SOME ARGUMENTS FOR THE WAVE EQUATION IN QUANTUM THEORY 7: THE HYPERBOLIC METHOD

TRISTRAM DE PIRO

Abstract.

Lemma 0.1. ....

Proof. The same results hold for $w \neq c$. If $w \neq c$, using Jefimenko's equations, we can prove the existence of fields $(\overline{E}_w, \overline{B}_w)$, for which the components depending on $\overline{J}_w$ have compact support at time t, the support increasing as $w \rightarrow c$ and uniformly bounded. ?$\frac{\partial \overline{J}}{\partial t}$ obeys a wave equation (with speed w)?. Also true that if $(\overline{E}, \overline{B})$ are defined from $(\rho, \overline{J})$, using Jefimenko's equations, then $(\frac{\partial \overline{E}}{\partial t}, \frac{\partial \overline{B}}{\partial t})$ are defined from $(\frac{\partial \rho}{\partial t}, \frac{\partial \overline{J}}{\partial t})$ using Jefimenko's equations, provided the causal solution exists.

For the sixth claim, following the method of [**?**], and the results in this paper, we can construct charge and current configurations $(\rho_w, \overline{J}_w)$ for $w \in \mathcal{R}_{>0}$, $w \neq c$, such that $\square^2_w(\rho_w) = 0$, $\square^2_w(\overline{J}_w) = \overline{0}$, $\bigtriangledown(\rho) + \frac{1}{w^2}\frac{\partial \overline{J}}{\partial t} = 0$, $\frac{\partial \rho}{\partial t} = -\bigtriangledown \cdot \overline{J}$, with the same initial conditions (f, g) and support V. All the arguments for charge and current we have used for c, hold in the case $w \neq c$, being careful to replace c with w in the definitions. In this case, the fields $(\overline{E}_w, \overline{B}_w)$ generated by Jefimenko's equations are well defined for $t \in \mathcal{R}$, with respect to charge, as, for given $\overline{x}_0$, the locus of $\{\overline{x} : B(\overline{x}, wt_r) \cap V \neq \emptyset\}$ is bounded, because $wt_r = w(t - \frac{|\overline{x} - \overline{x}_0|}{c})$ contains the factor $\frac{w}{c} \neq 1$, and for current, a similar idea, the proof being the same, as the current obeys the wave equation and has compact support, receding at speed w. Then, we have that $(\rho_w, \overline{J}_w, \overline{E}_w, \overline{B}_w)$ satisfy Maxwell's equations. If we use Kirchoff's formula for $\frac{\partial \rho}{\partial t}$, with initial conditions $(\frac{\partial \rho}{\partial t}|_0, \frac{\partial^2 \rho}{\partial^2 t}|_0) = (\frac{\partial \rho}{\partial t}|_0, -c^2(\bigtriangledown^2 \rho)|_0)$;

$$\frac{\partial \rho}{\partial t}(\overline{x}, t) = \frac{1}{4\pi c^2 t^2}\int_{\delta B(\overline{x}, ct)} (t\frac{\partial^2 \rho}{\partial t^2}|_0) + \frac{\partial \rho}{\partial t}_0(\overline{y}) +$$

$$D(\frac{\partial \rho}{\partial t}|_0)(\overline{y}) \cdot (\overline{y} - \overline{x}))dS(\overline{y}) \ (t > 0)$$

$\frac{\partial\rho}{\partial t}(\overline{x},t)=\frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(t\frac{\partial^2\rho}{\partial t^2}|_0)+\frac{\partial\rho}{\partial t}|_0(\overline{y})$

$+D(\frac{\partial\rho}{\partial t}|_0)(\overline{y})\cdot(\overline{y}-\overline{x}))dS(\overline{y})\ (t<0)$

We then have, using Jefimenko's equations;

$(\frac{1}{4\pi\epsilon_0}\int_V\frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|}d\tau')_1=\frac{1}{4\pi\epsilon_0}\int_V\frac{\partial\rho}{\partial t}(\overline{r}',t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$=\frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)+\frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$+\frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)+\frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

We can use then use the asymmetry (r_1-r_1') $r_1=0$, $r_1''=-r_1'$, together with the symmetry, in the integral;

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2(\rho)}{\partial t^2})(\overline{y},0)dS(\overline{y})=\int_{\delta B(\overline{r}'',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2(\rho)}{\partial t^2})(\overline{y},0)dS(\overline{y})\ (t=0)$

and vanishing in the integral of $\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}D(\frac{\partial\rho}{\partial t}|_0)(\overline{y})\cdot t\overline{z}dS(\overline{y})$ for large $\overline{r}'$, see Lemma 0.2, and the $\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}$ decay in the remaining term, to show that $lim_{w\rightarrow c}(\rho_w,\overline{J}_w,\overline{E}_w,\overline{B}_w)$ exists and define $(\rho_c,\overline{J}_c,\overline{E}_c,\overline{B}_c)$ as $lim_{w\rightarrow c}(\rho_w,\overline{J}_w,\overline{E}_w,\overline{B}_w)$, for the original charge and current combination $(\rho_c,\overline{J}_c)$. It is clear that $(\rho_c,\overline{J}_c,\overline{E}_c,\overline{B}_c)$ satisfies Maxwell's equations, and the configuration $(\overline{E}_c,\overline{B}_c)$ is defined by Jefimenko's equations as an indefinite integral. A detailed exposition of this claim is the the subject of the following.

We are mainly interested in the case $w=c$, but most of the calculations can be adapted to the case $w\neq c$, the important point being to keep the factor c in Jefimenko's equations, ([1]). Unless otherwise stated

[1] There may be a point that particles travelling at speed c in the base frame would contradict special relativity, but it is not clear with an extended charge distribution that there are any individual particles. In any case, the associated charge and current configuration $(\rho,\overline{J})$ exists and seems to define fields $(\overline{E},\overline{B})$ satisfying Maxwell's equations with special properties, at least in the case $w>c$. The case when inertial frames travel at speeds $w>c$ is developed in [**?**].

though, $w = c$. We can assume by the above and the proof in [?], that $\rho \in C^{\infty}(\mathcal{R}^4)$, for the components j_i, $1 \leq i \leq 3$, $j_i \in C^{\infty}(\mathcal{R}^4)$, for $t \in \mathcal{R}$, ρ_t and $j_{i,t}$ have compact support, and the components j_i satisfy the wave equation $\square^2 j_i = 0$, $1 \leq i \leq 3$. It follows that the derivatives $\frac{\partial \rho}{\partial t} \in C^{\infty}(\mathcal{R}^4)$ and $\frac{\partial j_i}{\partial t} \in C^{\infty}(\mathcal{R}^4)$, $1 \leq i \leq 3$, that $\frac{\partial \rho}{\partial t}$ and $\frac{\partial j_i}{\partial t}$, $1 \leq i \leq 3$ obey the wave equation and, for $t \in \mathcal{R}$, $\frac{\partial \rho}{\partial t}_t$ and $\frac{\partial j_{i,t}}{\partial t}$, $1 \leq i \leq 3$ have compact support. The fields $\{\overline{E}, \overline{B}\}$ defined by Jefimenko's equations are given by;

$$\overline{E}(\overline{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2} d\tau' + \int_V \frac{\dot{\rho}(\overline{r}', t_r)\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|} d\tau' - \int_V \frac{\dot{\overline{J}}(\overline{r}', t_r)}{c^2|\overline{r}-\overline{r}'|} d\tau'$$

$$\overline{B}(\overline{r}, t) = \frac{\mu_0}{2\pi} \int_V \frac{\overline{J}(\overline{r}', t_r) \times \hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2} d\tau' + \int_V \frac{\dot{\overline{J}}(\overline{r}', t_r) \times \hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|} d\tau$$

