},\kappa_2}}\frac{P_{1,1}}{2\pi^2}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)sin(\theta)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|}|dRd\theta d\phi$

$=\int_{W_{\bar{l},\kappa_1,\kappa_2}}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}-\bar{l}||\bar{k}|^2}$

$=\int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}$

$\leq\int_{B(\bar{0},\kappa)}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}+\int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}\setminus B(\bar{0},\kappa)}\frac{P_{1,1}}{2\pi^2}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}$

$\leq\frac{P_{1,1}}{2\pi^2}||\frac{\bar{b}_{12}(\bar{k})}{|\bar{k}+\bar{l}|^2}||_{\infty,B(\bar{0},\kappa)}\int_{0<R<\kappa}\frac{1}{R}R^2|sin(\theta)|dRd\theta d\phi+\frac{P_{1,1}}{2\pi^2}\int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}\setminus B(\bar{0},\kappa)}\frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}||\bar{k}+\bar{l}|^2}d\bar{k}$

$\leq\frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2}+\frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}\int_{(W_{\bar{l},\kappa_1,\kappa_2})_{\bar{l}}}|\frac{\bar{b}_{12}(\bar{k})}{|\bar{k}+\bar{l}|^2})|d\bar{k}$

$=\frac{2P_{1,1}}{\delta^2\pi^2}||\bar{b}_{12}(\bar{k})||_{\infty,B(\bar{0},\kappa)}\frac{\kappa^2}{2}+\frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}\int_{(W_{\bar{l},\kappa_1,\kappa_2})}|\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{R^2}|R^2sin(\theta)dRd\theta d\phi$

$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\overline{b}_{12}(\overline{k})||_{\infty,B(\overline{0},\kappa)}\frac{\kappa^2}{2}+\frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}|\theta_{0,\overline{l},\kappa_2}-\theta_{0,\overline{l},\kappa_1}||\phi_{0,\overline{l},\kappa_2}-\phi_{0,\overline{l},\kappa_2}|_{S^1(1)}\int_{\mathcal{R}_{>0}}|\overline{b}_{12,\overline{l}}(R)|dR$

$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\overline{b}_{12}(\overline{k})||_{\infty,B(\overline{0},\kappa)}\frac{\kappa^2}{2}+\frac{1}{\kappa}\frac{P_{1,1}}{2\pi^2}|\theta_{0,\overline{l},\kappa_2}-\theta_{0,\overline{l},\kappa_1}||\phi_{0,\overline{l},\kappa_2}-\phi_{0,\overline{l},\kappa_2}|_{S^1(1)}K$

$\leq \frac{2P_{1,1}}{\delta^2\pi^2}||\overline{b}_{12}(\overline{k})||_{\infty,B(\overline{0},\kappa)}\frac{\kappa^2}{2}+\frac{P_{1,1}}{2\pi^2}\kappa$

$\leq \frac{2P_{1,1}}{\pi^2}||\overline{b}_{12}(\overline{k})||_{\infty,B(\overline{0},\kappa)}\frac{\delta^2}{2}+\frac{P_{1,1}}{2\pi^2}\kappa=\kappa'\ (M)$

for $|\theta_{0,\overline{l},\kappa_2}-\theta_{0,\overline{l},\kappa_1}|=|\phi_{0,\overline{l},\kappa_2}-\phi_{0,\overline{l},\kappa_1}|_{S^1(1)}$, $|\theta_{0,\overline{l},\kappa_2}-\theta_{0,\overline{l},\kappa_1}|\leq\frac{\kappa}{\sqrt{K}}$ $(G)$

where;

$W_{\overline{l},\kappa_1,\kappa_2}=([\phi_{0,\overline{l},\kappa_1},\phi_{0,\overline{l},\kappa_2}]\times[\phi_{0,\overline{l},\kappa_1},\phi_{0,\overline{l},\kappa_2}]\times\mathcal{R}_{>0})$

$(W_{\overline{l},\kappa_1,\kappa_2})_{\overline{l}}=\{\overline{k}:\overline{k}+\overline{l}\in W_{\overline{l},\kappa_1,\kappa_2}\}$

and, we can assume that $|\overline{b}_{12,\overline{l}}(R)|$ is independent of $\{\theta,\phi\}$, with $||\overline{b}_{12,\overline{l}}(R)||_{L^1(\mathcal{R}_{>0})}\leq K$, independently of $\overline{l}$, due to the decay.

In particularly, choosing $\theta_{0,\overline{l},\kappa_2}=\theta_{0,\overline{l}}+\frac{\kappa}{2\sqrt{K}}$, $\theta_{0,\overline{l},\kappa_1}=\theta_{0,\overline{l}}-\frac{\kappa}{2\sqrt{K}}$, $\phi_{0,\overline{l},\kappa_2}=\phi_{0,\overline{l}}+\frac{\kappa}{2\sqrt{K}}$, $\phi_{0,\overline{l},\kappa_1}=\phi_{0,\overline{l}}-\frac{\kappa}{2\sqrt{K}}$, we have that $(G)$ holds and $d(\overline{l},V_{\overline{l},\kappa_1,\kappa_2})\geq lsin(\frac{\kappa}{2\sqrt{K}})\geq\frac{l\kappa}{4\sqrt{K}}$, for sufficiently small $\kappa$. We then have that;

$|\frac{\overline{b}_{12,\overline{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\overline{l}|}|_{V_{\overline{l},\kappa_1,\kappa_2}}|\leq\frac{4\sqrt{K}}{l\kappa}||\overline{b}_{12,\overline{l}}(R,\theta,\phi)||_\infty=\frac{4\sqrt{K}D}{l\kappa}$

where $D\in\mathcal{R}_{>0}$, independent of $\overline{l}$. From $(H),(M)$, we obtain that, for $l>1$;

$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\overline{l},t)e^{irR}dRd\theta d\phi|$

$\leq\frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}l}{2\pi^2}(\frac{4\sqrt{K}D}{l\kappa})|\frac{\overline{d}'_{12}(\overline{l})}{l}|+\frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\overline{l})}{l}|}{4\sqrt{3}l})$

$+\kappa'||\frac{\overline{d}'_{12}(\overline{l})}{l}|\ (l>\delta)$

and, for $0<l\leq 1$;

$|\int_{\mathcal{R}_{>0}}\int_{0\leq\theta<\pi}\int_{0\leq\phi\leq 2\pi}\alpha_4(R,\theta,\phi,\overline{l},t)e^{irR}dRd\theta d\phi|$

$\leq \frac{4\pi^2}{r\xi}(\frac{4\sqrt{6}P_{1,1}}{2\pi^2}(\frac{4\sqrt{K}D}{l\kappa})|\frac{\overline{d}'_{12}(\bar{l})}{l}| + \frac{C2^{\frac{7}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|}{4\sqrt{3}})$

$+\kappa'||\frac{\overline{d}'_{12}(\bar{l})}{l}|\ (l > \delta)$

Using the fact that $\{\frac{|\overline{d}'_{12}(\bar{l})|}{l^2}, \frac{|\overline{d}'_{12}(\bar{l})|}{l}\} \subset L^1(\mathcal{R}^3)$, and integrating $g(\overline{k}, \bar{l}, t)e^{ir|\overline{k}+\bar{l}|}$ over $\mathcal{R}^3 \times B(\overline{0}, \delta)$ separately, using Lemma 0.9, looking at all components, for sufficiently large $r \in \mathcal{R}_{>0}$, need uniformity in $\bar{l}$ version of Lemma 0.12, follows that,

$|\int_{\mathcal{R}^6} g(\overline{k}, \bar{l}, t)e^{ir|\overline{k}+\bar{l}|}d\overline{k}d\bar{l}| \leq A\delta + \frac{F(\kappa)}{r} + H\kappa'$

where $\{A, H\} \subset \mathcal{R}$. Follows that?(split again $Re(g), Im(g)$)

$|\int_{\mathcal{R}^6} g(\overline{k}, \bar{l}, t)sin(r|\overline{k} + \bar{l}|)d\overline{k}d\bar{l} \leq B\delta + \frac{T(\kappa)}{r} + S\kappa'$

for sufficiently large $r$, In particular as $\kappa' > 0, \delta > 0$ can be made arbitrarily small, and;

$|lim_{r\rightarrow\infty} \int_{\mathcal{R}^6} g(\overline{k}, \bar{l}, t)cos(r|\overline{k} + \bar{l}|)d\overline{k}d\bar{l}| < A\delta + H\kappa'$

$lim_{r\rightarrow\infty} \int_{\mathcal{R}^6} g(\overline{k}, \bar{l}, t)cos(r|\overline{k} + \bar{l}|)d\overline{k}d\bar{l} = 0$

so no radiation condition holds.

□

**Lemma 0.8.** *We have that;*

$|\alpha_4(R, \theta, \phi, t, \bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\overline{d}'_{12}(\bar{l})}{l}|$, *for* $R > 4l\sqrt{3}, l > 1$

$R > 4\sqrt{3}, 0 < l \leq 1$

$|Re(\alpha_4)(R, \theta, \phi, t, \bar{l}) \leq |\frac{C2^{\frac{5}{2}}}{R^2}|\frac{\overline{d}'_{12}(\bar{l})}{l}||$, *for* $R > 4l\sqrt{3}, l > 1$

$R > 4\sqrt{3}, 0 < l \leq 1$

$|Im(\alpha_4)(R, \theta, \phi, t, \bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\overline{d}'_{12}(\bar{l})}{l}|$, *for* $R > 4l\sqrt{3}, l > 1$

$R > 4\sqrt{3}, 0 < l \leq 1$

*where* $C\in\mathcal{R}_{>0}$

*In particularly, the families* $\{Re(\alpha_4)(R,\theta,\phi,t,\bar{l}):\bar{l}\in\mathcal{R}^3,\bar{l}\neq\bar{0},\theta\neq cos^{-1}(\frac{l_3}{l_1}),\phi\neq tan^{-1}(\frac{l_2}{l_1})\}$ *and* $\{Im(\alpha_4)(R,\theta,\phi,t,\bar{l}):\bar{l}\in\mathcal{R}^3,\bar{l}\neq\bar{0},\theta\neq cos^{-1}(\frac{l_3}{l_1}),\phi\neq tan^{-1}(\frac{l_2}{l_1})\}$ *are of moderate decrease* $n_{\bar{l},\theta,\phi}$*, with;*

$n_{\bar{l},\theta,\phi}=4l\sqrt{3}$, $l>1$

$n_{\bar{l},\theta,\phi}=4\sqrt{3}$, $0<l\leq 1$

*and* $D_{\bar{l},\theta,\phi}=C2^{\frac{5}{2}}|\frac{\overline{d}'_{12}(\bar{l})}{l}|$

*Proof.* We have that;

