SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 6: WAVES, CURRENT AND CHARGE

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Abstract. We develop the theory of current and charge $(\rho, \overline{J})$, with compact support, satisfying the wave equations, the continuity equation and the connecting relation $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$.

Definition 0.1. *We say that a scalar process $\rho \in C^\infty(\mathcal{R}^4)$ has compact support, if, for $t \in \mathcal{R}$, ρ_t has compact support and the support varies continuously with t. We say that a field $\overline{J} \in C^\infty(\mathcal{R}^4)$ if the components $j_i \in C^\infty(\mathcal{R}^4)$, for $1 \leq i \leq 3$ and has compact support, if the components have compact support.*

Lemma 0.2. *If $\rho \in C^\infty(\mathcal{R}^4)$ satisfies the wave equation, $\Box^2(\rho) = 0$, with the property that ρ has compact support, then ρ has the representation;*

For $t > 0$;

$$\rho(\overline{x}, t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}, ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x}))dS(\overline{y})$$

and, for $t < 0$;

$$\rho(\overline{x}, t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}, -ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x}))dS(\overline{y}) \quad (VV)$$

where $g(\overline{x}) = (\frac{\partial \rho}{\partial t})_{t=0}$ has compact support.

Conversely, given $\rho_0(\overline{x})$ and $g(\overline{x})$ with compact support, $\{\rho_0, g\} \subset C^\infty(\mathcal{R}^3)$, the formula (VV) defines a process $\rho \in C^\infty(\mathcal{R}^4)$ satisfying the wave equation $\Box^2(\rho) = 0$, with the property that ρ has compact support.

Proof. For the first claim, observe that the process $\rho(\overline{x},t)$, $t>0$ satisfies the wave equation $\square^2(\rho)=0$, $t>0$, with, by continuity;

$lim_{t\rightarrow 0+}\rho_t=\rho_0$

and;

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}=g(\overline{x})=(\frac{\partial\rho}{\partial t})_{t=0}$

where ρ_0 and $g(\overline{x})$ have compact support and $\{\rho_0,g\}\subset C^\infty(\mathcal{R}^3)$. The representation for $t>0$ then comes from Kirchoff's formula, see [1]. The process $\rho_1(\overline{x},t)=\rho(\overline{x},-t)$, for $t>0$, also satisfies the wave equation $\square^2(\rho_1)=0$, $t>0$, with, by continuity;

$lim_{t\rightarrow 0+}(\rho_1)_t=lim_{t\rightarrow 0-}\rho_t=\rho_0$

and;

$lim_{t\rightarrow 0+}(\frac{\partial\rho_1}{\partial t})_t=lim_{t\rightarrow 0-}-(\frac{\partial\rho_1}{\partial t})_t=-g(\overline{x})=-(\frac{\partial\rho}{\partial t})_{t=0}$

The representation for $t<0$ then comes from Kirchoff's formula again, noting that we have reversed the sign of $g(\overline{x})$, when $t<0$.

For the converse claim, suppose the initial conditions $\rho_0\in S(\mathcal{R}^3),\frac{\partial\rho}{\partial t}|_{t=0}\subset C^\infty(\mathcal{R}^3)$, have compact support, with ρ defined on $\mathcal{R}^4$ by Kirchoff's formula;

For $t>0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

and, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y})+\rho_0(\overline{y})+D\rho_0(\overline{y})\centerdot(\overline{y}-\overline{x}))dS(\overline{y})$

then, see [1] again, we have that, for $\overline{x}\in\mathcal{R}^3$;

$lim_{t\rightarrow 0+}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}(\overline{x},t)=g(\overline{x})$

$lim_{t\rightarrow 0+}\rho(\overline{x},-t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0+}\frac{\partial\rho}{\partial t}(\overline{x},-t)=-g(\overline{x})$

where $g(\overline{x})=\frac{\partial\rho}{\partial t}|_{t=0}$, so that;

$lim_{t\rightarrow 0-}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0-}\frac{\partial\rho}{\partial t}(\overline{x},t)=lim_{t\rightarrow 0+}-\frac{\partial\rho}{\partial t}(\overline{x},-t)$

$=--g(\overline{x})$

$=g(\overline{x})$

In particular;

$lim_{t\rightarrow 0}\rho(\overline{x},t)=\rho(\overline{x},0)$

$lim_{t\rightarrow 0}\frac{\partial\rho}{\partial t}(\overline{x},t)=g(\overline{x})$

Moreover, for fixed $t_0\in\mathcal{R}$, $t_0\neq 0$, as ρ_0 and g have compact support, we can see that $\delta B(\overline{x},c|t_0|)\cap Supp(\rho_0,g,D\rho_0)=\emptyset$, for $|\overline{x}_0|>C_{t_0}$, where $C_{t_0}\in\mathcal{R}_{>0}$, so that ρ_{t_0} has compact support as well. As $\{\rho_0,g\}\subset C^\infty(\mathcal{R}^3)$, we can show, by differentiating Kirchoff's formula, that, for $t_0\neq 0$, $\rho_{t_0}\in C^\infty(\mathcal{R}^3)$. We then have that $\rho_{t_0}\in S(\mathcal{R}^3)$ and we can then apply Lemma 0.5 to show that, for $t>0$;