We have using Kirchoff's formula, that, for $t > 0$;

$$\rho(\overline{x}, t) = \frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, ct)} (tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$$

and, for $t < 0$;

$$\rho(\overline{x}, t) = \frac{1}{4\pi c^2 t^2} \int_{\delta B(\overline{x}, -ct)} (tg(\overline{y}) + \rho_0(\overline{y}) + D\rho_0(\overline{y}) \cdot (\overline{y} - \overline{x})) dS(\overline{y})$$

so that;

$$\left(\frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2} d\tau'\right)_1 = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\overline{r}', t - \frac{|\overline{r}-\overline{r}'|}{c})(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^3} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_V \Big[\frac{1}{4\pi c^2 (t - \frac{|\overline{r}-\overline{r}'|}{c})^2} \int_{\delta B(\overline{r}', c(t - \frac{|\overline{r}-\overline{r}'|}{c}))} ((t - \frac{|\overline{r}-\overline{r}'|}{c}) \frac{\partial \rho(\overline{y}, 0)}{\partial t} + \rho(\overline{y}, 0)$$

$$+ D\rho(\overline{y}, 0) \cdot (\overline{y} - \overline{r}')) dS(\overline{y})$$

$$+ \frac{1}{4\pi c^2 (t - \frac{|\overline{r}-\overline{r}'|}{c})^2} \int_{\delta B(\overline{r}', -c(t - \frac{|\overline{r}-\overline{r}'|}{c}))} ((t - \frac{|\overline{r}-\overline{r}'|}{c}) \frac{\partial \rho(\overline{y}, 0)}{\partial t} + \rho(\overline{y}, 0)$$

$$+ D\rho(\overline{y}, 0) \cdot (\overline{y} - \overline{r}')) dS(\overline{y})\Big] \frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^3} d\tau'$$

Let;

$$W_1 = \{\overline{r}' : \delta B(\overline{r}', c(t - \frac{|\overline{r}-\overline{r}'|}{c})) \cap B(\overline{0}, w) \neq \emptyset\}$$

$$W_2 = \{\overline{r}' : \delta B(\overline{r}', -c(t - \frac{|\overline{r}-\overline{r}'|}{c}) \cap B(\overline{0}, w) \neq \emptyset\}$$

With the convention $(*)$ below, if $t>0$, we require that $c(t-\frac{|\overline{r}-\overline{r}'|}{c})>0$ iff $|\overline{r}-\overline{r}'|<ct$, so that $W_1\subset B(\overline{0},ct)$, if $t>0$ and $W_1=\emptyset$ if $t\leq 0$. Similarly, we require that $-c(t-\frac{|\overline{r}-\overline{r}'|}{c}>0$ iff $|\overline{r}-\overline{r'}|>ct$, so that, if $t\geq 0$, $W_2\subset\mathcal{R}^3\setminus B(\overline{0},ct)$ and if $t<0$, we obtain no restriction on W_2. In either case, we clearly have, by smoothness of the data, continuity and the fact that $B(\overline{0},ct)$ is bounded for $t>0$, that;

$$|\frac{1}{4\pi\epsilon_0}\int_{W_1}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$$

$$+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'|$$

$$\leq\int_{B(\overline{0},ct)}C_t|\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}|d\tau'$$

$$\leq\frac{C_t}{c}\int_{B(\overline{0},ct)}\frac{1}{|\overline{r}-\overline{r}'|^2}d\tau'$$

$$\leq\frac{C_t}{c}\int_{B(\overline{0},ct)}\frac{1}{|\overline{r}'|^2}d\tau'$$

$$\leq\frac{C_t}{c}\int_0^{\pi}\int_{-\pi}^{\pi}\int_0^{ct}\frac{1}{r^2}r^2|sin(\theta)|drd\theta d\phi$$

$$\leq\frac{2\pi^2C_t}{c}\int_0^{ct}dr$$

$$\leq 2\pi^2tC_t$$

so that;

$$(\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau')_1=f_1(\overline{r},t)$$

$$+\int_{W_2}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$$

$$+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'$$

We can assume in the calculation that $\overline{r}\neq\overline{0}$, by changing coordinates with a translation given by $\overline{r}_0$, see below for the corresponding time translation, as we can define a new pair $(\rho^{\overline{r}_0},\overline{J}^{\overline{r}_0})$ by $\rho^{\overline{r}_0}(\overline{x},s)=\rho(\overline{x}-\overline{r}_0,s)$ and $\overline{J}^{\overline{r}_0}(\overline{x},s)=\overline{J}(\overline{x}-\overline{r}_0,s)$, for $(\overline{x},s)\in\mathcal{R}^4$. The new pair $(\rho^{\overline{r}_0},\overline{J}^{\overline{r}_0})$ inherits the properties of $(\rho,\overline{J})$, in particular we have that $\rho^{\overline{r}_0}\in C^\infty(\mathcal{R}^4)$, the components of $\overline{J}^{\overline{r}_0}$, $j_i^{\overline{r}_0}\in C^\infty(\mathcal{R}^4)$, $1\leq i\leq 3$, $\square^2(\rho^{\overline{r}_0})=0$, for $1\leq i\leq 3$, the continuity equation $\frac{\partial\rho^{\overline{r}_0}}{\partial t}=-\bigtriangledown\centerdot\overline{J}^{\overline{r}_0}$ holds, and the connecting relation $\bigtriangledown(\rho^{\overline{r}_0})+\frac{1}{c^2}\frac{\partial\overline{J}^{\overline{r}_0}}{\partial t}=\overline{0}$. Moreover, we can use Kirchoff's formula with the initial data for $(\rho^{\overline{r}_0},\overline{J}^{\overline{r}_0})$ given by $(\rho_0^{\overline{r}_0},(\frac{\partial\rho^{\overline{r}_0}}{\partial t})_0,\overline{J}_0^{\overline{r}_0},(\frac{\partial\overline{J}^{\overline{r}_0}}{\partial t})_0)$ and we have that, making the substitution $\overline{r}''=\overline{r}_0+\overline{r}'$;

$(\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}',t_r)\hat{\hat{\mathfrak{r}}}}{|\overline{0}-\overline{r}'|^2}d\tau')_1=(\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\overline{r}''-\overline{r}_0,t_r')\hat{\hat{\mathfrak{r}}}}{|\overline{r}_0-\overline{r}''|^2}d\tau'')_1=$

$=(\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho^{\overline{r}_0}(\overline{r}'',t_r')\hat{\hat{\mathfrak{r}}}}{|\overline{r}_0-\overline{r}''|^2}d\tau'')_1$

for the corresponding retarded time $t_r'=t-\frac{|\overline{r}_0-\overline{r}''|}{c}$, and, similarly, for the corresponding terms in Jefimenko's equations.

We have, for $\overline{r}'\neq\overline{0}$, $\overline{r}\neq\overline{0}$, that;

$\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c})\cap B(\overline{0},w)\neq\emptyset$

iff $|\overline{r}'-[-c(t-\frac{|\overline{r}-\overline{r}'|}{c})]\frac{\overline{r}'}{|\overline{r}'|}|\leq w$

iff $|\overline{r}'|\overline{r}'|+(ct-|\overline{r}-\overline{r}'|)\overline{r}'|\leq w|\overline{r}'|$

iff $|\overline{r}'||\,|\overline{r}'|+(ct-|\overline{r}-\overline{r}'|)|\leq w|\overline{r}'|$

iff $||\overline{r}'|+ct-|\overline{r}-\overline{r}'||\leq w$

iff $-w-ct\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w-ct$

so that, if $t>0$;

$W_2=\{\overline{r}':-w-ct\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w-ct\}\cap\mathcal{R}^3\setminus B(\overline{0},ct)$

and, if $t\leq 0$;

$W_2=\{\overline{r}':-w-ct\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w-ct\}$

Letting;

$N=max_{\overline{y}\in B(0,\overline{w})}(|(\frac{\partial\rho}{\partial t})_0|,|\rho_0|,|D\rho_0|,|(D\rho)_0|)$

so that, for $\overline{r}'\in W_2$, using the fact the initial data is supported in $B(\overline{0},w)$;

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}|\frac{\partial\rho(\overline{y},0)}{\partial t}|dS(\overline{y})\leq 4\pi w^2 M$

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}|\rho(\overline{y},0)|dS(\overline{y})\leq 4\pi w^2 M$

$\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}|D\rho(\overline{y},0)\cdot(\overline{y}-\overline{r}')|dS(\overline{y})\leq\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))\cap B(\overline{0},w)}M|(\overline{y}-\overline{r}')|dS(\overline{y})$