$|\alpha_4|\leq|\frac{P_{1,1}}{2\pi^2}\frac{\overline{b}_{12,\bar{l}}(\overline{k})}{k^2|\overline{k}-\bar{l}|}||\frac{\overline{d}'_{12}(\bar{l})}{l}|$

$|\overline{b}_{12,\bar{l}}(\overline{k})|\leq\frac{D}{|\overline{k}-\bar{l}|^4}$, $|\overline{k}-\bar{l}|>0$ (change this)

where $D\in\mathcal{R}_{>0}$

so that;

$|\alpha_4(R,\theta,\phi,t,\bar{l})|\leq|\frac{\overline{d}'_{12}(\bar{l})}{l}|\frac{C}{|\overline{k}-\bar{l}|^5}$

$=C|\frac{\overline{d}'_{12}(\bar{l})}{l}|\frac{1}{[(Rsin(\theta)cos(\phi)-l_1)^2+(Rsin(\theta)sin(\phi)-l_2)^2+(Rcos(\theta)-l_3)^2]^{\frac{5}{2}}}$

$=\frac{C}{R^5}|\frac{\overline{d}'_{12}(\bar{l})}{l}|\frac{1}{[(sin(\theta)cos(\phi)-\frac{l_1}{R})^2+(sin(\theta)sin(\phi)-\frac{l_2}{R})^2+(cos(\theta)-\frac{l_3}{R})^2]^{\frac{5}{2}}}$

$=\frac{C}{R^5}|\frac{\overline{d}'_{12}(\bar{l})}{l}|\frac{1}{[1-\frac{2l_1sin(\theta)cos(\phi)}{R}-\frac{2l_2sin(\theta)sin(\phi)}{R}-\frac{2l_3cos(\theta)}{R}+\frac{l^2}{R^2}]^{\frac{5}{2}}}$

$=\frac{C}{R^5}|\frac{\overline{d}'_{12}(\bar{l})}{l}|\frac{1}{(1-x+\frac{l^2}{R^2})^{\frac{5}{2}}}$

where $C\in\mathcal{R}_{>0}$ and;

$|x|\leq\frac{2(|l_1|+|l_2|+|l_3|)}{R}\leq\frac{2l\sqrt{3}}{R}\leq\frac{1}{2}$, for $R>4l\sqrt{3}$

so that;

$|\alpha_4(R,\theta,\phi,t,\bar{l})|\leq\frac{C2^{\frac{5}{2}}}{R^5}|\frac{\overline{d}'_{12}(\bar{l})}{l}|\leq\frac{C2^{\frac{5}{2}}}{R^2}|\frac{\overline{d}'_{12}(\bar{l})}{l}$ (for $R>4l\sqrt{3},l>1$,

$R > 4\sqrt{3}, 0 < l \leq 1)$

In particularly;

$|Re(\alpha_4)(R,\theta,\phi,t,\bar{l})| \leq |\alpha_4(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

for $R > 4l\sqrt{3}, l > 1,\ R > 4\sqrt{3}, 0 < l \leq 1$

$|Im(\alpha_4)(R,\theta,\phi,t,\bar{l})| \leq |\alpha_4(R,\theta,\phi,t,\bar{l})| \leq \frac{C2^{\frac{5}{2}}}{R^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|$

for $R > 4l\sqrt{3}, l > 1,\ R > 4\sqrt{3}, 0 < l \leq 1$

□

**Lemma 0.9.** *We have that;*

$\frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}| \in L^1(\mathcal{R}^6),\ \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}| \in L^1(\mathcal{R}^6)$

*Proof.* For the first claim, fix $\bar{l} \neq \bar{0}$, then;

$\frac{1}{|\bar{k}|^2}|_{B(\bar{l},\frac{l}{2})} \leq \frac{4}{l^2},\ \frac{1}{|\bar{k}-\bar{l}|}|_{\mathcal{R}^3\setminus B(\bar{l},\frac{l}{2})} \leq \frac{2}{l}$

so that;

$\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|d\bar{k} = \int_{B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}d\bar{k} + \int_{\mathcal{R}^3\setminus B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}d\bar{k}$

$\leq \frac{4}{l^2}\int_{B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}-\bar{l}|}d\bar{k} + \frac{2}{l}\int_{\mathcal{R}^3\setminus B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$\leq \frac{4}{l^2}\int_{B(\bar{l},\frac{l}{2})} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}-\bar{l}|}d\bar{k} + \frac{2}{l}\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$= \frac{4}{l^2}\int_{B(\bar{0},\frac{l}{2})} \frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}|}d\bar{k} + \frac{2}{l}\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$= \frac{4}{l^2}\int_0^{\frac{l}{2}}\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi} \frac{|\bar{b}_{12}(R,\theta,\phi)|}{R}R^2sin(\theta)dRd\theta d\phi + \frac{2}{l}\int_{B(\bar{0},1)} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$+ \int_{\mathcal{R}^3\setminus B(\bar{0},1)} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2}d\bar{k}$

$\leq \frac{8\pi^2}{l^2}[\frac{R^2}{2}]_0^{\frac{l}{2}} + \frac{2}{l}\int_0^1\int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi} \frac{|\bar{b}_{12}(R,\theta,\phi)|}{R^2}R^2sin(\theta)dRd\theta d\phi + \int_{\mathcal{R}^3\setminus B(\bar{0},1)}|\bar{b}_{12,\bar{l}}(\bar{k})|d\bar{k}$

$\leq \pi^2 + \frac{4\pi^2}{l}[R]_0^1 + C$

$= \pi^2 + \frac{4\pi^2}{l} + C$

where $C = ||\bar{b}_{12,\bar{l}}||_{L^1(\mathcal{R}^3)}$ is independent of $\bar{l}$. It follows that;

$\int_{\mathcal{R}^6} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}|^2|\bar{k}-\bar{l}|}|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{k}d\bar{l} \leq \int_{\mathcal{R}^3}(\pi^2 + \frac{4\pi^2}{l} + C)|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{l}$

$= (\pi^2 + C)\int_{\mathcal{R}^3} \frac{|\bar{d}'_{12}(\bar{l})}{|\bar{l}|}d\bar{l} + 4\pi^2 \int_{\mathcal{R}^3} \frac{|\bar{d}'_{12}(\bar{l})}{|\bar{l}|^2}d\bar{l}$

$\leq (\pi^2 + C)(\int_{B(\bar{0},1)} \frac{|\bar{d}'_{12}(\bar{l})}{|\bar{l}|}d\bar{l} + \int_{\mathcal{R}^3\setminus B(\bar{0},1)} |\bar{d}'_{12}(\bar{l})|d\bar{l})$

$+4\pi^2(\int_{B(\bar{0},1)} \frac{|\bar{d}'_{12}(\bar{l})}{|\bar{l}|^2}d\bar{l} + \int_{\mathcal{R}^3\setminus B(\bar{0},1)} |\bar{d}'_{12}(\bar{l})|d\bar{l})$

$\leq (\pi^2 + C)(\int_0^1 \int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi} ||\bar{d}'_{12}(R,\theta,\phi)|Rsin(\theta)d\theta d\phi + D)$

$+4\pi^2(\int_0^1 \int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi} ||\bar{d}'_{12}(R,\theta,\phi)|sin(\theta)d\theta d\phi + D)$

$\leq (\pi^2 + C)(\pi^2 + D) + 4\pi^2(2\pi^2 + D)$

$= 9\pi^4 + \pi^2 C + 5\pi^2 D + CD$

where $D = ||\bar{d}'_{12}||_{L^1(\mathcal{R}^3)}$

For the second claim, fix $\bar{l} \neq \bar{0}$, then, using the substitution $\bar{k}' = \bar{k} - \bar{l}$ and the previous proof, we obtain that;

$\int_{\mathcal{R}^3} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|d\bar{k} = \int_{\mathcal{R}^3} \frac{|\bar{b}_{12}(\bar{k})|}{|\bar{k}|^2|\bar{k}+\bar{l}|}|d\bar{k} \leq \pi^2 + \frac{4\pi^2}{l} + C$

Following the above proof again, we have that;

$\int_{\mathcal{R}^6} \frac{|\bar{b}_{12,\bar{l}}(\bar{k})|}{|\bar{k}||\bar{k}-\bar{l}|^2}|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{k}d\bar{l} \leq \int_{\mathcal{R}^3}(\pi^2 + \frac{4\pi^2}{l} + C)|\frac{\bar{d}'_{12}(\bar{l})}{l}|d\bar{l}$

$\leq 9\pi^4 + \pi^2 C + 5\pi^2 D + CD$

□

**Definition 0.10.** *We say that $f \in C(\mathcal{R})$ is of moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|^2}$ for $|x| > 1$. We say that $f \in C(\mathcal{R}_{>0})$ is of moderate decrease if there exists a constant $D \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D}{|x|^2}$ for $|x| > 1$. We say that $f \in C(\mathcal{R})$ is of moderate decrease $n$, if there exists a constant $D_n \in \mathcal{R}_{>0}$ with $|f(x)| \leq \frac{D_n}{|x|^2}$ for $|x| > n$. We say that $f \in C(\mathcal{R}_{>0})$ is of moderate* *decrease n if there exists a constant* $D_n \in \mathcal{R}_{>0}$ *with* $|f(x)| \leq \frac{D_n}{|x|^2}$ *for* $|x| > n$. *We say that* $f \in C(\mathcal{R})$ *is of very moderate decrease if there exists a constant* $D \in \mathcal{R}_{>0}$ *with* $|f(x)| \leq \frac{D}{|x|}$ *for* $|x| > 1$. *We say that* $f \in C(\mathcal{R})$ *is of very moderate decrease n if there exists a constant* $D_n \in \mathcal{R}_{>0}$ *with* $|f(x)| \leq \frac{D}{|x|}$ *for* $|x| > n$. *We say that* $f \in C(\mathcal{R}_{>0})$ *is of very moderate decrease if there exists a constant* $D \in \mathcal{R}_{>0}$ *with* $|f(x)| \leq \frac{D}{|x|}$ *for* $|x| > 1$. *We say that* $f \in C(\mathcal{R}_{>0})$ *is of very moderate decrease n if there exists a constant* $D_n \in \mathcal{R}_{>0}$ *with* $|f(x)| \leq \frac{D_n}{|x|}$ *for* $|x| > n$. *We say that* $f \in C(\mathcal{R})$ *is non-oscillatory if there are finitely many points* $\{y_i : 1 \leq i \leq n\} \subset \mathcal{R}$ *for which* $f|_{(y_i,y_{i+1})}$ *is monotone,* $1 \leq i \leq n-1$, *and* $f|_{(-\infty,y_1)}$ *and* $f|_{(y_n,\infty)}$ *is monotone. We denote by* $val(f)$ *the minimum number of such points. We say that* $f \in C(\mathcal{R}_{>0})$ *is non-oscillatory if there are finitely many points* $\{y_i : 1 \leq i \leq n\} \subset \mathcal{R}_{>0}$ *for which* $f|_{(y_i,y_{i+1})}$ *is monotone,* $1 \leq i \leq n-1$, *and* $f|_{(0,y_1)}$ *and* $f|_{(y_n,\infty)}$ *is monotone. Similarily, we denote by* $val(f)$ *the minimum number. We say that* $f \in C(\mathcal{R})$ *is oscillatory if there exists an increasing sequence* $\{y_i : i \in \mathcal{Z}\} \subset \mathcal{R}$, *for which* $f|_{(y_i,y_{i+1})}$ *is monotone,* $i \in \mathcal{Z}$, *and there exists* $\delta > 0$, *with* $y_{i+1} - y_i > \delta$, *for* $i \in \mathcal{Z}$. *We say that* $f \in C(\mathcal{R}_{>0})$ *is oscillatory if there exists a sequence* $\{y_i : i \in \mathcal{N}\} \subset \mathcal{R}$, *for which* $f|_{(0,y_1)}$ *is monotone, and* $f|_{(y_i,y_{i+1})}$ *is monotone,* $i \in \mathcal{N}$, *and there exists* $\delta > 0$, *with* $y_1 > \delta$ *and* $y_{i+1} - y_i > \delta$, *for* $i \in \mathcal{N}$.