$\rho(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b(\overline{k})e^{ikct}+d(\overline{k})e^{-ikct})e^{i\overline{k}\cdot\overline{x}}d\overline{k}$

$\rho(\overline{x},-t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b^-(\overline{k})e^{ikct}+d^-(\overline{k})e^{-ikct})e^{i\overline{k}\cdot\overline{x}}d\overline{k}$ (X)

where;

$b(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})+\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$d(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})-\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$b^-(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})+\frac{1}{ikc}\mathcal{F}(-g)(\overline{k}))$

$=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})-\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

$d^{-}(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})-\frac{1}{ikc}\mathcal{F}(-g)(\overline{k}))$

$=\frac{1}{2}(\mathcal{F}(\rho_0)(\overline{k})+\frac{1}{ikc}\mathcal{F}(g)(\overline{k}))$

see also earlier in the paper, so that, for $t<0$;

$\rho(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b^{-}(\overline{k})e^{-ikct}+d^{-}(\overline{k})e^{ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$ (Y)

Differentiating under the integral sign in (X), we have that, for $t>0$;

$\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})e^{ikct}+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

where $(ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})\in L^1(\mathcal{R}^3)$ and $(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})\in L^1(\mathcal{R}^3)$, so that;

$lim_{t\rightarrow 0+}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)$

$=lim_{t\rightarrow 0+}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})e^{ikct}+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}((ik_1)^i(ik_2)^j(ik_3)^k b(\overline{k})+(ik_1)^i(ik_2)^j(ik_3)^k d(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k\mathcal{F}(\rho_0)(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

$=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$ $(X)'$

Similarly, differentiating under the integral sign in (Y), using the fact that $b^{-}(\overline{k})+d^{-}(\overline{k})=\mathcal{F}(\rho_0)(\overline{k})$;

$lim_{t\rightarrow 0-}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$ (Y')

and combining $(X)',(Y)'$, we obtain that;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},t)=\frac{\partial^{i+j+k}\rho}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$

By a similar argument, differentiating under the integral sign, and using the facts that $b(\overline{k})ikc-d(\overline{k})ikc=\mathcal{F}(g)(\overline{k})$ $-ikcb^{-}(\overline{k})+ikcd^{-}(\overline{k})=\mathcal{F}(g)(\overline{k})$;

$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+1}\rho}{\partial x^i\partial y^j\partial z^k\partial t}(\overline{x},t)=\frac{\partial^{i+j+k}g}{\partial x^i\partial y^j\partial z^k}(\overline{x},0)$

Similarly, using the fact that $\rho_0 \in S(\mathcal{R}^3)$, $\{b(\overline{k}), d(\overline{k})\} \subset L^1(\mathcal{R}^3)$, so we can apply the inversion theorem, we have that;

$$lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+2}\rho}{\partial x^i \partial y^j \partial z^k \partial t^2}(\overline{x},t)$$

$$= lim_{t\rightarrow 0+}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)b(\overline{k})e^{ikct}$$

$$+(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)d(\overline{k})e^{-ikct})e^{i\overline{k}.\overline{x}}d\overline{k}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)(b(\overline{k})+d(\overline{k}))e^{i\overline{k}.\overline{x}}d\overline{k}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(ik_1)^i(ik_2)^j(ik_3)^k(-k^2c^2)(\mathcal{F}(\rho_0)(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}c^2(\mathcal{F}(\frac{\partial^{i+j+k}\bigtriangledown^2(\rho_0)}{\partial x^i \partial y^j \partial z^k})(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$$

$$= c^2\frac{\partial^{i+j+k}\bigtriangledown^2(\rho_0)}{\partial x^i \partial y^j \partial z^k}(\overline{x})$$

and;

$$lim_{t\rightarrow 0-}\frac{\partial^{i+j+k+2}\rho}{\partial x^i \partial y^j \partial z^k \partial t^2}(\overline{x},t) = \frac{\partial^{i+j+k}c^2\bigtriangledown^2(\rho_0)}{\partial x^i \partial y^j \partial z^k}(\overline{x})$$

As $\rho|_{t>0}$, $\rho|_{t<0}$ obey the wave equation, so do the partial derivatives $\frac{\partial^{i+j+k+l}}{\partial x^i \partial y^j \partial z^k \partial t^l}|_{t>0}$, so that, for $l \geq 1$, l even, $t \neq 0$;

$$\frac{\partial^{i+j+k+l}\rho}{\partial x^i \partial y^j \partial z^k \partial t^l}|_{t\neq 0} = c^l(\bigtriangledown^2)^{\frac{l}{2}}(\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k})|_{t\neq 0}$$

and, for $l \geq 1$, l odd, $t \neq 0$;