$\leq 4\pi w^2 M|-c(t-\frac{|\overline{r}-\overline{r}'|}{c})|$

we have that, for s sufficiently large;

$|\int_{W_2\setminus B(\overline{0},s)}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}+\rho(\overline{y},0)$

$+D\rho(\overline{y},0)\cdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'|$

$\leq\int_{W_2\setminus B(\overline{0},s)}\frac{1}{4\pi c^2|t-\frac{|\overline{r}-\overline{r}'|}{c}|^2}(4\pi w^2 M|t-\frac{|\overline{r}-\overline{r}'|}{c}|+4\pi w^2 M$

$+4\pi w^2 M|-c(t-\frac{|\overline{r}-\overline{r}'|}{c})|)\frac{1}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$=w^2 M\int_{W_2\setminus B(\overline{0},s)}(\frac{1}{c^3|t-\frac{|\overline{r}-\overline{r}'|}{c}|}+\frac{1}{c^3|t-\frac{|\overline{r}-\overline{r}'|}{c}|^2}+\frac{1}{c^2|t-\frac{|\overline{r}-\overline{r}'|}{c}|})\frac{1}{|\overline{r}-\overline{r}'|^2}d\tau'$

We have that, for s sufficiently large;

$\int_{W_2\setminus B(\overline{0},s)}\frac{1}{|t-\frac{|\overline{r}-\overline{r}'|}{c}||\overline{r}-\overline{r}'|^2}d\tau'=\int_{W_2}\frac{1}{|t-\frac{(|\overline{r}|^2-2\overline{r}\cdot\overline{r}'+|\overline{r}'|^2)^{\frac{1}{2}}}{c}|(|\overline{r}|^2-2\overline{r}\cdot\overline{r}'+|\overline{r}'|^2)}d\tau'$

$=\int_{W_2\setminus B(\overline{0},s)}\frac{1}{|\overline{r}'|^3|\frac{t}{|\overline{r}|'}-\frac{(\frac{|\overline{r}|^2}{|\overline{r}'|^2}-2\frac{\overline{r}\cdot\overline{r}'}{|\overline{r}'|^2}+1)^{\frac{1}{2}}}{c}|(\frac{|\overline{r}|^2}{|\overline{r}'|^2}-2\frac{\overline{r}\cdot\overline{r}'}{|\overline{r}'|^2}+1)}d\tau'$

and, for $|\overline{r}'|>max(t,4|\overline{r}|)$;

$\frac{1}{|\overline{r}'|^3|\frac{t}{|\overline{r}|'}-\frac{(\frac{|\overline{r}|^2}{|\overline{r}'|^2}-2\frac{\overline{r}\cdot\overline{r}'}{|\overline{r}'|^2}+1)^{\frac{1}{2}}}{c}|(\frac{|\overline{r}|^2}{|\overline{r}'|^2}-2\frac{\overline{r}\cdot\overline{r}'}{|\overline{r}'|^2}+1)}\leq\frac{c}{(2c+4)|\overline{r}'|^3}$

so that;

$\int_{W_2\setminus B(\overline{0},max(t,4|\overline{r}|))}\frac{1}{|t-\frac{|\overline{r}-\overline{r}'|}{c}||\overline{r}-\overline{r}'|^2}d\tau'\leq\int_{W_2\setminus B(\overline{0},max(t,4|\overline{r}))}\frac{c}{(2c+4)|\overline{r}'|^3}d\tau'$

Similarly;

$\int_{W_2\setminus B(\overline{0},max(\sqrt{t},4|\overline{r}|))}\frac{1}{|t-\frac{|\overline{r}-\overline{r}'|}{c}|^2|\overline{r}-\overline{r}'|^2}d\tau'\leq\int_{W_2\setminus B(\overline{0},max(\sqrt{t},4|\overline{r}))}\frac{c}{(2c+8)|\overline{r}'|^4}d\tau'$

so that;

$$|\int_{W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r}|))}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}((t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{\partial\rho(\overline{y},0)}{\partial t}$$

$$+\rho(\overline{y},0)+D\rho(\overline{y},0)\centerdot(\overline{y}-\overline{r}'))dS(\overline{y})]\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^3}d\tau'|$$

$$\leq w^2M\int_{W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r}))}(\frac{(c+1)}{c^2(2c+4)|\overline{r}'|^3}+\frac{1}{c^2(2c+8)|\overline{r}'|^4})d\tau'$$

As above, we have that;

$(\frac{1}{4\pi\epsilon_0}\int_{W_2\cap B(\overline{0},max(t,\sqrt{t},4|\overline{r}))}\frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau')_1$ is finite and we claim that

$$\int_{W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r}))}(\frac{(c+1)}{c^2(2c+4)|\overline{r}'|^3}+\frac{1}{c^2(2c+8)|\overline{r}'|^4})d\tau'$$

is finite as well. In order to see this, note that up to a bounded region, W_2 is contained in a family of real quadratic surfaces, parametrised by a finite interval $[-\beta,\beta]\supset[-w-ct,w-ct]$ degenerating to the plane $|\overline{r}'|=|\overline{r}-\overline{r}'|$, if $0\in[-w-ct,w-ct]$, ([2]). Compactifying in $P(\mathcal{R}^3)\times$

[2] Noting, that for $d\in\mathcal{R}_{\neq 0}$, $|\overline{r}'|$ sufficiently large, with the interval $(-\beta,\beta)$ symmetric, we have that, denoting by $|\overline{r}'|-|\overline{r}-\overline{r}'|=|d|$, the union of $|\overline{r}'|-|\overline{r}-\overline{r}'|=d$ and $|\overline{r}'|-|\overline{r}-\overline{r}'|=-d$;

$|\overline{r}'|-|\overline{r}-\overline{r}'|=|d|$ or $|\overline{r}'|+|\overline{r}-\overline{r}'|=|d|$

iff $|\overline{r}-\overline{r}'|^2=|\overline{r}'|^2-2|d||\overline{r}'|+|d|^2$

so that, as $|\overline{r}'|+|\overline{r}-\overline{r}'|=|d|$ is bounded in $\mathcal{R}^3$;

$|\overline{r}'|-|\overline{r}-\overline{r}'|=|d|$

iff $|\overline{r}-\overline{r}'|^2=|\overline{r}'|^2-2|d||\overline{r}'|+|d|^2$

iff $R^2-(2r_1r_1'+2r_2r_2'+2r_3r_3')+|\overline{r}'|^2=|\overline{r}'|^2-2|d||\overline{r}'|+d^2$

iff $-(2r_1r_1'+2r_2r_2'+2r_3r_3')-(|d|^2-R^2)=-2|d||\overline{r}'|$

iff $[(2r_1r_1'+2r_2r_2'+2r_3r_3')+(|d|^2-R^2)]^2=4|d|^2(r_1'^2+r_2'^2+r_3'^2)$

iff $4(r_1r_1'+r_2r_2'+r_3r_3')^2+4(r_1r_1'+r_2r_2'+r_3r_3')(|d|^2-R^2)+(|d|^2-R^2)^2$

$=4d^2(r_1'^2+r_2'^2+r_3'^2)$

where $R=|\overline{r}|$. Note that the degenerate case of a single two dimensional plane in $\mathcal{R}^3$ corresponds to the idealised case when the initial charge distribution ρ_0 is supported at a single point.