**Lemma 0.11.** *Let* $f \in C(\mathcal{R})$ *and* $\frac{df}{dx} \in C(\mathcal{R})$ *be of moderate decrease, with* $\frac{df}{dx}$ *non-oscillatory, then defining the Fourier transform by;*

$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{R}} f(x) e^{-ikx} dx$$

*we have that, there exists a constant* $C \in \mathcal{R}_{>0}$, *such that;*

$$|\mathcal{F}(f)(k)| \leq \frac{C}{|k|^2}$$

*for sufficiently large* $k$. *Let* $f \in C(\mathcal{R})$ *and* $\frac{df}{dx} \in C(\mathcal{R})$ *be of moderate decrease, with* $\frac{df}{dx}$ *oscillatory, then, similarly;*

*we have that, there exists a constant* $C \in \mathcal{R}_{>0}$, *such that;*

$$|\mathcal{F}(f)(k)| \leq \frac{C}{|k|^2}$$

*for sufficiently large* $k$.

*The same result holds in the two claims, replacing moderate decrease with moderate decrease n.*

*Proof.* As $f$ is of moderate decrease, we have that $f \in L^1(\mathcal{R})$ and $lim_{|x|\rightarrow\infty} f(x) = 0$ Similarly, $\frac{df}{dx} \in L^1(\mathcal{R})$ and $\frac{df}{dx}$ is continuous. We have, using integration by parts, that;

$\mathcal{F}(\frac{df}{dx})(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathcal{R}} \frac{df}{dx}(y) e^{-iky} dy$

$= [f(y)e^{-iky}]_{-\infty}^{\infty} + ik \int_{\mathcal{R}} f(y) e^{-iky} dy$

$= ik \int_{\mathcal{R}} f(y) e^{-iky} dy$

$= ik\mathcal{F}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|}$, $(\dagger)$

As $\frac{df}{dx}$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_\epsilon \in \mathcal{N}$ such that;

$|\mathcal{F}(\frac{df}{dx})(k) - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N_\epsilon}^{N_\epsilon} \frac{df}{dx}(y) e^{-iky} dy| < \epsilon$ $(*)$

As $\frac{df}{dx}|_{-N_\epsilon, N_\epsilon}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [7], using underflow, we can find $\{D_\epsilon, E_\epsilon\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N_\epsilon}^{N_\epsilon} \frac{df}{dx}(y) e^{-iky} dy| < \frac{E_\epsilon}{|k|}$, $(**)$

It is easy to see from the proof, that $\{D_\epsilon, E_\epsilon\}$ can be chosen uniformly in $\epsilon$. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k| > D_\epsilon$;

$|\mathcal{F}(\frac{df}{dx})(k)|$

$\leq |\mathcal{F}(\frac{df}{dx})(k) - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N_\epsilon}^{N_\epsilon} \frac{df}{dx}(y) e^{-iky} dy| + |\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N_\epsilon}^{N_\epsilon} \frac{df}{dx}(y) e^{-iky} dy|$

$< \epsilon + \frac{E_\epsilon}{|k|}$

so that, as $\{D_\epsilon, E_\epsilon\}$ were uniform and $\epsilon$ was arbitrary, we obtain that;

$|\mathcal{F}(\frac{df}{dx})(k)| < \frac{E}{|k|}$, for $|k| > D$

and, from (†), for $|k| > D$, that;

$|\mathcal{F}(f)(k)| \leq \frac{|\mathcal{F}(\frac{df}{dx})(k)|}{|k|} < \frac{E}{|k|^2}$

For the second claim, we can follow the proof of the second claim in Lemma 0.13. The final claim is a simple adaptation of the first two claims. □

**Lemma 0.12.** *Let $f \in C(\mathcal{R}_{>0})$ be of moderate decrease, with $f$ non-oscillatory, and $lim_{x\rightarrow 0}f(x) = M$, with $M \in \mathcal{R}$, then defining the half Fourier transform $\mathcal{G}$, by;*

$\mathcal{G}(f)(k) = \int_0^\infty f(x)e^{-ikx}dx$

*we have that, there exists a constant $E \in \mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|}$

*for sufficiently large $|k|$. Moreover, we can choose;*

$E = 2||f||_\infty val(f)$

*Let $f \in C(\mathcal{R}_{>0})$ be of moderate decrease, with $f$ oscillatory, and $lim_{x\rightarrow 0}f(x) = M$, with $M \in \mathcal{R}$, then, similarly;*

*we have that, there exists a constant $E \in \mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|}$

*for sufficiently large $|k|$. Moreover, we can choose $E = \frac{(4||f||_\infty + D)}{\delta}$, where $D$ and $\delta$ are given in Definition 0.10.*

*The first claim is the same, replacing moderate decrease with moderate decrease $n$. The second claim is the same, replacing moderate decrease with moderate decrease $n$, with the modification that we can choose $E = \frac{2n||f||_\infty}{\delta} + \frac{2D_n}{n\delta}$.*

*Proof.* As $f$ is of moderate decrease and $lim_{x\rightarrow 0}f(x) = M$, we have that $f \in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty}f(x) = 0$.

As $f$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_\epsilon \in \mathcal{N}$ such that;

$|\mathcal{G}(f)(k) - \int_0^{N_\epsilon} f(y)e^{-iky}dy| < \epsilon$ $(*)$

As $f|_{0,N_\epsilon}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [7], using underflow, we can find $\{D_\epsilon, E_\epsilon\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\int_0^{N_\epsilon} f(y)e^{-iky}dy| < \frac{E_\epsilon}{|k|}$, $(**)$

It is easy to see from the proof, that $\{D_\epsilon, E_\epsilon\}$ can be chosen uniformly in $\epsilon$, Splitting the calculation into real and imaginary components, it is straightfoward to see that it is possible to choose $E_\epsilon$ with $E_\epsilon = 2||f||_\infty val(f)$, noting that the infinitesimal correction existing after the use of underflow, drops out after taking the standard part. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k| > D_\epsilon$;

$|\mathcal{G}(f)(k)|$

$\leq |\mathcal{G}(f)(k) - \int_0^{N_\epsilon} f(y)e^{-iky}dy| + |\int_0^{N_\epsilon} f(y)e^{-iky}dy|$

$< \epsilon + \frac{E_\epsilon}{|k|}$

so that, as $\{D_{\epsilon,\rho}, E_\epsilon\}$ were uniform and $\epsilon$ was arbitrary, we obtain that;

$|\mathcal{G}(f)(k)| < \frac{E}{|k|}$, for sufficiently large $|k|$

For the second claim, after choosing $N \in \mathcal{N}$, we have that $f|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta}$ monotone intervals. As in $(**)$, and inspection of the proof in [7], we get;

$|\int_0^N fe^{-iky}dy| < \frac{E_N}{|k|}$

for sufficiently large $|k|$, where $E_N = \frac{2NC}{\delta}$ and $C = max_{x\in\mathcal{R}_{>0}}|f|$.

Choosing $N > 1$, as $f$ is of moderate decrease, we can assume that $|f| \leq \frac{D}{x^2}$, for $x > N$. Then, using the proof in [7] again, the definition of oscillatory, and noting that ${}^*\sum_{y_i^* > N} \frac{D}{y_i^2} \simeq \sum_{y_i > N} \frac{D}{y_i^2}$, we have that, for sufficiently large $|k|$;

$|\int_N^\infty f e^{-iky} dy| < (\frac{2}{|k|} \sum_{y_i > N} \frac{D}{y_i^2})$

$\leq (\frac{2}{|k|} \sum_{n \in \mathcal{Z}_{\geq 0}} \frac{D}{(y_{i_0}+n\delta)^2})$

$\leq \frac{2D}{\delta|k|} \int_{y_{i_0}}^\infty \frac{dx}{x^2}$

$= \frac{2D}{\delta|k| y_{i_0}}$

$\leq \frac{2D}{\delta|k|N}$

where $y_{i_0} \geq N$ and $y_{i_0} \leq y_i$, for all $y_i \geq N$. It follows that;

$|\mathcal{G}(f)(k)| = |\int_0^N f e^{-iky} dy + \int_N^\infty f e^{-iky} dy|$

$\leq |\int_0^N f e^{-iky} dy| + |\int_N^\infty f e^{-iky} dy|$

$\leq \frac{E_N}{|k|} + \frac{2D}{\delta|k|N}$

$\leq \frac{2}{|k|}(\frac{NC}{\delta} + \frac{D}{\delta N})$

It follows, using (†), that;

$|\mathcal{G}(f)(k) \leq \frac{E}{|k|}$

where $E = 2(\frac{NC}{\delta} + \frac{D}{\delta N})$

In particular, choosing $N = 2$, we can take;

$E = 2(\frac{2C}{\delta} + \frac{D}{2\delta}) = \frac{(4C+D)}{\delta} = \frac{(4||f||_\infty + D)}{\delta}$

For the final claim, the modification for the first part is the same. In the second part, choose $N \geq n$, rather than $N > 1$ in the proof, and replace $D$ with $D_n$, to get $E = 2(\frac{NC}{\delta} + \frac{D_n}{\delta N})$, then, taking $N = n$, we obtain $E = 2(\frac{nC}{\delta} + \frac{D_n}{\delta n})$.