$$\frac{\partial^{i+j+k+l}\rho}{\partial x^i \partial y^j \partial z^k \partial t^l}|_{t\neq 0} = c^{l-1}(\bigtriangledown^2)^{\frac{l-1}{2}}(\frac{\partial^{i+j+k+1}\rho}{\partial x^i \partial y^j \partial z^k \partial t})|_{t\neq 0}$$

and, using the above, for l even;

$$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+l}\rho(\overline{x},t)}{\partial x^i \partial y^j \partial z^k \partial t^l} = c^l(\bigtriangledown^2)^{\frac{l}{2}}(\frac{\partial^{i+j+k}\rho_0}{\partial x^i \partial y^j \partial z^k})$$

and, for l odd;

$$lim_{t\rightarrow 0}\frac{\partial^{i+j+k+l}\rho(\overline{x},t)}{\partial x^i \partial y^j \partial z^k \partial t^l} = c^{l-1}(\bigtriangledown^2)^{\frac{l-1}{2}}(\frac{\partial^{i+j+k}g}{\partial x^i \partial y^j \partial z^k})$$

In particularly, as all the partial derivatives of ρ extend continuously to the boundary $t = 0$, we have that $\rho \in C^\infty(\mathcal{R}^4)$, and the wave equation is satisfied at $t = 0$, $\frac{\partial^2\rho}{\partial t^2} = c^2 \bigtriangledown^2 (\rho)$, (NB). This last claim follows from the fact that, using the integral representation of a solution to the wave equation, $\bigtriangledown^2(f) - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$ in $\mathcal{R}^3 \times [0,\infty)$, generated by the initial data (g,h), that $lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+l} f_t}{\partial x^i \partial x^j \partial z^k \partial t^l} = (c^2\bigtriangledown^2)^{\frac{l}{2}}\frac{\partial^{i+j+k+l} g}{\partial x^i \partial x^j \partial z^k}$ for l even and that $lim_{t\rightarrow 0+}\frac{\partial^{i+j+k+l} f_t}{\partial x^i \partial x^j \partial z^k \partial t^l} = (c^2\bigtriangledown^2)^{\frac{l-1}{2}}\frac{\partial^{i+j+k+l} h}{\partial x^i \partial x^j \partial z^k}$ for l odd. By uniqueness of the wave equation with specified initial conditions (g,h), the same must be true for Kirchoff's representation. The same result holds for the backward wave equation with initial data $(g,-h)$, so the limit of the partial derivatives is same for $t > 0$ as $t < 0$, and the limit, as $t \rightarrow 0$, of $\frac{\partial^2 \rho}{\partial t^2} - c^2 \bigtriangledown^2 (\rho)$ is zero. Using Kirchoff's formula, as we noted above, for $t \in \mathcal{R}$, ρ_t has compact support, and it is clear that the support varies continuously with t.

$\square$

Lemma 0.3. *If a solution to the wave equation for $t \in \mathcal{R}$ is generated by the data $\{\rho_0, g\} \subset C^\infty(\mathcal{R}^3)$ with compact support, and Kirchoff's formula, then we have that, for $t > 0$;*

$$\rho(\overline{x},t) = \rho(\overline{x},-t) \text{ iff } g(\overline{x}) = 0$$

$$\rho(\overline{x},t) = -\rho(\overline{x},-t) \text{ iff } \rho_0(\overline{x}) = 0$$

Proof. We have, if;

$$\rho(\overline{x},t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y}) \ (t > 0)$$

$$\rho(\overline{x},t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},-ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y}) \ (t < 0)$$

Then, for $t > 0$, $\rho(\overline{x},t) = \rho(\overline{x},-t)$ iff;

$$\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$$

$$= \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)}(-tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$$

$$\text{iff } \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)} 2tg(\overline{y})dS(\overline{y}) = 0$$

$$\text{iff } \int_{\delta B(\overline{x},ct)} g(\overline{y})dS(\overline{y}) = 0$$

$$\text{iff } g(\overline{y}) = 0$$

as if $g(\overline{y}_0) \neq 0$, without loss of generality, by continuity, we can choose $t_0 > 0$ sufficiently small with $g|_{\delta B(\overline{y}_0,ct)} > 0$, so that $\int_{\delta B(\overline{y}_0,ct_0)} g(\overline{y})dS(\overline{y}) >$

0

and, for $t > 0$, $\rho(\overline{x},t) = -\rho(\overline{x},-t)$ iff;

$\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$

$= \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)}(tg(\overline{y}) - \rho_0(\overline{y}) - D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})$

iff $\frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x},ct)} 2[\rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x})]dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)}[\rho_0(\overline{y}) + D\rho_0(\overline{y}) \centerdot (\overline{y}-\overline{x})]dS(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)} \rho_0(\overline{y})dS(\overline{y}) + ct\int_{\delta B(\overline{x},ct)} \bigtriangledown(\rho_0) \centerdot d\overline{S} = 0$

iff $\int_{\delta B(\overline{x},ct)} \rho_0(\overline{y})dS(\overline{y}) + ct\int_{B(\overline{x},ct)} div(\bigtriangledown(\rho_0))dV(\overline{y}) = 0$

iff $\int_{\delta B(\overline{x},ct)} \rho_0(\overline{y})dS(\overline{y}) + ct\int_{B(\overline{x},ct)} \bigtriangledown^2(\rho_0)dV(\overline{y}) = 0$

iff $\rho_0(\overline{y}) = 0$

as if $\rho_0(\overline{y}_0) \neq 0$, by continity, without loss of generality, there exists $\epsilon > 0$, such that, for sufficiently small t_0;