In coordinates (x,y,z), if we intersect a real generic quadratic surface defined by;

$\alpha x^2+\beta y^2+\gamma z^2+\delta xy+\epsilon xz+\zeta yz+\eta x+\theta y+\iota z+\kappa=0$, $(*)$

where $\{\alpha,\beta,\gamma,\delta,\epsilon,\theta,\eta,\xi,\eta,\iota,\kappa\}\subset\mathcal{R}$, with a real generic plane $\lambda x+\mu y+\nu z=\xi$, we obtain that $x=\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z$, so that substituting in $(*)$;

$\alpha(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)^2+\beta y^2+\gamma z^2+\delta(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)y+\epsilon(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)z+\zeta yz+$

$\eta(\frac{\xi}{\lambda}-\frac{\mu}{\lambda}y-\frac{\nu}{\lambda}z)+\theta y+\iota z+\kappa=0$

which defines a real quadratic curve in the coordinates (y,z). If the curve is generic and unbounded, it cannot be a parabola, a circle or an ellipse, so by the classification of conic sections, must be a hyperbola. By a result in [**?**], the standard form of a hyperbola is given by;

$\frac{y^2}{a^2}-\frac{z^2}{b^2}=(\frac{y}{a}+\frac{z}{b})(\frac{y}{a}-\frac{z}{b})=1$

so that by a further change of coordinates $\xi=\frac{y}{a}+\frac{z}{b}$, $\eta=\frac{y}{a}-\frac{z}{b}$, we can write this in the standard form $\xi\eta=1$, with asymptotes $\xi=0$, $\eta=0$, defining a curve C' with asymptotes $\{l_1',l_2'\}$. If the original hyperbola C has asymptotes $\{l_1,l_2\}$, and is defined using a set of coefficients $\{c_i:1\leq i\leq 5\}$, with a fixed bound $|c_i|\leq f$, $f\in\mathcal{R}_{>0}$, then there exists a linear transformation $T:\mathcal{R}^2\rightarrow\mathcal{R}^2$ and a shift map $S:\mathcal{R}^2\rightarrow\mathcal{R}^2$ such that $(ST)(C')=C$, $(ST)(l_1')=l_1$, $(ST)(l_2')=l_2$. If $\overline{x}\in C'$ and $\overline{x}'$ is the nearest point to $\overline{x}$ on $l_1'\cup l_2'$, then $|\overline{x}-\overline{x}'|<\frac{\sqrt{2}}{|\overline{x}}$ for $|\overline{x}|>2$. It follows that;

$|(ST)(\overline{x})-(ST)(\overline{x}')|\leq||T|||\overline{x}-\overline{x}'|$

$<||T||\frac{\sqrt{2}}{|\overline{x}|}$

$=||T||\frac{\sqrt{2}}{|(ST)^{-1}(ST)\overline{x}|}$

so that for $\overline{y}\in C$, we have that, for the nearest point $\overline{y}'\in l_1\cup l_2$;

$|\overline{y}-\overline{y}'|<\frac{\sqrt{2}||T||}{|(ST)^{-1}\overline{y}|}$

$\leq\frac{\sqrt{2}||T||(||T||+1)}{|\overline{y}}$

provided $|(ST)^{-1}\overline{y}|\geq max(|\overline{s}|,2)$, $(*)$, where $\overline{s}$ defines S, as;

$|\overline{y}|=|(ST)(ST)^{-1}(\overline{y})$

$=|T(ST)^{-1}+\overline{s}|$

$\leq(||T||+1)|(ST)^{-1}(\overline{y}|$

provided $|(ST)^{-1}\overline{y}|\geq|\overline{s}|$, in which case;

$[-\beta,\beta]$, and using the implicit function theorem, we could choose a finite cover $\{U_1,\ldots,U_n\}$ of $\mathcal{R}^2\setminus B(\overline{0},1)\times[-\beta,\beta]$ and a sequence of maps $f_i : U_i \rightarrow W_2 \setminus B(\overline{0}, max(t,\sqrt{t},4|\overline{r}))$ with constants $C_i \in \mathcal{R}_{>0}$ such that $|f_i(\overline{x},t')| \geq C_i|\overline{x}|$, $|det(Jac(f_i))|$ is bounded uniformly in t' by constants $N_i \in \mathcal{R}_{>0}$, and the maps f_i cover $W_2 \setminus B(\overline{0}, max(t,\sqrt{t},4|\overline{r}))$. We then have that;

$$|\int_{W_2\setminus B(\overline{0},max(t,\sqrt{t},4|\overline{r}))}(\frac{(c+1)}{c^2(2c+4)|\overline{r}'|^3}+\frac{1}{c^2(2c+8)|\overline{r}'|^4})d\tau'|$$

$$\leq \sum_{i=1}^n|\int_{U_i} f_i^*(\frac{(c+1)}{c^2(2c+4)|\overline{r}'|^3}+\frac{1}{c^2(2c+8)|\overline{r}'|^4})|det(Jac(f_i))|dxdxydt'$$

$$\leq \sum_{i=1}^n\int_{U_i} N_i(\frac{(c+1)}{C_i^3c^2(2c+3)|(x,y)|^3}+\frac{1}{C_i^3c^2(2c+8)|(x,y)|^4})dxdydt'$$

$$\leq \sum_{i=1}^n\frac{2N_i\beta}{C_i^3}\int_{\mathcal{R}^2\setminus B(\overline{0},1)}(\frac{(c+1)}{c^2(2c+3)|(x,y)|^3}+\frac{1}{c^2(2c+8)|(x,y)|^4})dxdy$$

$$\leq \sum_{i=1}^n\frac{4\pi N_i\beta}{C_i^3}\int_{r>1}(\frac{(c+1)r}{c^2(2c+3)r^3}+\frac{r}{c^2(2c+8)r^4})dr$$

$|(ST)^{-1}(\overline{y}| \geq \frac{|\overline{y}|}{||T||+1}$

and;

$\frac{1}{|(ST)^{-1}(\overline{y}|)} \leq \frac{(||T||+1)}{|\overline{y}|}$

We can achieve the condition $(*)$ with $|\overline{y}| \geq ||T||(|T^{-1}\overline{s}|+max(2,|\overline{s}|))$, as;

$|(ST)^{-1}\overline{y}| \geq max(|\overline{s}|,2)$

iff $|T^{-1}(\overline{y})-T^{-1}\overline{s}| \geq max(|\overline{s}|,2)$

which we can achieve if $|T^{-1}(\overline{y})| \geq |T^{-1}\overline{s}|+max(|\overline{s}|,2)$

but as $|\overline{y}| \leq ||T|||T^{-1}(\overline{y})|$

we have that, $|T^{-1}(\overline{y})| \geq \frac{|\overline{y}|}{||T||}$, so if $|\overline{y}| \geq ||T||(|T^{-1}\overline{s}|+max(2,|\overline{s}|))$, then $|T^{-1}(\overline{y})| \geq |T^{-1}\overline{s}|+max(|\overline{s}|,2)$

We then obtain that, for $\overline{y}\in C$, for the nearest point $\overline{y}'\in l_1\cup l_2$;

$|\overline{y}-\overline{y}'| \leq \frac{E}{|\overline{y}}$

for $|\overline{y}| \geq D$, where $D=||T||(|T^{-1}\overline{s}|+max(2,|\overline{s}|))$, $E=\sqrt{2}||T||(||T||+1)$.

$= \sum_{i=1}^{n} \frac{4\pi N_i\beta}{C_i^3} \int_{r>1} (\frac{(c+1)}{c^2(2c+3)r^2} + \frac{1}{c^2(2c+8)r^3})dr$

$= \sum_{i=1}^{n} \frac{4\pi N_i\beta}{C_i^3} (\frac{(c+1)}{c^2(2c+3)} + \frac{1}{2c^2(2c+8)})$

This proves that $(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|^2}d\tau')_1$ is finite and well defined.