□

**Lemma 0.13.** *Let $f \in C(\mathcal{R}_{>0})$ and $\frac{df}{dx} \in C(\mathcal{R}_{>0})$ be of moderate decrease, with $\frac{df}{dx}$ non-oscillatory, and $lim_{x \to 0} f(x) = 0$, $lim_{x \to 0} \frac{df}{dx}(x) = M$, with $M \in \mathcal{R}$, then defining the half Fourier transform $\mathcal{G}$, by;*

$\mathcal{G}(f)(k) = \int_0^\infty f(x)e^{-ikx}dx$

*we have that, there exists a constant* $E \in \mathcal{R}_{>0}$*, such that;*

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|^2}$

*for sufficiently large* $k$*. Moreover, we can choose* $E = 2||\frac{df}{dx}||_\infty val(\frac{df}{dx})$

*Let* $f \in C(\mathcal{R}_{>0})$ *and* $\frac{df}{dx} \in C(\mathcal{R}_{>0})$ *be of moderate decrease, with* $\frac{df}{dx}$ *oscillatory, and* $lim_{x\rightarrow 0}f(x) = 0$, $lim_{x\rightarrow 0}\frac{df}{dx}(x) = M$*, with* $M \in \mathcal{R}$*, then, similarly;*

*we have that, there exists a constant* $E \in \mathcal{R}_{>0}$*, such that;*

$|\mathcal{G}(f)(k)| \leq \frac{E}{|k|^2}$

*for sufficiently large* $k$*, Moreover, we can choose* $E = \frac{(4||\frac{df}{dx}||_\infty + D)}{\delta}$.

*The first claim is the same, replacing moderate decrease with moderate decrease* $n$*. The second claim is the same, replacing moderate decrease with moderate decrease* $n$*, with the modification that we can choose* $E = \frac{2n||\frac{df}{dx}||_\infty}{\delta} + \frac{2D_n}{n\delta}$.

*Proof.* As $f$ is of moderate decrease and $lim_{x\rightarrow 0}f(x) = 0$, we have that $f \in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty}f(x) = 0$. Similarly, $\frac{df}{dx} \in L^1(\mathcal{R}_{>0})$ and $\frac{df}{dx}$ is continuous. We have, using integration by parts, that;

$\mathcal{G}(\frac{df}{dx})(k) = \int_0^\infty \frac{df}{dx}(y)e^{-iky}dy$

$= [f(y)e^{-iky}]_0^\infty + ik\int_0^\infty f(y)e^{-iky}dy$

$= ik\int_0^\infty f(y)e^{-iky}dy$

$= ik\mathcal{G}(f)(k)$

so that, for $|k| > 1$;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|}$, (†)

As $\frac{df}{dx}$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_\epsilon \in \mathcal{N}$ such that;

$|\mathcal{G}(\frac{df}{dx})(k) - \int_0^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy| < \epsilon$ $(*)$

As $\frac{df}{dx}|_{0,N_\epsilon}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [7], using underflow, we can find $\{D_\epsilon, E_\epsilon\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\int_0^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy| < \frac{E_\epsilon}{|k|}$, $(**)$

It is easy to see from the proof, that $\{D_\epsilon, E_\epsilon\}$ can be chosen uniformly in $\epsilon$. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k| > D_\epsilon$;

$|\mathcal{G}(\frac{df}{dx})(k)|$

$\leq |\mathcal{G}(\frac{df}{dx})(k) - \int_0^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy| + |\int_0^{N_\epsilon} \frac{df}{dx}(y)e^{-iky}dy|$

$< \epsilon + \frac{E_\epsilon}{|k|}$

so that, as $\{D_\epsilon, E_\epsilon\}$ were uniform and $\epsilon$ was arbitrary, we obtain that;

$|\mathcal{G}(\frac{df}{dx})(k)| < \frac{E}{|k|}$, for $|k| > D$

and, from (†), for $|k| > D$, that;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|} < \frac{E}{|k|^2}$

The choice of $E$ is the same as in the proof of Lemma 0.12. For the second claim, the proof up to (†) is the same. After choosing $N \in \mathcal{N}$, we have that $\frac{df}{dx}|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta}$ monotone intervals. As in $(**)$, and inspection of the proof in [7], we get;

$|\int_0^N \frac{df}{dx}e^{-iky}dy| < \frac{E_N}{|k|}$

where $E_N \leq \frac{2NC}{\delta}$ and $C = max_{x \in \mathcal{R}_{>0}}|\frac{df}{dx}|$.

Choosing $N > 1$, as $\frac{df}{dx}$ is of moderate decrease, we can assume that $|\frac{df}{dx}| \leq \frac{D}{x^2}$, for $x > N$. Then, using the proof in [7] again, and the definition of oscillatory, we have that, for sufficiently large $|k|$;

$|\int_N^\infty \frac{df}{dx}e^{-iky}dy| < (\frac{2}{|k|}\sum_{y_i>N}\frac{D}{y_i^2})$

$\leq (\frac{2}{|k|}\sum_{n\in\mathcal{Z}_{\geq 0}}\frac{D}{(y_{i_0}+n\delta)^2})$

$\leq \frac{2D}{\delta|k|}\int_{y_{i_0}}^\infty \frac{dx}{x^2}$

$= \frac{2D}{\delta|k|y_{i_0}}$

$\leq \frac{2D}{\delta|k|N}$

where $y_{i_0} \geq N$ and $y_{i_0} \leq y_i$, for all $y_i \geq N$. It follows that;

$|\mathcal{G}(\frac{df}{dx})(k)| = |\int_0^N \frac{df}{dx}e^{-iky}dy + \int_N^\infty \frac{df}{dx}e^{-iky}dy|$

$\leq |\int_0^N \frac{df}{dx}e^{-iky}dy| + |\int_N^\infty \frac{df}{dx}e^{-iky}dy|$

$\leq \frac{E_N}{|k|} + \frac{2D}{\delta|k|N}$

$\leq \frac{2}{|k|}(\frac{NC}{\delta} + \frac{D}{\delta N})$

It follows, using (†), that;

$|\mathcal{G}(f)(k)| \leq \frac{|\mathcal{G}(\frac{df}{dx})(k)|}{|k|} < \frac{E_N}{|k|^2}$

where $E_N = 2(\frac{NC}{\delta} + \frac{D}{\delta N})$

As in Lemma 0.12, we can choose $E$ as in the final claim of the two parts.

For the final claim, the modification for the first part is the same. In the second part, choose $N \geq n$, rather than $N > 1$ in the proof, and replace $D$ with $D_n$, to get $E_N = 2(\frac{NC}{\delta} + \frac{D_n}{\delta N})$, then, taking $N = n$, we obtain $E = 2(\frac{nC}{\delta} + \frac{D_n}{\delta n})$. □

**Definition 0.14.** *We say that a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$, with $f_{\overline{v}} \in C(\mathcal{R}_{>0})$ and $V \subset \mathcal{R}^n$ open, is of moderate decrease if there exists constants $D_{\overline{v}} \in \mathcal{R}_{>0}$ with $|f_{\overline{v}}(x)| \leq \frac{D_{\overline{v}}}{|x|^2}$ for $|x| > 1$. We say that a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$, with $f_{\overline{v}} \in C(\mathcal{R}_{>0})$ and $V \subset \mathcal{R}^n$ open, is of moderate decrease $n_{\overline{v}}$ if there exists constants $D_{\overline{v}} \in \mathcal{R}_{>0}$ with $|f_{\overline{v}}(x)| \leq \frac{D_{\overline{v}}}{|x|^2}$ for $|x| > n_{\overline{v}}$, where $n : V \to \mathcal{R}_{>0}$ is continuous. We* *say that the family $\{f_{\overline{v}}:\overline{v}\in V\}$ is non-oscillatory if there are finitely many points $\{y_{i,\overline{v}}:1\leq i\leq n\}\subset\mathcal{R}$ for which $f_{\overline{v}}|_{(y_{i,\overline{v}},y_{i+1,\overline{v}})}$ is monotone, $1\leq i\leq n-1$, and $f|_{(-\infty,y_{1,\overline{v}})}$ and $f|_{(y_{n,\overline{v}},\infty)}$ is monotone. We denote by $val(W)$ the minimum number of such points. We say that a family $W=\{f_{\overline{v}}:\overline{v}\in V\}$, with $f_{\overline{v}}\in C(\mathcal{R}_{>0})$ is oscillatory if there exists a sequence $\{y_{i,\overline{v}}:i\in\mathcal{N}\}\subset\mathcal{R}$, for which $f|_{(0,y_{1,\overline{v}})}$ is monotone, and $f|_{(y_{i,\overline{v}},y_{i+1,\overline{v}})}$ is monotone, $i\in\mathcal{N}$, and there exists $\delta_{\overline{v}}>0$, with $y_1>\delta_{\overline{v}}$ and $y_{i+1}-y_i>\delta_{\overline{v}}$, for $i\in\mathcal{N}$.*

**Lemma 0.15.** *Let a family $W=\{f_{\overline{v}}:\overline{v}\in V\}$ be of moderate decrease, with $W$ non-oscillatory, and $lim_{x\rightarrow 0}f_{\overline{v}}(x)=M_{\overline{v}}$, with $M_{\overline{v}}\in\mathcal{R}$, then we have that, there exists constants $E_{\overline{v}}\in\mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{E_{\overline{v}}}{|k|}$

*for sufficiently large $|k|$, independent of $\overline{v}$. Moreover, we can choose;*

$E_{\overline{v}}=2||f_{\overline{v}}||_{\infty}val(W)$

*Let a family $W=\{f_{\overline{v}}:\overline{v}\in V\}$ be of moderate decrease and oscillatory, and $lim_{x\rightarrow 0}f_{\overline{v}}(x)=M_{\overline{v}}$, with $M_{\overline{v}}\in\mathcal{R}$, then, similarly;*

*we have that, there exists constants $E_{\overline{v}}\in\mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f)(k)|\leq\frac{E_{\overline{v}}}{|k|}$

*for sufficiently large $|k|$, independent of $\overline{v}$. Moreover, we can choose*

$E_{\overline{v}}=\frac{(4||f_{\overline{v}}||_{\infty}+D_{\overline{v}})}{\delta_{\overline{v}}}$

*where $D_{\overline{v}}$ and $\delta_{\overline{v}}$ are given in Definition 0.20.*

*The first claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$. The second claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$, with the modification that we can choose $E_{\overline{v}}=\frac{2n_{\overline{v}}||f_{\overline{v}}||_{\infty}}{\delta_{\overline{v}}}+\frac{2D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}}$.*

*Proof.* As each $f_{\overline{v}}$ is of moderate decrease and $lim_{x\rightarrow 0}f_{\overline{v}}(x)=M_{\overline{v}}$, we have that each $f_{\overline{v}}\in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty}f_{\overline{v}}(x)=0$.