$\int_{\delta B(\overline{y}_0,ct_0)} \rho_0(\overline{y})dS(\overline{y}) > 4\pi\epsilon c^2 t_0^2$

and, if M is a uniform bound on $\bigtriangledown^2(\rho_0)$

$|ct_0\int_{B(\overline{y}_0,ct_0)} \bigtriangledown^2(\rho_0)dV(\overline{y})| < \frac{4M\pi c^4 t_0^4}{3}$

so that, if $4\pi\epsilon c^2 t_0^2 > \frac{4M\pi c^4 t_0^4}{3}$ iff $\frac{3\epsilon}{Mc^2} > t_0^2$, we can choose $0 < t_0 < \frac{(3\epsilon)^{\frac{1}{2}}}{\sqrt{M}c}$, to obtain;

$\int_{\delta B(\overline{y}_0,ct_0)} \rho_0(\overline{y})dS(\overline{y}) + ct_0\int_{B(\overline{y}_0,ct_0)} \bigtriangledown^2(\rho_0)dV(\overline{y}) > 0$

$\square$

Lemma 0.4. *If $\rho \in C^\infty(\mathcal{R}^4)$ has compact support and satisfies the wave equation $\square^2(\rho) = 0$, then if we define $\overline{J}$ by;*

$\overline{J}(\overline{x},t) = -c^2\int_{-\infty}^{t} \bigtriangledown(\rho)ds$

then $\overline{J} \in C^\infty(\mathcal{R}^4)$ *has compact support and satisfies the wave equation* $\square^2(\overline{J} = 0$. *Moreover, the combination* $(\rho, \overline{J})$ *satisfies;*

(i). $\frac{\partial \rho}{\partial t} = -div(\overline{J})$

(ii). $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

Proof. Letting;

$\overline{J}(\overline{x}, t) = -c^2 \int_{-\infty}^t \bigtriangledown(\rho) ds$

see [2] for the existence of the integral. We have, differentiating under the integral sign, and using the fundamental theorem of calculus, that, for $(i, j, k) \in \mathcal{Z}^3_{\geq 0}$;

$\frac{\partial^{i+j+k} j_1}{\partial x^i \partial y^j \partial z^k} = -c^2 \int_{-\infty}^t \frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j \partial z^k} ds \ (Z)$

$\frac{\partial^{i+j+k+1} j_1}{\partial x^i \partial y^j \partial z^k \partial t} = -c^2 \frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j \partial z^k}$

and for $l \geq 2$;

$\frac{\partial^{i+j+k+l} j_1}{\partial x^i \partial y^j \partial z^k \partial t^l} = -c^2 \frac{\partial^{i+j+k+1}\rho}{\partial x^{i+1}\partial y^j \partial z^k \partial t^{l-1}}$

As $(\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k})_0 \in S(\mathcal{R}^3)$, and $\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k}$ satisfies the wave equation on $\mathcal{R}^4$, by the proof in [2], we have that the integral (Z) is well defined. Then, as $\rho \in C^\infty(\mathcal{R}^4)$, we have that $j_1 \in C^\infty(\mathcal{R}^4)$. A similar argument shows that the components $\{j_2, j_3\} \subset C^\infty(\mathcal{R}^4)$. By the fundamental theorem of calculus, we have that;

$\frac{\partial \overline{J}}{\partial t} = -c^2 \bigtriangledown(\rho)$

By the previous claim, for $t_0 \in \mathcal{R}$, ρ_{t_0} has compact support, so that $(\bigtriangledown(\rho))_{t_0}$ has compact support and $(\frac{\partial \overline{J}}{\partial t})_{t_0}$ has compact support. It is clear from the above that the compact support V_t of ρ_t and $(\bigtriangledown(\rho))_t$ varies continuously with t, so on the interval $(t_0-\epsilon, t_0+\epsilon)$, $(\frac{\partial \overline{J}}{\partial t})|_{(t_0-\epsilon, t_0+\epsilon)}$ has compact support $W_{t_0,\epsilon}$ in $\mathcal{R}^4$.