We then have that, using Kirchoff's formula for $\frac{\partial\rho}{\partial t}$, with initial conditions $(\frac{\partial\rho}{\partial t}|_0, \frac{\partial^2\rho}{\partial^2 t}|_0) = (\frac{\partial\rho}{\partial t}|_0, -c^2(\bigtriangledown^2\rho)|_0)$;

$\frac{\partial\rho}{\partial t}(\overline{x},t) = \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},ct)}(t\frac{\partial^2\rho}{\partial t^2}|_0) + \frac{\partial\rho}{\partial t}_0(\overline{y}) +$

$D(\frac{\partial\rho}{\partial t}|_0)(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})\ (t>0)$

$\frac{\partial\rho}{\partial t}(\overline{x},t) = \frac{1}{4\pi c^2t^2}\int_{\delta B(\overline{x},-ct)}(t\frac{\partial^2\rho}{\partial t^2}|_0) + \frac{\partial\rho}{\partial t}|_0(\overline{y})$

$+D(\frac{\partial\rho}{\partial t}|_0)(\overline{y}) \centerdot (\overline{y}-\overline{x}))dS(\overline{y})\ (t<0)$

that, using Jefimenko's equations;

$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho}(\overline{r}',t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|}d\tau')_1 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\partial\rho}{\partial t}(\overline{r}', t-\frac{|\overline{r}-\overline{r}'|}{c})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$= \frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0) + \frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0) \centerdot (\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$

$+\frac{1}{4\pi\epsilon_0}\int_V[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0) + \frac{\partial\rho}{\partial t}(\overline{y},0)$

$+D(\frac{\partial\rho}{\partial t})(\overline{y},0) \centerdot (\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'\ (QQ)$

with the convention that $\delta B(\overline{x}_0,r_0) = \emptyset$, when $r_0 \leq 0$, $(*)$, using the fact that, for fixed $t \in \mathcal{R}_{<0}$, $t-\frac{|\overline{r}-\overline{r}'|}{c} < 0$, and for $t \in \mathcal{R}_{\geq 0}$, $t-\frac{|\overline{r}-\overline{r}'|}{c} = 0$ iff $\overline{r}' \in \delta B(\overline{r},ct)$, with $d\tau'(\delta B(\overline{r},ct)) = 0$. Without loss of generality, we have that $\{(\frac{\partial^2\rho}{\partial^2 t})_0, (\frac{\partial\rho}{\partial t})_0, \rho_0, (D\rho)_0(D\frac{\partial\rho}{\partial t})_0\}$ are supported on $B(\overline{0},\overline{w})$, for some $w \in \mathcal{R}_{>0}$, and, using continuity, we let;

$M = max_{\overline{y}\in B(0,\overline{w})}(|(\frac{\partial^2\rho}{\partial t^2})_0|, |(\frac{\partial\rho}{\partial t})|_0|, |D(\frac{\partial\rho}{\partial t})|_0|)$

We can change the time coordinate, as we can define a new pair $(\rho^t, \overline{J}^t)$ by $\rho^t(\overline{x},s) = \rho(\overline{x},s+t)$ and $\overline{J}^t(\overline{x},s) = \overline{J}(\overline{x},s+t)$, for $(\overline{x},s) \in \mathcal{R}^4$. The new pair $(\rho^t,\overline{J}^t)$ inherits the properties of $(\rho,\overline{J})$, in particular we have that $\rho^t \in C^\infty(\mathcal{R}^4)$, the components of $\overline{J}^t$, $j_i^t \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, $\Box^2(\rho^t) = 0$, $\Box^2 j_i^t = 0$, for $1 \leq i \leq 3$, the continuity equation $\frac{\partial\rho^t}{\partial t} = -\bigtriangledown \centerdot \overline{J}^t$ holds, and the connecting relation $\bigtriangledown(\rho^t) + \frac{1}{c^2}\frac{\partial\overline{J}^t}{\partial t} = \overline{0}$. Moreover, we can use Kirchoff's formula with the initial data for $(\rho^t, \overline{J}^t)$ given by $(\rho_0^t, (\frac{\partial\rho^t}{\partial t})_0, \overline{J}_0^t, (\frac{\partial\overline{J}^t}{\partial t})_0) = (\rho_t, (\frac{\partial\rho}{\partial t})_t, \overline{J}_t, (\frac{\partial\overline{J}}{\partial t})_t)$ and we have that;

$$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho}(\overline{r}', t_r)\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau')_1 = (\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho^t}(\overline{r}', t_r')\hat{\mathfrak{r}}}{c|\overline{r}-\overline{r}'|}d\tau')_1$$

for the corresponding retarded time $t_r' = -\frac{|\overline{r}-\overline{r}'|}{c}$, and, similarly, for the corresponding terms in Jefimenko's equations.

We can assume in this calculation, that $\overline{r}$ is disjoint from the a ball $B(\overline{0}, s)$ containing the support of $\{(\frac{\partial^2\rho}{\partial t^2})_0, (\frac{\partial\rho}{\partial t})|_0, D(\frac{\partial\rho}{\partial t})|_0\}$. This is because, if t is fixed, then we have for a sufficiently large $t' > t$, that $\overline{r}$ is disjoint from a ball $B(\overline{x}_0, s)$ containing the support of $\{(\frac{\partial^2\rho}{\partial t^2})_{t'}, (\frac{\partial\rho}{\partial t})|_{t'}, D(\frac{\partial\rho}{\partial t})|_{t'}\}$. Then, using the uniqueness property, we have that $\rho(\overline{x,t})$ is determined by the shifted initial conditions $\{(\frac{\partial^2\rho}{\partial t^2})_{t'}, (\frac{\partial\rho}{\partial t})|_{t'}, D(\frac{\partial\rho}{\partial t})|_{t'}\}$. By a change of coordinates, $\overline{x}' = \overline{x} + \overline{x}_0$, and considering $\rho^{\overline{x}_0}$, we can assume that $\overline{x}_0 = \overline{0}$, $\overline{r}$ is disjoint from $B(\overline{0}, s)$, with the support of $\{(\frac{\partial^2\rho}{\partial t^2})_{t'}, (\frac{\partial\rho}{\partial t})|_{t'}, D(\frac{\partial\rho}{\partial t})|_{t'}\}$ contained in $B(\overline{0}, s)$. By a further change of coordinates, $t'' = t + t'$, and considering $\rho^{t'}$, we can assume that $t' = 0$, with the original t moving to $t - t'$, so that we can assume $t < 0$, but we can't assume that $t = 0$.

It follows, as $t < 0$, that in (QQ), we can ignore the term;

$$(\frac{1}{4\pi\epsilon_0}\int_V \frac{\dot{\rho}(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r}-\overline{r}'|}d\tau')_1 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\partial\rho}{\partial t}(\overline{r}', t - \frac{|\overline{r}-\overline{r}'|}{c})\frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V [\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}', c(t-\frac{|\overline{r}-\overline{r}'|}{c}))}(t - \frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y}, 0) + \frac{\partial\rho}{\partial t}(\overline{y}, 0)$$

$$+D(\frac{\partial\rho}{\partial t})(\overline{y}, 0) \centerdot (\overline{y} - \overline{r}')]dS(\overline{y})\frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau'$$

and, we are left, simplifying the radius, from (QQ) with;

$$+\frac{1}{4\pi\epsilon_0}\int_V [\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}', -ct+|\overline{r}-\overline{r}'|)}(t - \frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y}, 0) + \frac{\partial\rho}{\partial t}(\overline{y}, 0)$$

$$+D(\frac{\partial\rho}{\partial t})(\overline{y}, 0) \centerdot (\overline{y} - \overline{r}')]dS(\overline{y})\frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}d\tau' \ (QQQ)$$

If $\overline{d} \in B(\overline{0}, s)$, we let;

$V_{\overline{d},t}=\{\overline{r}'\in\mathcal{R}^3:\overline{d}\in\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\}$

$=\{\overline{r}'\in\mathcal{R}^3:|\overline{d}-\overline{r}'|=-ct+|\overline{r}-\overline{r}'|\}$

so that, in (QQQ), we have that $V=\bigcup_{\overline{d}\in B(\overline{0},s)}V_{\overline{d},t}$

As $B(\overline{0},s)$ is open, we can choose $\delta_{\overline{d}}>0$ such that $B(\overline{d},\delta_{\overline{d}})\subset B(\overline{0},s)$. By the calculation above, we can assume that the real unbounded hypersurface $V_{\overline{d},t}$ is a real quadratic surface and, by the calculation below, that the asymptotic cone $Z_{\overline{d},t}$ is a union of lines parametrised over a finite interval. For a line l appearing in the asymptotic cone, fixing $0<\epsilon<\delta_{\overline{d}}$, and $r(\epsilon)$ sufficiently large, we can assume that for $\overline{r}'\in l\cap(\mathcal{R}^3\setminus B(\overline{0},r(\epsilon)))$, there exists $\overline{r}''\in V_{\overline{d},t}$ with $\overline{r}'-\overline{r}''|<\epsilon$, see footnote 2, so that;