As each $f_{\overline{v}}$ is of moderate decrease, for any $\epsilon>0$, we can find $N_{\epsilon,\overline{v}}\in\mathcal{N}$ such that;

$|\mathcal{G}(f_{\overline{v}})(k)-\int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy|<\epsilon$ $(*)$

As each $f_{\overline{v}}|_{0,N_{\epsilon,\overline{v}}}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [7], quantifying over the nonstandard parameter space ${}^*V$, linking the parameters with $N_{\epsilon,\overline{v}}$, and using underflow again, we can find $\{D_\epsilon, E_{\epsilon,\overline{v}}\}\subset\mathcal{R}_{>0}$, such that, for all $|k|>D_\epsilon$, we have that;

$|\int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy|<\frac{E_{\epsilon,\overline{v}}}{|k|}$, $(**)$

It is easy to see from the proof, that $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ can be chosen uniformly in $\epsilon$, as the number of monotone intervals in the interval $(0, N_{\epsilon,\overline{v}})$ is always bounded by $val(W)$. Splitting the calculation into real and imaginary components, it is again straightfoward to see that it is possible to choose $E_{\epsilon,\overline{v}}$ with $E_{\epsilon,\overline{v}}=2||f_{\overline{v}}||_\infty val(W)$. Again, note that the infinitesimal correction existing after the use of underflow, drops out after taking the standard part, for each $f_{\overline{v}}$. Then, from $(*), (**)$, and the triangle inequality, we obtain that, for $|k|>D_\epsilon$;

$|\mathcal{G}(f_{\overline{v}})(k)|$

$\leq|\mathcal{G}(f_{\overline{v}})(k)-\int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy|+|\int_0^{N_{\epsilon,\overline{v}}} f_{\overline{v}}(y)e^{-iky}dy|$

$<\epsilon+\frac{E_{\epsilon,\overline{v}}}{|k|}$

so that, as $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ were uniform and $\epsilon$ was arbitrary, we obtain that;

$|\mathcal{G}(f_{\overline{v}})(k)|<\frac{E_{\overline{v}}}{|k|}$, for sufficiently large $|k|$, independently of $\overline{v}$.

For the second claim, after choosing $N\in\mathcal{N}$, we have that each $f_{\overline{v}}|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta_{\overline{v}}}$ monotone intervals. As in $(**)$, and inspection of the proof in [7], we get;

$|\int_0^N f_{\overline{v}}e^{-iky}dy|<\frac{E_N}{|k|}$

for sufficiently large $|k|$, independent of $\overline{v}$, where $E_N=\frac{2NC_{\overline{v}}}{\delta_{\overline{v}}}$ and $C_{\overline{v}}=max_{x\in\mathcal{R}_{>0}}|f_{\overline{v}}|$.

Choosing $N>1$, as each $f_{\overline{v}}$ is of moderate decrease, we can assume that $|f_{\overline{v}}|\leq\frac{D_{\overline{v}}}{x^2}$, for $x>N$. Then, using the proof in [7] again, and the definition of oscillatory, we have that, for sufficiently large $|k|$, independent of $\overline{v}$;

$|\int_N^\infty f_{\overline{v}} e^{-iky} dy| < (\frac{2}{|k|} \sum_{y_{i,\overline{v}} > N} \frac{D_{\overline{v}}}{y_{i,\overline{v}}^2})$

$\leq (\frac{2}{|k|} \sum_{n \in \mathcal{Z}_{\geq 0}} \frac{D_{\overline{v}}}{(y_{i_0,\overline{v}} + n\delta_{\overline{v}})^2})$

$\leq \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|} \int_{y_{i_0,\overline{v}}}^\infty \frac{dx}{x^2}$

$= \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k| y_{i_0,\overline{v}}}$

$\leq \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

where $y_{i_0,\overline{v}} \geq N$ and $y_{i_0,\overline{v}} \leq y_{i,\overline{v}}$, for all $y_{i,\overline{v}} \geq N$. It follows that;

$|\mathcal{G}(f_{\overline{v}})(k)| = |\int_0^N f_{\overline{v}} e^{-iky} dy + \int_N^\infty f_{\overline{v}} e^{-iky} dy|$

$\leq |\int_0^N f_{\overline{v}} e^{-iky} dy| + |\int_N^\infty f_{\overline{v}} e^{-iky} dy|$

$\leq \frac{E_N}{|k|} + \frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

$\leq \frac{2}{|k|}(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{\delta_{\overline{v}}N})$

It follows, using (†), that;

$|\mathcal{G}(f_{\overline{v}})(k) \leq \frac{E_N}{|k|}$

where $E_N = 2(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{N\delta_{\overline{v}}})$

In particular, choosing $N = 2$, we can take;

$E = E_2 = 2(\frac{2C_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{2\delta_{\overline{v}}}) = \frac{(4C_{\overline{v}} + D_{\overline{v}})}{\delta_{\overline{v}}} = \frac{(4||f_{\overline{v}}||_\infty + D_{\overline{v}})}{\delta_{\overline{v}}}$

For the final claim, the modification for the first part is the same. In the second part, choose $N \geq n_{\overline{v}}$, rather than $N > 1$ in the proof, then, taking $N = n_{\overline{v}}$, we obtain $E = E_{n_{\overline{v}}} = 2(\frac{n_{\overline{v}}C_{\overline{v}}}{\delta_{\overline{v}}} + \frac{D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}})$

□

**Lemma 0.16.** *Let a family $W = \{f_{\overline{v}} : \overline{v} \in V\}$ be of moderate decrease such that the family $W' = \{\frac{df}{dx}_{\overline{v}} : \overline{v} \in V\}$ is of moderate decrease and non-oscillatory, with $lim_{x \to 0} f_{\overline{v}}(x) = 0$, $lim_{x \to 0} \frac{df_{\overline{v}}}{dx}(x) = M_{\overline{v}}$, with*

*$M_{\overline{v}}\in\mathcal{R}$, for $\overline{v}\in V$, then we have that, there exists constants $E_{\overline{v}}\in\mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{E_{\overline{v}}}{|k|^2}$

*for sufficiently large $k$, independent of $\overline{v}$. Moreover, we can choose $E_{\overline{v}}=2||\frac{df_{\overline{v}}}{dx}||_\infty val(W')$*

*Let the families $W=\{f_{\overline{v}}:\overline{v}\in V\}$ and $W'=\{\frac{df}{dx}_{\overline{v}}:\overline{v}\in V\}$ be of moderate decrease with $W'$ oscillatory as well, with $lim_{x\rightarrow 0}f_{\overline{v}}(x)=0$, $lim_{x\rightarrow 0}\frac{df_{\overline{v}}}{dx}(x)=M_{\overline{v}}$, with $M_{\overline{v}}\in\mathcal{R}$, then, similarly, we have that, there exists constants $E_{\overline{v}}\in\mathcal{R}_{>0}$, such that;*

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{E_{\overline{v}}}{|k|^2}$

*for sufficiently large $k$, independent of $\overline{v}$. Moreover, we can choose;*

$E_{\overline{v}}=\frac{(4||\frac{df_{\overline{v}}}{dx}||_\infty+D_{\overline{v}})}{\delta_{\overline{v}}}$

*where $D_{\overline{v}}$ and $\delta_{\overline{v}}$ are given in Definition 0.20.*

*The first claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$. The second claim is the same, replacing moderate decrease with moderate decrease $n_{\overline{v}}$, with the modification that we can choose $E_{\overline{v}}=\frac{2n_{\overline{v}}||\frac{df_{\overline{v}}}{dx}||_\infty}{\delta_{\overline{v}}}+\frac{2D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}}$.*

*Proof.* As each $f_{\overline{v}}$ is of moderate decrease and $lim_{x\rightarrow 0}f_{\overline{v}}(x)=0$, we have that each $f_{\overline{v}}\in L^1(\mathcal{R}_{>0})$ and $lim_{|x|\rightarrow\infty}f_{\overline{v}}(x)=0$. Similarly, each $\frac{df_{\overline{v}}}{dx}\in L^1(\mathcal{R}_{>0})$ and each $\frac{df_{\overline{v}}}{dx}$ is continuous. We have, using integration by parts, that;

$\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)=\int_0^\infty\frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy$

$=[f_{\overline{v}}(y)e^{-iky}]_0^\infty+ik\int_0^\infty f_{\overline{v}}(y)e^{-iky}dy$

$=ik\int_0^\infty f_{\overline{v}}(y)e^{-iky}dy$

$=ik\mathcal{G}(f_{\overline{v}})(k)$

so that, for $|k|>1$;

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|}{|k|}$, (†)

As $\frac{df_{\overline{v}}}{dx}$ is of moderate decrease, for any $\epsilon > 0$, we can find $N_{\epsilon,\overline{v}} \in \mathcal{N}$ such that;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k) - \int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy| < \epsilon$ $(*)$

As $\frac{df_{\overline{v}}}{dx}|_{0,N_{\epsilon,\overline{v}}}$ is continuous and non-oscillatory, by the proof of Lemma 0.9 in [7], using underflow and quantifying over the nonstandard parameter space again, linked to the parameters $N_{\epsilon,\overline{v}}$, we can find $\{D_\epsilon, E_{\epsilon,\overline{v}}\} \subset \mathcal{R}_{>0}$, such that, for all $|k| > D_\epsilon$, we have that;

$|\int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy| < \frac{E_{\epsilon,\overline{v}}}{|k|}$, $(**)$

Again, as in the proof of Lemma 0.15, $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ can be chosen uniformly in $\epsilon$. Then, from $(*)$, $(**)$, and the triangle inequality, we obtain that, for $|k| > D_\epsilon$;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|$

$\leq |\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k) - \int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy| + |\int_0^{N_{\epsilon,\overline{v}}} \frac{df_{\overline{v}}}{dx}(y)e^{-iky}dy|$

$< \epsilon + \frac{E_{\epsilon,\overline{v}}}{|k|}$

so that, as $\{D_\epsilon, E_{\epsilon,\overline{v}}\}$ were uniform and $\epsilon$ was arbitrary, we obtain that;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)| < \frac{E_{\overline{v}}}{|k|}$, for $|k| > D$, independent of $\overline{v}$

and, from (†), for $|k| > D$, that;

$|\mathcal{G}(f_{\overline{v}})(k)| \leq \frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|}{|k|} < \frac{E_{\overline{v}}}{|k|^2}$

where the choice of $E_{\overline{v}}$ is the same as in the proof of Lemma 0.15. For the second claim, the proof up to (†) is the same. After choosing $N \in \mathcal{N}$, we have that each $\frac{df_{\overline{v}}}{dx}|_{(0,N)}$ is non-oscillatory, and, moreover, there are at most $\frac{N}{\delta_{\overline{v}}}$ monotone intervals. As in $(**)$, and inspection of the proof in [7], we get;

$|\int_0^N \frac{df_{\overline{v}}}{dx}e^{-iky}dy| < \frac{E_N}{|k|}$

where $E_N\leq \frac{2NC_{\overline{v}}}{\delta_{\overline{v}}}$ and $C_{\overline{v}}=max_{x\in\mathcal{R}_{>0}}|\frac{df_{\overline{v}}}{dx}|$.