$\overline{J}$ satisfies the wave equation on $\mathcal{R}^4$, as, using the fundamental theorem of calculus and the fact that $\bigtriangledown(\rho)$ satisfies the wave equation;

$\square^2(\overline{J}) = \bigtriangledown^2(\overline{J}) - \frac{1}{c^2}\frac{\partial^2 \overline{J}}{\partial t^2}$

$= -c^2(\int_{-\infty}^{t} \bigtriangledown^2(\bigtriangledown(\rho))ds) - \frac{1}{c^2}(-c^2\frac{\partial \bigtriangledown(\rho)}{\partial t})$

$= -c^2(\int_{-\infty}^{t} \frac{1}{c^2}\frac{\partial^2 \bigtriangledown(\rho)}{\partial t^2}ds) + \frac{\partial \bigtriangledown(\rho)}{\partial t}$

$= -\frac{\partial \bigtriangledown(\rho)}{\partial t}) + \frac{\partial \bigtriangledown(\rho)}{\partial t}$

$= \overline{0}$

By the connecting relation;

$\bigtriangledown\rho + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$

we have that $\frac{\partial \overline{J}}{\partial t}$ vanishes outside $Supp(\rho_t)$, and for any $\overline{x} \in \mathcal{R}^3$, there exists two uniformly bounded intervals $[t_{1,\overline{x},-}, t_{2,\overline{x},-}]$, $[t_{1,\overline{x},+}, t_{2,\overline{x},+}]$, for which $\overline{x} \in Supp(\rho_t)$, for $t \in [t_{1,\overline{x},-}, t_{2,\overline{x},-}] \cup [t_{1,\overline{x},+}, t_{2,\overline{x},+}]$. Using the fact that $Supp(\rho_t)$ is moving and $\bigtriangledown(\rho)$ satisfies the wave equation, so uniformly bounded, we can define;

$\overline{J}_0(\overline{x}) = \int_{t_{1,\overline{x},-}}^{t_{2,\overline{x},-}} \frac{\partial \overline{J}}{\partial t}dt + \int_{t_{1,\overline{x},+}}^{t_{2,\overline{x},+}} \frac{\partial \overline{J}}{\partial t}dt$

$= \int_{-\infty}^{\infty} \frac{\partial \overline{J}}{\partial t}dt$ (the ultimate value of $\overline{J}(\overline{x}, t)$)

with $\overline{J}_0$ bounded. On any ball $B(\overline{0}, r)$, we have that $\overline{J} - \overline{J}_0$ eventually vanishes, and, as $div(\overline{J}) - div(\overline{J}_0) = 0$ ultimately on the ball, and $div(\overline{J}) = -\frac{\partial \rho}{\partial t} = 0$, ultimately, otherwise charge would build up, we have that $div(\overline{J}_0) = 0$. It follows that $(\rho, \overline{J} - \overline{J}_0)$ satisfies the continuity equation, and the linkage relation;

$\bigtriangledown\rho + \frac{1}{c^2}\frac{\partial(\overline{J}-\overline{J}_0)}{\partial t} = \overline{0}$

is still satisfied, as $\overline{J}_0$ is time independent. On any ball $B(\overline{0}, r)$, we have that ultimately $\overline{J} - \overline{J}_0 = \overline{0}$, so that, as $\square^2(\overline{J}) = \overline{0}$ and $\overline{J}_0$ is time independent, ultimately;

$\bigtriangledown^2(\overline{J}_0) = \square^2(\overline{J}_0) = \square^2(\overline{J}) = \overline{0}$

and $\overline{J}_0$ is harmonic. As the components $\bigtriangledown(\rho)_i$, for $1 \leq i \leq 3$, satisfy the wave equation, we have that that there exists constants $C_i \in \mathcal{R}_{>0}$, for which $|\bigtriangledown(\rho)_i(\overline{x}, t)| \leq \frac{C_i}{|t|}$ for $1 \leq i \leq 3$, so that;

$|\bigtriangledown(\rho)(\overline{x},t)| \leq \frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t|}$

and;

$|\overline{J}_0(\overline{x})| = |\int_{t_{1,\overline{x},-}}^{t_{2,\overline{x},-}} -c^2\bigtriangledown(\rho)dt + \int_{t_{1,\overline{x},+}}^{t_{2,\overline{x},+}} -c^2\bigtriangledown(\rho)dt|$

$\leq c^2[(t_{2,\overline{x},-} - t_{1,\overline{x},-}) + (t_{2,\overline{x},+} - t_{1,\overline{x},+})]|\bigtriangledown(\rho)|_{[t_{1,\overline{x},-},t_{2,\overline{x},-}]\cup[t_{1,\overline{x},-},t_{2,\overline{x},-}]}|$

$\leq c^2(t_{2,\overline{x},-} - t_{1,\overline{x},-})\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t_{1,\overline{x},-}|} + c^2(t_{2,\overline{x},+} - t_{1,\overline{x},+})\frac{\sqrt{C_1^2+C_2^2+C_3^2}}{|t_{1,\overline{x},+}|}$

$\leq \frac{C}{|\overline{x}|}$

as the intervals $[t_{1,\overline{x},-}, t_{2,\overline{x},-}]$, $[t_{1,\overline{x},+}, t_{2,\overline{x},+}]$ are uniformly bounded, and the hitting times $\{t_{1,\overline{x},-}, t_{1,\overline{x},+}\}$ are proportional to the distance $\overline{x}$. It follows, as bounded harmonic functions are constant, that $\overline{J}_0 = \overline{0}$, and $\overline{J}$ has compact supports.