$\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)=(\delta B(\overline{r}'',-ct+|\overline{r}-\overline{r}'|)+(\overline{r}'-\overline{r''}))\cap B(\overline{0},s)$

and, as $\overline{d}'=\overline{d}+(\overline{r}'-\overline{r''})\in B(\overline{d},\delta_{\overline{d}})\subset B(\overline{0},s)$, that $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)\neq\emptyset$ and passes through $\overline{d}'\in B(\overline{0},s)$ with $|\overline{d}'-\overline{d}|<\epsilon$. Let $P_{\overline{d}}$ be the plane passing through $\overline{d}$, with $P_{\overline{d}}$ perpendicular to l and intersecting l at $\overline{p}_{\overline{d}}$. Let $T_{\overline{d}'}$ be the tangent plane to $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$ at $\overline{d}'$, intersecting l at $\overline{p}_{\overline{d}'}$, so that we can assume, for sufficiently large $r(\epsilon)$, that $|\overline{p}_{\overline{d}}-\overline{p}_{\overline{d}'}|<\epsilon$. Let $\overline{r}'_{opp}=\overline{p}_{\overline{d}}-(\overline{r}'-\overline{p}_{\overline{d}}=2\overline{p}_{\overline{d}}-\overline{r}'$. Then, for sufficiently large $r(\epsilon)$, we have that $\overline{r}'_{opp}\in l\cap(\mathcal{R}^3\setminus B(\overline{0},r(\epsilon)))$, $\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)\cap B(\overline{0},s)\neq\emptyset$ and passes through $\overline{d}'_{opp}\in B(\overline{0},s)$ with $|\overline{d}'_{opp}-\overline{d}'|<\epsilon$. We have that;

(i). Using the facts that $|\frac{\partial\rho}{\partial t}|_0|\leq M$ on $B(\overline{0},s)$, the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ is at most $2\pi s^2$, $\overline{r}'_{opp}=2\overline{p}_{\overline{d}}-\overline{r}'$, we have, for sufficiently large $r(\epsilon)$, that;

$$|\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\tfrac{|\overline{r}-\overline{r}'|}{c})(\tfrac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\tfrac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(t-\tfrac{|\overline{r}-\overline{r}'_{opp}|}{c})(\tfrac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\tfrac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$$

$$=|\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})}\tfrac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\tfrac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})$$

$$+\tfrac{1}{4\pi\epsilon_0}[\tfrac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\tfrac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\tfrac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})|$$

$$= |[\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})}\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}+\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})$$

$$+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}](\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})$$

$$-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$= |\frac{1}{16\pi^2\epsilon_0c^3}[\frac{(r_1-r_1')((t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}+\frac{(r_1-r_1')+(r_1-r'_{1,opp})}{(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y}) + [\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$= |\frac{1}{16\pi^2\epsilon_0c^3}[\frac{(r_1-r_1')((t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}+\frac{2r_1-2p_{\overline{d},1}}{(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}]$$

$$\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y}) + [\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})}\frac{(r_1+r_1'-2p_{\overline{d},1})}{c|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$\leq \frac{Ms^2}{8\pi\epsilon_0c^3}|\frac{(r_1-r_1')((t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}|+\frac{Ms^2}{8\pi\epsilon_0c^3}|\frac{2r_1-2p_{\overline{d},1}}{(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}|$$

$$+|\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})}\frac{(r_1+r_1'-2p_{\overline{d},1})}{c|\overline{r}+\overline{r}'-2p_{\overline{d}}|^2}|$$

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

$$\leq \frac{Ms^2}{\pi\epsilon_0c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0c^3}\frac{1}{|(t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})||\overline{r}+\overline{r}'-2p_{\overline{d}}|}$$

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

(P)

(following the method in (ii), noting the $O(|\overline{r}'|^3)$ term cancels in the first long term to obtain $\frac{O(|\overline{r}'|)O(|\overline{r}'|^2)}{O(|\overline{r}'|^6)}=\frac{1}{O(|\overline{r}'|^3)}$)

Change coordinates, so that the azimuth angle θ of the sphere $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, is centred on the line passing through $\{\overline{r}',\overline{d}'\}$, giving coordinates;

$$\overline{r}'+sin(\theta)cos(\phi)\overline{x}+sin(\theta)sin(\phi)\overline{y}+cos(\theta)(\overline{d}'-\overline{r}')$$

$(0 \leq \theta \leq \pi, -\pi \leq \phi \leq \pi)$

for a choice of orthogonal vectors $\{\overline{x}, \overline{y}, \overline{d}' - \overline{r}'\}$ with modulus $-ct + |\overline{r} - \overline{r}'|$. Similarly, choose the azimuth angle θ_{opp} of the sphere $\delta B(\overline{r}'_{opp}, -ct + |\overline{r} - \overline{r}'_{opp}|)$, is centred on the line passing through $\{\overline{r}'_{opp}, \overline{d}'_{opp}\}$, giving coordinates;

$\overline{r}' + sin(\theta_{opp})cos(\phi_{opp})\overline{x}_{opp} + sin(\theta_{opp})sin(\phi_{opp})\overline{y}_{opp} + cos(\theta_{opp})(\overline{d}'_{opp} - \overline{r}'_{opp})$

$(0 \leq \theta_{opp} \leq \pi, -\pi \leq \phi_{opp} \leq \pi)$

for a choice of orthogonal vectors $\{\overline{x}_{opp}, \overline{y}_{opp}, \overline{d}'_{opp} - \overline{r}'_{opp}\}$ with modulus $-ct + |\overline{r} - \overline{r}'_{opp}|$. We have, for points $\{\overline{q}', \overline{q}'_{opp}\}$ of intersection between $B(\overline{0}, s)$ and $\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|)$, $B(\overline{0}, s)$ and $\delta B(\overline{r}'_{opp}, -ct + |\overline{r} - \overline{r}'_{opp}|)$ that;

$\theta(\overline{q}') \simeq sin(\theta(\overline{q}')) \leq \frac{2s}{-ct + |\overline{r} - \overline{r}'|}$

$\theta_{opp}(\overline{q}'_{opp}) \simeq sin(\theta_{opp}(\overline{q}'_{opp})) \leq \frac{2s}{-ct + |\overline{r} - \overline{r}'_{o}pp|}$ (TT)

and, for sufficiently large $r(\epsilon)$, choosing $\{\overline{x}, \overline{y}, \overline{x}_{opp}, \overline{y}_{opp}\}$ compatibly, we may assume that;

$|\overline{q}' - \overline{q}'_{opp}| \leq 2\epsilon$

for $\{\overline{q}', \overline{q}'_{opp}\}$ defined by coordinates $\theta = \theta_{opp}$, $\phi = \phi_{opp}$ with $0 \leq \theta \leq max(\theta_{max}, \theta_{max,opp})$, where;

$\theta_{max} = max_{0 \leq \phi \leq 2\pi} \theta(\overline{q}')$

for $\overline{q}'$ in $B(\overline{0}, s) \cap \delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|)$, with coordinates $\{\theta, \phi\}$, and;

$\theta_{max,opp} = max_{0 \leq \phi \leq 2\pi} \theta_{opp}(\overline{q}'_{opp})$

for $\overline{q}'_{opp}$ in $B(\overline{0}, s) \cap \delta B(\overline{r}'_{opp}, -ct + |\overline{r} - \overline{r}'_{opp}|)$, with coordinates $\{\theta_{opp}, \phi_{opp}\}$

It follows that, for sufficiently large $r(\epsilon)$, using the surface measure $dS = r^2 sin(\theta)$, the fact (TT) and $r^2(1 - cos(\frac{1}{r})) = O(1)$, and footnote 2, for sufficiently large r;

$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$

$\leq 2\epsilon|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(-ct+|\overline{r}-\overline{r}'_{opp}|)^2\int_0^{max(\theta_{max},\theta_{max,opp})}sin(\theta)d\theta$

$= 2\epsilon|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(-ct+|\overline{r}-\overline{r}'_{opp}|)^2(1-cos(max(\theta_{max},\theta_{max,opp})))$

$\leq C\epsilon$

$\leq \frac{D}{|\overline{r}'+1|}$

where $\{C,D\}\subset\mathcal{R}_{>0}$.