Choosing $N>1$, as $\frac{df_{\overline{v}}}{dx}$ is of moderate decrease, we can assume that $|\frac{df_{\overline{v}}}{dx}|\leq\frac{D_{\overline{v}}}{x^2}$, for $x>N$. Then, using the proof in [7] again, and the definition of oscillatory, we have that, for sufficiently large $|k|$, independent of $\overline{v}$;

$|\int_N^\infty \frac{df_{\overline{v}}}{dx}e^{-iky}dy|<(\frac{2}{|k|}\sum_{y_{i,\overline{v}}>N}\frac{D_{\overline{v}}}{y_{i,\overline{v}}^2})$

$\leq(\frac{2}{|k|}\sum_{n\in\mathcal{Z}_{\geq 0}}\frac{D_{\overline{v}}}{(y_{i_0,\overline{v}}+n\delta_{\overline{v}})^2})$

$\leq\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|}\int_{y_{i_0,\overline{v}}}^\infty\frac{dx}{x^2}$

$=\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|y_{i_0,\overline{v}}}$

$\leq\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

where $y_{i_{0,\overline{v}}}\geq N$ and $y_{i_{0,\overline{v}}}\leq y_{i,\overline{v}}$, for all $y_{i,\overline{v}}\geq N$. It follows that;

$|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|=|\int_0^N\frac{df}{dx}e^{-iky}dy+\int_N^\infty\frac{df_{\overline{v}}}{dx}e^{-iky}dy|$

$\leq|\int_0^N\frac{df_{\overline{v}}}{dx}e^{-iky}dy|+|\int_N^\infty\frac{df_{\overline{v}}}{dx}e^{-iky}dy|$

$\leq\frac{E_N}{|k|}+\frac{2D_{\overline{v}}}{\delta_{\overline{v}}|k|N}$

$\leq\frac{2}{|k|}(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{\delta_{\overline{v}}N})$

It follows, using (†), that;

$|\mathcal{G}(f_{\overline{v}})(k)|\leq\frac{|\mathcal{G}(\frac{df_{\overline{v}}}{dx})(k)|}{|k|}<\frac{E_{\overline{v}}}{|k|^2}$

where $E_{\overline{v}}=2(\frac{NC_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{\delta_{\overline{v}}N})$

As in Lemma 0.15, we can choose $E_{\overline{v}}$ as in the final claim of the two parts.

For the final claim, the modification for the first part is the same. In the second part, choose $N\geq n_{\overline{v}}$, rather than $N>1$ in the proof, then, taking $N=n_{\overline{v}}$, we obtain $E_{\overline{v}}=2(\frac{n_{\overline{v}}C_{\overline{v}}}{\delta_{\overline{v}}}+\frac{D_{\overline{v}}}{n_{\overline{v}}\delta_{\overline{v}}})$

□

**Lemma 0.17.** *For fixed $\bar{l} \in \mathcal{R}^3$, $t \in \mathcal{R}_{>0}$, we have that the polar representation of $e^{i(k-l)ct}$, $\bar{k} \in \mathcal{R}^3$, $k = |\bar{k}|$, $l = |\bar{l}|$, is given by;*

$$e^{-ilct}e^{irct}$$

*for $r \in \mathcal{R}_{>0}$, $0 \leq \theta < \pi$, $-\pi \leq \phi \leq \pi$*

*Moreover, the real and imaginary parts of $e^{-ilct}e^{irct}$ are oscillatory, with spacings;*

$$\delta_{real,\bar{l}} = \delta_{real,\bar{l}} = \frac{\pi}{ct}$$

*If $f$ is non-oscillatory, analytic, of moderate decrease, with $lim_{r\rightarrow\infty} ln(f)''(r) = 0$, then $fRe(e^{-ilct}e^{irct})$ and $fIm(e^{-ilct}e^{irct})$ are oscillatory, with a fixed lower bound $\delta$ on the spacing, independent of $\bar{l}$.*

*Proof.* The first claim is clear. We have that;

$$Re(e^{-ilct}e^{irct}) = cos((r-l)ct)$$

$$Im(e^{-ilct}e^{irct}) = sin((r-l)ct)$$

We have that the maxima of $cos((r-l)ct)$ occur when $sin((r-l)ct) = 0$ and $-cos((r-l)ct) < 0$, so when $r = l + \frac{\pi}{2ct} + \frac{2n\pi}{ct}$, for $n \in \mathcal{Z}$. The minima of $cos((r-l)ct)$ occur when $sin((r-l)ct) = 0$ and $cos((r-l)ct) < 0$, so when $r = l + \frac{\pi}{2ct} + \frac{(2n+1)\pi}{ct}$, for $n \in \mathcal{Z}$. It follows that $cos((r-l)ct)$ is monotone in the intervals $[l + \frac{\pi}{2ct} + \frac{2n\pi}{ct}, l + \frac{\pi}{2ct} + \frac{(2n+1)\pi}{ct}]$, for $n \in \mathcal{Z}$, and the spacing is given by;

$$(l + \frac{\pi}{2ct} + \frac{(2n+1)\pi}{ct}) - (l + \frac{\pi}{2ct} + \frac{2n\pi}{ct}) = \frac{\pi}{ct}$$

A similar calculation follows for $sin((r-l)ct)$. For the final claim, we have that;

$$(fcos((r-l)ct))' = 0$$

iff $f'cos((r-l)ct) - fsin((r-l)ct) = 0$

iff $\frac{f'}{f} = tan((r-l)ct)$ (†)

Let $G(r,\overline{(l)})=\frac{f'}{f}-tan((r-l)ct)$, then, we have that, for $\bar{l}\neq 0$, the differential;

$dG=(\frac{\partial G}{\partial r},\frac{\partial G}{\partial\lambda_1},\frac{\partial G}{\partial\lambda_2},\frac{\partial G}{\partial\lambda_3})$

$=(ln(f)''-ctsec^2((r-1)ct),\frac{\lambda_1 ct}{\lambda}sec^2((r-1)ct),\frac{\lambda_2 ct}{\lambda}sec^2((r-1)ct)$

$,\frac{\lambda_3 ct}{\lambda}sec^2((r-1)ct))\neq 0$ $(C)$

We have that;

$|\frac{\partial tan((r-l)ct)}{\partial r}|=|ctsec^2((r-l)ct)|\geq ct$

With the assumption that $lim_{r\rightarrow\infty}ln(f)''(r)=0$, we have that that there exists $L\in\mathcal{R}_{>0}$, such that $|\frac{f'}{f}'||_{(L,\infty)}<ct$. It follows that the spacing between solutions to $(\dagger)$ in $(L,\infty)$ is at least $\frac{\pi}{2ct}$. We have that, for $\bar{l}\neq\bar{0}$, $(f'cos((r-l)ct)-fsin((r-l)ct))|_{(0,L)}$ is analytic, so, for fixed $\bar{l}\neq\bar{0}$, there exist finitely many solutions to $(\dagger)$ in $(0,L]$. Let;

$\delta_L=\inf(\delta_{\bar{l},L}:\bar{l}\neq\bar{0})$

where $\delta_{\bar{l}}$ is the spacing between solutions to $(\dagger)$ on $(0,L]$, for fixed $\bar{l}$. Then, if $\delta_L=0$, we would have obtain a branch point in the zero set of $G(r,\overline{(l)})$, contradicting $(C)$. It follows that $\delta_L>0$. Let $\delta=min(\delta_L,\frac{\pi}{2ct})$, then as $fcos((r-l)ct)|_{y_i,y_{i+1}}$ is monotone, for $i\in\mathcal{Z}$, where $y_i$ is a solution to $(\dagger)$, we have that $fcos((r-l)ct)$ is oscillatory with a lower bound on the spacing given by $\delta>0$, independent of $\bar{l}$. A similar calculation hods for $fsin(r-l)ct$.

□

**Lemma 0.18.** *For fixed $\bar{l}\in\mathcal{R}^3$, $t\in\mathcal{R}_{>0}$, we have that the polar representation of $e^{i(k-l)ct}$, $\bar{k}\in\mathcal{R}^3$, $k=|\bar{k}|$, $l=|\bar{l}|$, is given by;*

$e^{irct\nu(r,\theta,\phi,\bar{l})}$, $r\in\mathcal{R}_{>0}$, $0\leq\theta<\pi$, $-\pi\leq\phi\leq\pi$

*where;*

$lim_{r\rightarrow\infty}\nu(r,\theta,\phi,\bar{l})=1$

*uniformly in $\{\theta,\phi\}$. Moreover, for $\theta\neq cos^{-1}(\frac{l_3}{l})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$, the real and imaginary parts of $e^{irct\nu(r,\theta,\phi,\bar{l})}$ are oscillatory.*

*If $f$ is non-oscillatory, analytic, of moderate decrease, with $lim_{x\rightarrow\infty}ln(f)''(x)=0$ then $fcos(rct\nu(r,\theta,\phi,\bar{l}))$ and $fsin(rct\nu(r,\theta,\phi,\bar{l}))$ are oscillatory, for $\theta\neq cos^{-1}(\frac{l_3}{l})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$.*

*Proof.* Making the substitution, $k_1=rsin(\theta)cos(\phi)$, $k_2=rsin(\theta)sin(\phi)$, $k_3=rcos(\theta)$, we obtain;

$e^{i(k-l)ct}=e^{i[(rsin(\theta)cos(\phi)-l_1)^2+(rsin(\theta)sin(\phi)-l_2)^2+(rcos(\theta)-l_3)^2]^{\frac{1}{2}}ct}$

$=e^{i(r^2-(2l_1sin(\theta)cos(\phi)+2l_2sin(\theta)sin(\phi)+2l_3cos(\theta))+l^2)^{\frac{1}{2}}ct}$

$=e^{irct\nu(r,\theta,\phi,\bar{l})}$

where;

$\nu(r,\theta,\phi,\bar{l})=(1-\frac{1}{r}(2l_1sin(\theta)cos(\phi)+2l_2sin(\theta)sin(\phi)+2l_3cos(\theta))+\frac{l^2}{r^2})^{\frac{1}{2}}$