□

Lemma 0.5. *For any $\{\rho, \overline{J}\} \subset C^\infty(\mathcal{R}^3 \times \mathcal{R}_{>0})$ with compact support satisfying the wave equations $\square^2(\rho) = 0$, $\square^2(\overline{J}) = \overline{0}$ $lim_{t\to 0}\rho_t = \rho_0$, $lim_{t\to 0}(\frac{\partial\rho}{\partial t})_t = g$, $lim_{t\to 0}\overline{J}_t = \overline{J}_0$, $lim_{t\to 0}(\frac{\partial\overline{J}}{\partial t})_t = \overline{g}$, we have the explicit representation;*

$\rho(\overline{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b(\overline{k})e^{ikct} + d(\overline{k})e^{-ikct})e^{i\overline{k}\cdot\overline{x}}d\overline{k}$

$\overline{J}(\overline{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b}(\overline{l})e^{ilct} + \overline{d}(\overline{l})e^{-ilct})e^{i\overline{l}\cdot\overline{x}}d\overline{l}$

where $\{b, d, \overline{b}, \overline{d}\} \subset L^1(\mathcal{R}^3)$.

Proof. As;

$\square^2(\rho) = 0$, $\square^2(\overline{J}) = \overline{0}$, $(*)$

We have that;

$\bigtriangledown^2(\rho) - \frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2} = 0$, $\bigtriangledown^2(\overline{J}) - \frac{1}{c^2}\frac{\partial^2\overline{J}}{\partial t^2} = 0$

We have that $\rho_t \in S(\mathcal{R}^3)$, as it is smooth and has compact support, so that, we can apply the three dimensional Fourier transform $\mathcal{F}$, and using integration by parts, differentiating under the integral sign, we have that, for $t > 0$;

$$\mathcal{F}(\bigtriangledown^2(\rho)(\overline{k},t)) - \frac{1}{c^2}\mathcal{F}(\frac{\partial^2\rho}{\partial t^2})(\overline{k},t)$$

$$= -k^2\mathcal{F}(\rho)(\overline{k},t) - \frac{1}{c^2}\frac{\partial^2(\mathcal{F}(\rho)(\overline{k},t))}{\partial t^2}$$

$$= -k^2 a(\overline{k},t) - \frac{1}{c^2}\frac{\partial^2 a(\overline{k},t)}{\partial t^2}$$

$$= 0$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$, $a = \mathcal{F}(\rho)$. For fixed $\overline{k}$, we obtain the ordinary differential equation;

$$\frac{d^2 a_{\overline{k}}}{dt^2} = -c^2k^2 a_{\overline{k}}$$

so that;

$$a_{\overline{k}}(t) = C_0(\overline{k})e^{ikct} + D_0(\overline{k})e^{-ikct}$$

with;

$$a_{\overline{k}}(0) = lim_{t\rightarrow 0}a_{\overline{k}}(t) = \mathcal{F}(\rho_0) = C_0(\overline{k}) + D_0(\overline{k})$$

$$a'_{\overline{k}}(0) = lim_{t\rightarrow 0}a'_{\overline{k}}(t) = \mathcal{F}(g) = ikcC_0(\overline{k}) - ikcD_0(\overline{k})\ (\dagger\dagger)$$

and, solving the simultaneous equations $(\dagger\dagger)$, we obtain that;

$$C_0(\overline{k}) = \frac{1}{2}(a_{\overline{k}}(0) + \frac{1}{ikc}a'_{\overline{k}}(0))$$

$$D_0(\overline{k}) = \frac{1}{2}(a_{\overline{k}}(0) - \frac{1}{ikc}a'_{\overline{k}}(0))$$

and;

$$\mathcal{F}(\rho)(\overline{k},t) = a(\overline{k},t)$$

$$= \frac{1}{2}(a_{\overline{k}}(0) + \frac{1}{ikc}a'_{\overline{k}}(0))e^{ikct} + \frac{1}{2}(a_{\overline{k}}(0) - \frac{1}{ikc}a'_{\overline{k}}(0))e^{-ikct}$$

$$= b(\overline{k})e^{ikct} + d(\overline{k})e^{-ikct}$$

where;

$b(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho|_{(\overline{x},0)})|_{(\overline{k},0)}+\frac{1}{ikc}\mathcal{F}(\frac{\partial\rho}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

$d(\overline{k})=\frac{1}{2}(\mathcal{F}(\rho|_{(\overline{x},0)})|_{(\overline{k},0)}-\frac{1}{ikc}\mathcal{F}(\frac{\partial\rho}{\partial t}|_{(\overline{x},0)})|_{(\overline{k},0)})$

Similarly;

$\mathcal{F}(\overline{J})(\bar{l},t)=\overline{a}(\bar{l},t)=\overline{b}(\bar{l})e^{ilct}+\overline{d}(\bar{l})e^{-ilct}$

where;

$\overline{b}(\bar{l})=\frac{1}{2}(\mathcal{F}((\overline{J})|_{(\overline{x},0)})|_{(\bar{l},0)}+\frac{1}{ilc}\mathcal{F}(\frac{\partial\overline{J}}{\partial t}|_{(\overline{x},0)})|_{(\bar{l},0)})$