It follows from (P), for sufficiently large $r(\epsilon)$, following the method of (ii), that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{Ms^2}{\pi\epsilon_0 c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0 c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0 c^3}\frac{D}{|\overline{r}'+1|}\frac{1}{|(t-\frac{|\overline{r}+\overline{r}'-2\overline{p_{\overline{d}}}|}{c})||\overline{r}+\overline{r}'-2\overline{p_{\overline{d}}}|}$

$\leq \frac{E_1}{|\overline{r}'|^3}$

where $E_1\in\mathcal{R}_{>0}$.

(ii). Using the facts that $|\frac{\partial\rho}{\partial t}|_0|\leq M$ on $B(\overline{0},s)$, the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ is at most $2\pi s^2$, $\overline{r}'_{opp}=2\overline{p_{\overline{d}}}-\overline{r}'$, we have, for sufficiently large $r(\epsilon)$, that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{1}{4\pi\epsilon_0 c}\frac{2\pi Ms^2}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}+\frac{1}{4\pi\epsilon_0 c}\frac{2\pi Ms^2}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}$

$=\frac{Ms^2}{8\pi c\epsilon_0(ct-|\overline{r}-\overline{r}'|)^2|\overline{r}-\overline{r}'|}+\frac{Ms^2}{8\pi c\epsilon_0(ct-|\overline{r}_1+\overline{r}'|)^2|\overline{r}_1+\overline{r}'|}$

$=\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}-\overline{r}'|^3|\frac{ct}{|\overline{r}-\overline{r}'|}+1|^2}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3|(\frac{ct}{|\overline{r}_1+\overline{r}'|}-1)|^2}$

$\leq \frac{Ms^2}{4\pi c\epsilon_0|\overline{r}-\overline{r}'|^3}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3}$

$\leq \frac{3Ms^2}{8\pi c\epsilon_0|\overline{r}'|^3}$

$= \frac{E_2}{|\overline{r}'|^3}$

where $\overline{r}_1 = \overline{r} - 2\overline{p_{\overline{d}}}$, $E_2 \in \mathcal{R}_{>0}$.

(iii). We have that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0) \centerdot (\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$= |\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}(-ct+|\overline{r}-\overline{r}'|)\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y}))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}(-ct+|\overline{r}-\overline{r}'_{opp}|)\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot$
$(\overline{z}_{opp}(\overline{y})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}')}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}_{opp}(\overline{y})dS(\overline{y})|$
(NN)

Letting $\overline{z}_0 = \frac{(\overline{d}'-\overline{r}')}{-ct+|\overline{r}-\overline{r}'|}$, so that $|\overline{z}_0| = 1$, R the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$, using Lemma 0.2, following the method of (i), we have that, for sufficiently large $r(\epsilon)$;

$|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|$

$= |\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})+\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot$

$\overline{z_0}dS(\overline{y})|$

$\leq |\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})|+|[\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)]\centerdot$

$\overline{z_0}dS(\overline{y})|$

$\leq Rmax_{\overline{y}\in B(\overline{0},s)}|D(\frac{\partial\rho}{\partial t})(\overline{y},0)||\overline{z}(\overline{y})-\overline{z_0}|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)} D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})\centerdot$
$\overline{z_0}|$

$\leq RMmax_{\overline{y}\in B(\overline{0},s)}|\overline{z}(\overline{y})-\overline{z_0}|+|\overline{z_0}||\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RM|(1-cos(\theta_{max},sin(\theta_{max})|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})$

$-\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|+|\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$=\sqrt{2}RM(1-cos(\theta_{max}))^{\frac{1}{2}}+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})-\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RMF\theta_{max}+G\epsilon$

$\leq\frac{2sH}{-ct+|\overline{r}-\overline{r}'|}+\frac{W}{|1+\overline{r}'|}$

$=\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|}$

where $\{F,G,W,H,A_1,B_1\}\subset\mathcal{R}_{>0}$. Similarly, there exist $\{A_2,B_2\}\subset\mathcal{R}_{>0}$, such that

$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|\leq\frac{A_2}{-ct+|\overline{r}-\overline{r}'_{op}|}+\frac{B_2}{|1+\overline{r}'_{op}|}$

$=\frac{A_2}{-ct+|\overline{r}+\overline{r}'-2\overline{p}_{\overline{d}}|}+\frac{B_2}{|1+2\overline{p}_{\overline{d}}-\overline{r}'|}$

so that, from (NN), following the method of (ii)

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}')}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{-ct+|\overline{r}+\overline{r}'-2\overline{p}_{\overline{d}}|}+\frac{B_2}{|1+2\overline{p}_{\overline{d}}-\overline{r}'|})$

$=\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|(t-\frac{|\overline{r}-\overline{r}'|}{c})||\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$

$+\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c}||\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{-ct+|\overline{r}+\overline{r}'-2\overline{p}_{\overline{d}}|}+\frac{B_2}{|1+2\overline{p}_{\overline{d}}-\overline{r}'|})$

$\leq\frac{E_3}{|\overline{r}'|^3}$

where $E_3\in\mathcal{R}_{>0}$ $((i),(ii),(iii))$

By the calculation below, we can assume that the asymptotic cone $Z_{\overline{d},t}$ of the real unbounded hypersurface $V_{\overline{d},t}$ is a union of lines parametrised over a finite interval $[\alpha,\beta]$. It follows that we can define maps $\theta_1 : \mathcal{R}\times[\alpha,\beta)\rightarrow Z_{\overline{d},t}$, $\theta_2 : \mathcal{R}\times[\alpha,\beta)\rightarrow Z_{\overline{d},t}$, such that for fixed $\gamma\in[\alpha,\beta)$, $\theta_1(r,\gamma)\in l_{\gamma,\overline{d},1}$, $\theta_2(r,\gamma)\in l_{\gamma,\overline{d},2}$, $r\in\mathcal{R}$, where the intersection curve $C_{\gamma,\overline{d}}$ has the two real asymptotes $\{l_{\gamma,\overline{d},1}, l_{\gamma,\overline{d},2}\}$, and, such that, for $i\in\{1,2\}$;

(i). $\theta_i(0,\gamma)=p_{\overline{d},\gamma,i}$, (using the notation above)

(ii). $\theta_i(r,\gamma)_{opp}=\theta_1(-r,\gamma)$

(iii). There exist $R_i\subset\mathcal{R}_{>0}$ with θ_i diffeomorphisms outside $[-R_i,R_i]\times[\alpha,\beta)$, with the partial derivatives uniformly bounded.

(iv). $Im(\theta_1|_{\mathcal{R}\setminus[-R_1,R_1]\times[\alpha,\beta)}\cap Im(\theta_2|_{\mathcal{R}\setminus[-R_2,R_2]\times[\alpha,\beta)})=\emptyset$

(v). For $r_2>r_1>R_i$, $|\theta_i(r_2,\gamma)-\theta_i(r_1,\gamma)|=r_2-r_1$

It follows from $(iii),(v)$ that the pullback;

$$\theta_1|^*_{\mathcal{R}\setminus[-R_1,R_1]\times[\alpha,\beta)}(dLeb|_{Z_{\overline{d},t}})=|\frac{\partial\theta_1}{\partial r}\times\frac{\partial\theta_1}{\partial\gamma}|drd\gamma=f(r,\gamma)drd\gamma$$

has the property that $f(r,\gamma)$ has order $O(r)$, uniformly in γ and $f(r,\gamma)=f(-r,\gamma)$, for $r\in\mathcal{R}_{>0}$. For $R\in\mathcal{R}_{>0}$, with $R>R_i$, can define the regions $S_{R,i}\subset\mathcal{R}\times[\alpha,\beta)$, by;

$$S_{R,i}=\{(r',\gamma):R_i\leq|r'|\leq R,\gamma\in[\alpha,\beta)\}$$

with corresponding regions $\theta_i(S_{R,i})\subset Z_{\overline{d},t}$

Then, by the calculation above, letting;

$$H(\overline{r}')=\frac{1}{4\pi\epsilon_0}\Big[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')\Big]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

we have that, for $r>R_i$;

$$|\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma)|\leq\frac{C}{r^3}$$

$$|f(r,\gamma)|\leq Dr$$

$$|(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)|\leq\frac{CD}{r^2}$$

and;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}H(\overline{r}')d\overline{r}' = lim_{R\rightarrow\infty,R>R_i}\int_{S_{R,i}}(\theta_1^*H)(r,\gamma)f(r,\gamma)drd\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}[\int_{R_i}^{R}\theta_1^*H(r,\gamma)f(r,\gamma)dr+\int_{-R}^{-R_i}\theta_1^*H(r,\gamma)f(r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}[\int_{R_i}^{R}\theta_1^*H(r,\gamma)f(r,\gamma)dr+\int_{-R}^{-R_i}\theta_1^*H(-r,\gamma)f(-r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}\int_{R_i}^{R}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma$