It is clear, as $|2l_1sin(\theta)cos(\phi)+2l_2sin(\theta)sin(\phi)+2l_3cos(\theta)|\leq 2(|l_1|+|l_2|+|l_3|)$, that $lim_{r\rightarrow\infty}\nu(r,\theta,\phi,\bar{l})=1$, uniformly in $\{\theta,\phi\}$. For the next claim, we show that $cos(rct\nu(r,\theta,\phi,\bar{l}))$ is oscillatory, leaving the other case to the reader. We have that;

$\frac{\partial cos(rct\nu(r,\theta,\phi,\bar{l}))}{\partial r}=0$

iff $-sin(rct\nu(r,\theta,\phi,\bar{l}))(ct\nu(r,\theta,\phi,\bar{l})+rct\frac{\partial\nu(r,\theta,\phi,\bar{l})}{\partial r})=0$

iff $sin(rct\nu(r,\theta,\phi,\bar{l}))=0$ or $ct\nu(r,\theta,\phi,\bar{l})+rct\frac{\partial\nu(r,\theta,\phi,\bar{l})}{\partial r}=0$

iff $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n\pi$, $(n\in\mathcal{Z})$

or $ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})=0$

where;

$\gamma(\theta,\phi,\bar{l})=2l_1sin(\theta)cos(\phi)+2l_2sin(\theta)sin(\phi)+2l_3cos(\theta)$

We have;

$lim_{r\rightarrow\infty}[ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})]=ct\neq 0$

so that, by continuity, the zeros of;

$ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})$

are located in a compact interval $[0,K]$, for some $K\in\mathcal{R}_{>0}$. With the assumption on $\{\theta,\phi\}$, we have that;

$ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})$

is analytic, so it can only have a finite number of zeros located in the interval $[0,K]$, $(*)$. We have that $lim_{r\rightarrow\infty}rct\nu(r,\theta,\phi,\bar{l})=\infty$ and $lim_{r\rightarrow 0}rct\nu(r,\theta,\phi,\bar{l})=ctl$, so, by the intermediate value theorem, we can find an infinite number of solutions to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n\pi$, $n\in\mathcal{Z}$, located in $\mathcal{R}_{>0}$. As;

$lim_{r\rightarrow\infty}[ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})]=ct$

and;

$lim_{r\rightarrow 0}[ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})]$

$=lim_{r\rightarrow 0}\frac{\partial rct\nu(r,\theta,\phi,\bar{l})}{\partial r}$

$=lim_{r\rightarrow 0}\frac{\partial ct|\bar{k}(r,\theta,\phi)-\bar{l}|}{\partial r}$

is finite, we have that $\frac{\partial rct\nu(r,\theta,\phi,\bar{l})}{\partial r}$ is bounded by $M\in\mathcal{R}_{>0}$ on $\mathcal{R}_{>0}$.

Using the mean value theorem, if $r_n$ is a solution to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n\pi$, and $r_m$ is a solution to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+m\pi$, then

$|r_n-r_m|\geq\frac{|(\frac{\pi}{2}+n\pi)-(\frac{\pi}{2}+n\pi)|}{M}$

$=\frac{|(n-m)|\pi}{M}$

$\geq\frac{\pi}{M}$, $(n\neq m)$

By the observation $(*)$, and the fact that;

$[ct\nu(r,\theta,\phi,\bar{l})+\frac{rct}{2\nu(r,\theta,\phi,\bar{l})}(\frac{1}{r^2}\gamma(\theta,\phi,\bar{l})-\frac{2l^2}{r^3})]$

is monotone on $(K,\infty)$, there can be at most a finite number $\{n_{i_1},\ldots,n_{i_p}\}$ for which there exist multiple solutions $r_{n,n_{i_j}}\in\mathcal{R}_{>0}$ to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n_i\pi$. Let $Z$ denote the $\{r_i:i\in\mathcal{N}\}$ for which there exists a solution to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n\pi$, $n\in\mathcal{Z}$, and $Z_0$ the finite set consisting of solutions to $rct\nu(r,\theta,\phi,\bar{l})=\frac{\pi}{2}+n_{i_j}\pi$, $1\leq j\leq p$ and the zeros on $[0,K]$, corresponding to $(*)$. Ordering $Z\cup Z_0$ as a set $\{r_i:i\in\mathcal{N}\}$, it is clear that $cos(rct\nu(r,\theta,\phi,\bar{l}))|_{(r_i,r_{i+1})}$ is monotone. Choosing $\delta=min(\frac{\pi}{M},d(Z\setminus Z_0,Z_0),Sep(Z_0))>0$, where $Sep(Z_0)=min(d(r,r'):\{r,r'\}\subset Z_0,r\neq r')$, we obtain the result that $cos(rct\nu(r,\theta,\phi,\bar{l}))$ is oscillatory.

For the final claim, we can, without loss of generality, assume that there exists $L\in\mathcal{R}_{>0}$ for which $f|_{(L,\infty)}$ is monotone decreasing and $f|_{(L,\infty)}>0$. Then, by the product rule, we have that;

$(fcos(rct\nu(r,\theta,\phi,\bar{l})))'=0$

iff $f'cos(rct\nu(r,\theta,\phi,\bar{l})))-fsin(rct\nu(r,\theta,\phi,\bar{l}))(rct\nu(r,\theta,\phi,\bar{l}))'=0$

iff $\frac{f'}{f}=tan(rct\nu(r,\theta,\phi,\bar{l}))(rct\nu(r,\theta,\phi,\bar{l}))'$ (†)

We have that $lim_{r\rightarrow\infty}(rct\nu(r,\theta,\phi,\bar{l}))'=ct$, in particularly, we can assume that $(rct\nu(r,\theta,\phi,\bar{l}))'>0$ in $(L,\infty)$, so that $rct\nu(r,\theta,\phi,\bar{l})$ is increasing in $(L,\infty)$. By the hypotheses, $\frac{f'}{f}|_{(L,\infty)}<0$, so that for a solution $r_1$ to (†) in $(L,\infty)$, we must have that $tan(r_1ct\nu(r_1,\theta,\phi,\bar{l}))<0$, $(*****)$. Moreover, by the assumption;

$lim_{x\rightarrow\infty}ln(f)''(x)=lim_{x\rightarrow\infty}(\frac{f'}{f})'(x)=0$ $(***)$

As $tan'(x)\geq 1$, for $x\in\mathcal{R}$, and $lim_{r\rightarrow\infty}(rct\nu(r,\theta,\phi,\bar{l}))'=ct$, by the chain rule, we can assume that $|\frac{\partial(tan(rct\nu(r,\theta,\phi,\bar{l})))}{\partial r}|\geq\frac{ct}{2}$, in $(L,\infty)$, $(****)$. Combining, $(***),(****),(*****)$, it follows that for $\{r_1,r_2\}$ solving (†) in $(L,\infty)$, the separation $|r_2-r_1|\geq\frac{\pi}{2}$. By the assumptions, we have that $fcos(rct\nu(r,\theta,\phi,\bar{l}))$ is analytic on $[0,L+1)$, so that $(fcos(rct\nu(r,\theta,\phi,\bar{l})))'$ is analytic on $[0,L+1)$. It follows there can only be finitely many solutions to (†) in $(0,L)$, and, therefore, similarly to the above, $fcos(rct\nu(r,\theta,\phi,\bar{l}))$ is oscillatory. The argument for $fsin(rct\nu(r,\theta,\phi,\bar{l}))$ is similar and left to the reader.

□

**Lemma 0.19.** *With notation as in Lemmas 0.18 and 0.7, if;*

$\alpha(\bar{k},\bar{l},t)=\alpha(R,\theta,\phi,\bar{l},t)=\frac{iP_{1,1}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})\times(\bar{d}'_{11}(\bar{l})+\frac{\bar{d}'_{12}(\bar{l})}{l})]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)sin(\theta)$

*and;*

$\beta(\bar{k},\bar{l},t)=\beta(R,\theta,\phi,\bar{l},t)=\frac{-iQ_{0,0}}{2\pi^2}[(\bar{b}_{11,\bar{l}}(R,\theta,\phi)+\frac{\bar{b}_{12,\bar{l}}(R,\theta,\phi)}{|(Rsin(\theta)cos(\phi),Rsin(\theta)sin(\phi),Rcos(\theta))-\bar{l}|})\times(\bar{d}'_{11}(\bar{l})+\frac{\bar{d}'_{12}(\bar{l})}{l})]\centerdot\bar{u}_{\bar{l}}(R,\theta,\phi,\bar{l})]\mu(R,\theta,\phi,\bar{l},t)sin(\theta)$

*then;*

$\alpha(R,\theta,\phi,\bar{l},t)=\alpha_1(R,\theta,\phi,\bar{l},t)\mu(R,\theta,\phi,\bar{l},t)=e^{-ilct}\alpha_1(R,\theta,\phi,\bar{l},t)e^{iRct\nu(R,\theta,\phi,\bar{l})}$

$\beta(R,\theta,\phi,\bar{l},t)=\beta_1(R,\theta,\phi,\bar{l},t)\mu(R,\theta,\phi,\bar{l},t)=e^{-ilct}\beta_1(R,\theta,\phi,\bar{l},t)e^{iRct\nu(R,\theta,\phi,\bar{l})}$

*For fixed $\bar{l}\neq\bar{0}$ and $\theta\neq cos^{-1}(\frac{l_3}{l})$, $\phi\neq tan^{-1}(\frac{l_2}{l_1})$, if the real and imaginary components of $e^{-ilct}\alpha_1(R,\theta,\phi,\bar{l},t)$ satisfy the conditions of Lemma 0.18, then the real and imaginary components of $\alpha$ are oscillatory. Similarly, if the real and imaginary components of;*

$\{e^{-ilct}\beta_1(R,\theta,\phi,\bar{l},t),e^{-ilct}R\frac{\partial\beta_1(R,\theta,\phi,\bar{l},t)}{\partial R},icte^{-ilct}R\beta_1(R,\theta,\phi,\bar{l},t)(\nu(R,\theta,\phi,\bar{l})+R\frac{\partial\nu(R,\theta,\phi,\bar{l})}{\partial R})\}$

*satisfy the conditions of Lemma 0.18, then the real and imaginary components of $\frac{\partial R\beta(R,\theta,\phi,\bar{l},t)}{\partial R}$ are oscillatory.*

*Proof.* We have that;

$Re(\alpha)=Re(e^{-ilct}\alpha_1e^{iRct\nu})=Re(e^{-ilct}\alpha_1cos(Rct\nu))+Re(ie^{-ilct}\alpha_1sin(Rct\nu))$

$=Re(e^{-ilct}\alpha_1)cos(Rct\nu)+Im(e^{-ilct}\alpha_1)sin(Rct\nu)$

$Im(\alpha)=Im(e^{-ilct}\alpha_1e^{iRct\nu})=Im(e^{-ilct}\alpha_1cos(Rct\nu))+Im(ie^{-ilct}\alpha_1sin(Rct\nu))$

$=Im(e^{-ilct}\alpha_1)cos(Rct\nu)+Re(e^{-ilct}\alpha_1)sin(Rct\nu)$

so the first claim, follows from Lemma 0.18.