$\overline{d}(\bar{l})=\frac{1}{2}(\mathcal{F}((\overline{J})|_{(\overline{x},0)})|_{(\bar{l},0)}-\frac{1}{ilc}\mathcal{F}(\frac{\partial\overline{J}}{\partial t}|_{(\overline{x},0)})|_{(\bar{l},0)})$

and $l^2=l_1^2+l_2^2+l_3^2$. Observe that;

$\{b,d,\overline{b},\overline{d}\}\subset L^1(\mathcal{R}^3)$, (FG)

as by the classical theory;

$\{\mathcal{F}(\rho_0),\mathcal{F}((\frac{\partial\rho}{\partial t})_0),\mathcal{F}(\overline{J}_0),\mathcal{F}((\frac{\partial\overline{J}}{\partial t})_0)\}\subset S(\mathcal{R}^3)\subset L^1(\mathcal{R}^3)$

and, using the fact that;

$\{\mathcal{F}((\frac{\partial\rho}{\partial t})_0),\mathcal{F}((\frac{\partial\overline{J}}{\partial t})_0)\}\subset C^\infty(B(\overline{0},1))\subset L^2(B(\overline{0},1))$

and, by a polar coordinates calculation, $\{\frac{1}{ikc},\frac{1}{ilc}\}\subset L^2(B(\overline{0},1))$, by the Cauchy Schwarz inequality;

$\{\frac{\mathcal{F}((\frac{\partial\rho}{\partial t})_0)}{ikc},\frac{\mathcal{F}((\frac{\partial\overline{J}}{\partial t})_0)}{ilc}\}\subset L^1(B(\overline{0},1))$

whereas, by the rapid decay of $S(\mathcal{R}^3)$ and a simple polar coordinate calculation;

$\{\frac{\mathcal{F}((\frac{\partial\rho}{\partial t})_0)}{ikc},\frac{\mathcal{F}((\frac{\partial\overline{J}}{\partial t})_0)}{ilc}\}\subset L^1(\mathcal{R}^3\setminus B(\overline{0},1))$

Using the fact that $\{b(\overline{k})e^{ikct}+d(\overline{k})e^{-ikct},\overline{b}(\bar{l})e^{ilct}+\overline{d}(\bar{l})e^{-ilct}\}\subset S(\mathcal{R}^3)$ for $t\in\mathcal{R}$, by the fact that the Fourier transform preserves the Schwartz class, see [3], we can apply the inversion theorem, to obtain;

$\rho(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(b(\overline{k})e^{ikct}+d(\overline{k})e^{-ikct})e^{i\overline{k}\centerdot\overline{x}}d\overline{k}$

$\overline{J}(\overline{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(\overline{b}(\overline{l})e^{ilct}+\overline{d}(\overline{l})e^{-ilct})e^{i\overline{l}\centerdot\overline{x}}d\overline{l}$

By the observation (FG), we can split the integral into two integrals.

□

Lemma 0.6. *Let* $(\rho,\overline{J})$ *be defined as in Lemma 0.4, then if* V_t *defines the support of* ρ_t*, we have that;*

$\frac{d}{dt}(\int_{V_t}\rho_t dV)=0$

$\int_{V_t}\bigtriangledown^2(\rho)=0$

We have that $\frac{\partial\overline{J}}{\partial t}$ *has compact support, and* $\overline{J}$ *is generated by Kirchoff's formula with initial data* $(\overline{J}_0,-c^2\bigtriangledown(\rho_0))$ *and the representation of Lemma 0.5 holds for* $\overline{J}$*.*

Proof. If $t_1<t_2$, with $\{t_1,t_2\}\subset\mathcal{R}$, and $\{V_{t_1},V_{t_2}\}$ denote the compact supports of $\{\rho_{t_1},\rho_{t_2}\}$, then as the supports vary continuously, and $\overline{J}_t$ and ρ_t are compactly supported for each $t\in[t_1,t_2]$, $\overline{J}_t$ and ρ_t are uniformly compacted supported for $t\in[t_1,t_2]$ in a ball $B(\overline{0},p)$, for some $p\in\mathcal{R}_{>0}$. In particularly;

$\int_{V_{t_1}}\rho_{t_1}dV=\int_{B(\overline{0},p)}\rho_{t_1}dV$

$\int_{V_{t_2}}\rho_{t_2}dV=\int_{B(\overline{0},p)}\rho_{t_2}dV$

For $t\in[t_1,t_2]$, using the continuity equation, the divergence theorem and the fact $\overline{J}_t$ is uniformly compacted supported for $t\in[t_1,t_2]$ in $B(\overline{0},p)$, we have that;

$\frac{d}{dt}(\int_{B(\overline{0},p)}\rho_t dV)=\int_{B(\overline{0},p)}\frac{\partial\rho}{\partial t}dV$

$=-\int_{B(\overline{0},p)}div(\overline{J})_t dV$

$=-\int_{\delta B(\overline{0},p)}\overline{J}_t\centerdot d\overline{S}dV$

$=0$

so that;