We also have that;

$Re(\frac{\partial(R\beta)}{\partial R})=Re(\frac{\partial(Re^{-ilct}\beta_1e^{iRct\nu})}{\partial R})=Re(e^{-ilct}\beta_1e^{iRct\nu})+Re(R\frac{\partial(e^{-ilct}\beta_1e^{iRct\nu})}{\partial R})$

$=Re(e^{-ilct}\beta_1e^{iRct\nu})+Re(e^{-ilct}R\frac{\partial\beta_1}{R}e^{iRct\nu})+Re(icte^{-ilct}R\beta_1(\nu+R\frac{\partial\nu}{\partial R})e^{iRct\nu})$

$Im(\frac{\partial(R\beta)}{\partial R})=Im(\frac{\partial(Re^{-ilct}\beta_1e^{iRct\nu})}{\partial R})=Re(e^{-ilct}\beta_1e^{iRct\nu})+Re(R\frac{\partial(e^{-ilct}\beta_1e^{iRct\nu})}{\partial R})$

$=Im(e^{-ilct}\beta_1e^{iRct\nu})+Re(e^{-ilct}R\frac{\partial\beta_1}{R}e^{iRct\nu})+Re(icte^{-ilct}R\beta_1(\nu+R\frac{\partial\nu}{\partial R})e^{iRct\nu})$

and the second claim follows, using the previous calculation and Lemma 0.18.

□

**Definition 0.20.** *We say that* $f\in C(\mathcal{R}\setminus\{0\})$ *is of moderate decrease if there exists a constant* $D\in\mathcal{R}_{>0}$ *with* $|f(x)|\leq\frac{D}{|x|^2}$ *for* $|x|>1$. *We say that* $f\in C(\mathcal{R}\setminus\{0\})$ *is of very moderate decrease if there exists a constant* $D\in\mathcal{R}_{>0}$ *with* $|f(x)|\leq\frac{D}{|x|}$ *for* $|x|>1$.*We say that* $f\in C(\mathcal{R}\setminus\{0\})$ *is non-oscillatory if there are finitely many points* $\{y_i:1\leq i\leq n\}\subset\mathcal{R}$ *for which* $f|_{(y_i,y_{i+1})}$ *is monotone,* $1\leq i\leq n-1$, *and* $f|_{(-\infty,y_1)}$ *and* $f|_{(y_n,\infty)}$ *is monotone. We say that* $f\in C(\mathcal{R}\setminus\{0\})$ *is symmetrically asymptotic if* $f$ *and* $\frac{df}{dx}$ *are of moderate decrease,* $\frac{df}{dx}$ *is non-oscillatory,* $\{f,\frac{df}{dx}\}\subset L^1((-\epsilon,\epsilon))$, *and for* $\epsilon>0$;

$lim_{y\rightarrow 0-}f(y)=lim_{y\rightarrow 0+}f(y)=M$

*and*

$lim_{y\rightarrow 0-}\frac{df}{dx}(y)=-lim_{y\rightarrow 0+}\frac{df}{dx}(y)=L\ (*)$

*where* $L\in\{+\infty,-\infty\}$, $M\in\mathcal{R}$. *We say that* $f\in C(\mathcal{R}\setminus\{0\})$ *is light symmetrically asymptotic if* $f$ *and* $\frac{df}{dx}$ *are of very moderate decrease,* $f$ *and* $\frac{df}{dx}$ *are non-oscillatory,* $\{f,\frac{df}{dx}\}\subset L^1((-\epsilon,\epsilon))$, *and for* $\epsilon>0$, *the condition* $(*)$ *holds.*

**Lemma 0.21.** *Let* $f$ *be symmetrically asymptotic, then we have that, for any* $\delta>0$, *there exist constants* $\{C_\delta,D_\delta\}\subset\mathcal{R}_{>0}$, *such that;*

$|\mathcal{F}(f)(k)|\leq\frac{\delta}{|k|}+\frac{C_\delta}{|k|^2}$, *for* $|k|>D_\delta$

*Proof.* As $f$ is symmetrically asymptotic, we have that $lim_{x\rightarrow 0-}f(x)=lim_{x\rightarrow 0+}f(x)=M$, where $M\in\mathcal{R}$. In either case, we can apply integration by parts, to obtain $(\dagger)$ in Lemma 0.11. The step $(*)$ follows from the fact that $\frac{df}{dx}$ is of moderate decrease. As $\frac{df}{dx}$ is non-oscillatory, we can find $x_0<0<x_1$, with $\frac{df}{dx}|_{x_0,0}$ and $\frac{df}{dx}|_{0,x_0}$ monotone. In particular, for any $\delta>0$, we can find $x_0<y_0<0<y_1<x_1$ such that $\int_{(y_0,y_1)}|\frac{df}{dx}(y)|dy<\delta((2\pi)^{\frac{1}{2}})$ and $\frac{df}{dx}(y_0)=L_{1,0}$, $\frac{df}{dx}(y_1=L_{2,0}$, with $\{L_{1,0},L_{2,0}\}\subset\mathcal{R}$. Then;

$$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N_\epsilon}^{N_\epsilon}\frac{df}{dx}(y)e^{-iky}dy-\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{(-N_\epsilon,y_0)\cup(y_1,N_\epsilon)}\frac{df}{dx}(y)e^{-iky}dy|$$

$$\leq\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{(y_0,y_1)}|\frac{df}{dx}(y)|dy$$

$$<\delta$$

Again, by the proof of Lemma 0.9 in [7], using underflow, we can find $\{D_{\epsilon,y_0,y_1},E_{\epsilon,y_0,y_1}\}\subset\mathcal{R}_{>0}$, such that, for all $|k|>D_{\epsilon,y_0,y_1}$, we have that;

$$|\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{(-N_\epsilon,y_0)\cup(y_1,N_\epsilon)}\frac{df}{dx}(y)e^{-iky}dy|<\frac{E_{\epsilon,y_0,y_1}}{|k|},\ (**)$$

It is easy to see from the proof, that $\{D_{\epsilon,y_0,y_1},E_{\epsilon,y_0,y_1}\}$ can be chosen uniformly in $\epsilon$, so that using the triangle inequality again, we obtain;

$$|\mathcal{F}(\frac{df}{dx})(k)|\leq\epsilon+\delta+\frac{E_{\epsilon,y_0,y_1}}{|k|}$$

for $|k|>D_{\epsilon,y_0,y_1}$

As $\epsilon$ was arbitrary, and $E_{\epsilon,y_0,y_1}$ is uniform in $\epsilon$, we obtain that;

$$|\mathcal{F}(\frac{df}{dx})(k)|\leq\delta+\frac{E_{y_0,y_1}}{|k|}$$

for $|k|>D_{y_0,y_1}$.

so that, using $(\dagger)$ again;

$$|\mathcal{F}(f)(k)|\leq\frac{\delta}{|k|}+\frac{E_{y_0,y_1}}{|k|^2},\ (\dagger)$$

$$=\frac{\delta}{|k|}+\frac{C_\delta}{|k|^2}$$

for $|k| > D_\delta$, where $C_\delta = E_{y_0,y_1}$ and $D_\delta = D_{y_0,y_1}$.

□

**Lemma 0.22.** *There exists a unique fundamental solution $(\overline{E},\overline{0})$, with $\overline{E}$ decaying in the sense of [8], for given $(\rho,\overline{J})$, not vacuum. Without any decay condition, the difference $\overline{E}-\overline{E}'$ of two such solutions $\{\overline{E},\overline{E}'\}$, is either $\overline{0}$ or static and unbounded with $\bigtriangledown \cdot \overline{E} = 0$ and $\bigtriangledown \times \overline{E} = \overline{0}$, $(*)$, with the possibility $(*)$ being satisfiable. If $(\overline{E}_0,\overline{B}_0)$ is a solution to Maxwell's equation in vacuum, then we cannot have that $\overline{E}+\overline{E}_0 = \overline{0}$.*

*Proof.* Suppose there exist two fundamental solutions $(\overline{E},\overline{0})$ and $(\overline{E}',\overline{0})$, then $(0,\overline{0},\overline{E}-\overline{E}',\overline{0})$ is a solution to Maxwell's equations in vacuum. It follows from Maxwell's fourth equation, that;

$$\frac{\partial(\overline{E}-\overline{E}')}{\partial t} = \overline{0}$$

and, from the relations in Lemma 4.1 of [9], that;

$$\Box^2(\overline{E}-\overline{E}') = \bigtriangledown^2(\overline{E}-\overline{E}') = 0$$

By the decaying condition and properties of harmonic functions, we have that $\overline{E}-\overline{E}' = \overline{0}$, so that $\overline{E} = \overline{E}'$. Without the decay condition, we must have that $\overline{E}-\overline{E}'$ is unbounded or $\overline{E}-\overline{E}' = \overline{0}$, and from Maxwell's first and second equations, we must have that $\bigtriangledown \cdot \overline{E} = 0$ and $\bigtriangledown \times \overline{E} = \overline{0}$ as well. The satisfiable claim follows from the fact that we can construct a solution $(0,\overline{0},\overline{E}_0,\overline{0})$ to Maxwell's equations in free space, by the requirements that;

(i). $\bigtriangledown \cdot \overline{E}_0 = 0$

(ii). $\frac{\partial \overline{E}_0}{\partial t} = \overline{0}$

(iii). $\bigtriangledown \times \overline{E}_0 = \overline{0}$

It is possible to satisfy the requirements $(i),(iii)$, for a function $\overline{f} : \mathcal{R}^3 \rightarrow \mathcal{R}$, so that we can define $\overline{E}_0(\overline{x},t) = \overline{f}(\overline{x})$ to satisfy the conditions $(i),(ii),(iii)$. For the last claim, suppose that $\overline{E}+\overline{E}_0 = \overline{0}$, then $\overline{E} = -\overline{E}_0$ and we have that, by Maxwell's equations, and $(\overline{E}_0,\overline{B}_0)$ a vacuum solution;

$\bigtriangledown \cdot \overline{E} = -\bigtriangledown \cdot \overline{E}_0 = \frac{\rho}{\epsilon_0} = 0$

so that $\rho = 0$. Using the fact that $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$ and $\square^2 \overline{J} = \overline{0}$, we have that $\frac{\partial \overline{J}}{\partial t} = \overline{0}$ and $\bigtriangledown^2 \overline{J} = \overline{0}$, so that, as $\overline{J} \in S(\mathcal{R}^3)$, we must have that $\overline{J} = \overline{0}$ and $(\rho, \overline{J})$ is a vacuum solution, contradicting the hypotheses. $\square$

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