$\int_{B(\overline{0},p)} \rho_{t_1} dV = \int_{B(\overline{0},p)} \rho_{t_2} dV$

$\int_{V_{t_1}} \rho_{t_1} dV = \int_{V_{t_2}} \rho_{t_2} dV$

In particularly, $\frac{d}{dt}(\int_{V_t} \rho_t dV) = 0$. For the second claim, we have that, by the divergence theorem and the fact that $\bigtriangledown(\rho_t)$ vanishes on the boundary δV_t;

$\int_{V_t} \bigtriangledown^2(\rho_t) dV = \int_{\delta V_t} \bigtriangledown \cdot (\bigtriangledown(\rho_t)) dV$

$= \int_{\delta V_t} \bigtriangledown(\rho_t) \cdot d\overline{S}$

$= 0$

By the connecting relation, we have that $\frac{\partial \overline{J}}{\partial t} = -c^2 \bigtriangledown (\rho)$, which has compact support, because ρ does. As shown in Lemma 0.4, $\square^2(\overline{J}) = \overline{0}$, so, by Lemma 0.2, applied to the components of $\overline{J}$, $\overline{J}$ is generated by Kirchoff's formula with initial data $(\overline{J}_0, (\frac{\partial \overline{J}}{\partial t})_0) = (\overline{J}_0, -c^2 \bigtriangledown (\rho_0))$. Similarly, we can apply Lemma 0.5 to obtain the representation there for $\overline{J}$.

$\square$

Lemma 0.7. $(\rho, \overline{J})$ *be defined as in Lemma 0.4, then we can define antiderivatives, by letting;*

$\rho^a(\overline{x}, t) = \int_{-\infty}^t p(\overline{x}, s) ds$

$\overline{J}^a(\overline{x}, t) = \int_{-\infty}^t \overline{J}(\overline{x}, s) ds$

$(\rho^a, \overline{J}^a) \subset C^\infty(\mathcal{R}^4)$ *and satisfy the wave equations, the continuity equation and the connecting relation again. Moreover, if* $\rho, \overline{J}, \overline{E}, \overline{B})$ *is a solution to Maxwell's equations, then* $(-\frac{\rho^a}{\epsilon_0}, \overline{E})$ *satisfy the continuity equation.*

Proof. The definition follows from Lemma 0.6 as $\overline{J}$ can be represented by Kirchoff's formula. As is easily checked, if $p \in C^\infty(\mathcal{R}^4)$ and the components $j_i \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, then $\rho^a \in C^\infty(\mathcal{R}^4)$ and the components $j_i^a \in C^\infty(\mathcal{R}^4)$, for $1 \leq i \leq 3$. The wave equation holds for ρ^a and $\overline{J}^a$, as, using the fundamental theorem of calculus, differentiating under the integral sign, the result about he left hand limit in [2], and using the fact that ρ satisfies the wave equation;

$\square^2(\rho^a) = \int_{-\infty}^t \bigtriangledown^2(\rho)ds - \frac{1}{c^2}\frac{\partial\rho}{\partial t}$

$= \int_{-\infty}^t \frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2}ds - \frac{1}{c^2}\frac{\partial\rho}{\partial t}$

$= \frac{1}{c^2}\frac{\partial\rho}{\partial t} - \frac{1}{c^2}\frac{\partial\rho}{\partial t}$

$= 0$

and;

$\square^2(\overline{J}^a) = \int_{-\infty}^t \bigtriangledown^2(\overline{J})ds - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \int_{-\infty}^t \frac{1}{c^2}\frac{\partial^2\overline{J}}{\partial t^2}ds - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t} - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \overline{0}$

Differentiating under the integral sign and using the fundamental theorem of calculus, the fact that the continuity equation holds for $(\rho, \overline{J})$, the continuity equation holds as;

$\frac{\partial\rho^a}{\partial t} + \bigtriangledown \cdot \overline{J}^a$

$= \rho + \int_{-\infty}^t \bigtriangledown \cdot \overline{J}ds$

$= \rho + \int_{-\infty}^t + \int_{-\infty}^t -\frac{\partial\rho}{\partial s}ds$

$= \rho - \rho = 0$

and, differentiating under the integral sign, using the fundamental calculus of calculus and the connecting relation for $(\rho, \overline{J})$, the connecting relation holds;

$\bigtriangledown(\rho^a) + \frac{1}{c^2}\frac{\partial\overline{J}^a}{\partial t}$

$= \int_{-\infty}^t \bigtriangledown(\rho)ds + \frac{1}{c^2}\overline{J}$

$= \int_{-\infty}^t -\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}ds + \frac{1}{c^2}\overline{J}$

$= -\frac{1}{c^2}\overline{J} + \frac{1}{c^2}\overline{J}$

$= \overline{0}$

The last claim follows, using the FTC and Maxwell's first equation, that;

$\frac{\partial(-\frac{\rho^a}{\epsilon_0})}{\partial t} + div(\overline{E}) = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0}$

$= 0$ □

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