$= \int_{[\alpha,\beta)}\int_{R_i}^{\infty}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma$

where, letting $G(\gamma)=\int_{R_i}^{\infty}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)dr$;

$|G(\gamma)|\leq\int_{R_i}^{\infty}\frac{CD}{r^2}dr=[\frac{-CD}{r}]_{R_i}^{\infty}$

$=\frac{CD}{R_i}$

so that;

$lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}\int_{R_i}^{\infty}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma=\int_{[\alpha,\beta)}G(\gamma)d\gamma$

exists and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{[\alpha,\beta)}\int_{R_i}^{\infty}(\theta_1^*H(r,\gamma)+\theta_1^*H(-r,\gamma))f(r,\gamma)drd\gamma|\leq\frac{CD(\beta-\alpha)}{R_i}$

It follows;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}H(\overline{r}')d\overline{r}'$

exisrs, and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}H(\overline{r}')d\overline{r}'|\leq\frac{CD(\beta-\alpha)}{R_i}$

as well. (UU)

..... Let the lines appearing in the asymptotic cone $Z_{\overline{d},t}$, parametrised by $[\alpha,\beta)$, correspond to the system of hyperplanes H_γ, $\gamma\in[\alpha,\beta)$ with fixed locus $Q_{\overline{d},t}$. Then, for $\overline{x}\in V_{\overline{d},t}\setminus Q_{\overline{d},t}$, $|\overline{x}|$ sufficiently large, let $\overline{x}_{near}$ be the nearest point on the asymptotic line $l_{\overline{d},t,\gamma}$, and $\overline{x}_{opp}$ be the nearest point on $V_{\overline{d},t}\cap H_\alpha$ to $(\overline{x}_{near})_{opp}$. By a simple adaptation of the above argument $((i),(ii),(iii))$, we have that, for $|\overline{r}'|$ sufficiently large, there exists $C\in\mathcal{R}_{>0}$, with;

$|h(\overline{r}')+h(\overline{r}'_{opp})|\leq\frac{C}{|\overline{r}'|^3}$

where $h(\overline{r}')=(\frac{1}{4\pi\epsilon_0}\frac{\dot{\rho}(\overline{r}',t_r)\hat{\overline{\mathfrak{r}}}}{|\overline{r}-\overline{r}'|})_1$

(Follow argument of (UU), using the facts, that for sufficiently large $|\overline{r}'|$, $dV_{\overline{d},t}\simeq dZ_{\overline{d},t}$, $dV_{\overline{d},t}=g(\overline{r}')dZ_{\overline{d},t}$, with $g(\overline{r}')\simeq g(\overline{r}'_{opp})$, for the surface measures on $V_{\overline{d},t}$ and $Z_{\overline{d},t}$ respectfully, as;

$\frac{\partial\theta_1}{\partial r}(\overline{r}',\gamma)\simeq\frac{\partial\theta_1}{\partial r}(\overline{r}'_{opp},\gamma)$

for the appropriate parametrisation θ_1, so that;

$|h(\overline{r}')g(\overline{r}')dZ_{\overline{d},t}+h(\overline{r}'_{opp})g(\overline{r}'_{opp})dZ_{\overline{d},t}|$

$\leq|(h(\overline{r}')+h(\overline{r}'_{opp}))g(\overline{r}')dZ_{\overline{d},t}|+|h(\overline{r}'_{opp})(g(\overline{r}'_{opp})-g(\overline{r}'))dZ_{\overline{d},t}|$

$=O(\frac{1}{R^3})O(R)drd\gamma+O(\frac{1}{R^2})O(\frac{1}{R})drd\gamma=O(\frac{1}{R^2})drd\gamma.$

Same idea for asymptotic cones defined below in Lemma 0.4, reflecting the branch at infinity.)

.... If $t=0$, then $W_2=\{\overline{r}':-w\leq|\overline{r}'|-|\overline{r}-\overline{r}'|\leq w\}$, and, by the calculation in footnote 2, we can, for sufficiently large $\overline{r}'$, characterise W_2 as a family of quadratic surfaces, parametrised by $[0,w]$, degenerating to the plane $\overline{r}'=|\overline{r}-\overline{r}'|$. We denote by W_2^s, for $0\leq s\leq w$ the locus;

$\{\overline{r}':|\overline{r}'|-|\overline{r}-\overline{r}'|=s\}\cup\{\overline{r}':|\overline{r}'|-|\overline{r}-\overline{r}'|=-s\}$

characterised, for $s\neq 0$, by the quadratic real surface V_s in footnote 2, with W_0 being the plane $\{\overline{r}':|\overline{r}'|=|\overline{r}-\overline{r}'|\}$. Fixing $s_0\neq 0$, for a real generic hyperplane H_{s_0}, using footnote 2, the intersection $V_{s_0}\cap H_{s_0}$ is a real unbounded generic quadratic curve $C_{s_0}\subset H_{s_0}$. In particularly, by the classification of real quadratic curves as conic sections, C_{s_0} is generic hyperbolic and has two real asymptotes $\{l_{s_0,1},l_{s_0,2}\}$. If we take a generic real 1-dimensional pencil of hyperplanes $\{H_{s_0,r}:r\in\mathcal{R}\}$, such that $\bigcup_{r\in\mathcal{R}}H_{s_0,r}=\mathcal{R}^3$, with base locus l_{s_0}, then clearly;

$\bigcup_{r\in\mathcal{R}}(V_{s_0}\cap H_{s_0,r})=V_{s_0}$

and, using O-minimality, there exists finitely many open bounded intervals $\{I_j:1\leq j\leq n\}$ for which $V_{s_0}\cap H_{s_0,r}$ is finite, $r\in\bigcup_{1\leq j\leq n}I_j$. Let $P_{s_0}=\mathcal{R}\setminus\bigcup_{1\leq j\leq n}I_j$, and we still have that;

$\bigcup_{r\in P_{s_0}}(V_{s_0}\cap H_{s_0,r})=V_{s_0}$

We define the two dimensional asymptotic cone Z_{s_0} of V_{s_0} to be $\bigcup_{r\in P_{s_0}}l_{s_0,r,1}\cup l_{s_0,r,2}$ where the intersection curve $C_{s_0,r}$ has the two real asymptotes $\{l_{s_0,r,1},l_{s_0,r,2}\}$. By choosing the base locus l_{s_0} to intersect V_{s_0} in a finite number of points and noting that for a sufficiently generic family, $\overline{V_{s_0}}\cap H_{s_0,r}\cap W=0$, in coordinates $[X,Y,Z,W]$, where $\overline{V_{s_0}}$ is the projective closure of V_{s_0} in $P(\mathcal{R}^3)$, is mobile, and compact, so can be paramertised analytically by a finite interval. we can assume that P_{s_0} is a finite interval I_{s_0} when parametrising Z_{s_0} and V_{s_0}, so that;

$\bigcup_{r\in I_{s_0}}(V_{s_0}\cap H_{s_0,r})=V_{s_0}$

Let $d\tau'_{s_0}$ be the surface measure on Z_{s_0} obtained from the pullback of Lebesgue measure with the inclusion of Z_{s_0} in $\mathcal{R}^3$ and, similarly, let $d\tau'_{s_0,r,1}$ and $d\tau'_{s_0,r,2}$ be the line measures on $l_{s_0,r,1}$ and $l_{s_0,r,2}$, obtained from the pullback of Lebesgue measure, and let $d\tau'_{s_0,r,1,2}$ be the union of the measures on $l_{s_0,r,1}\cup l_{s_0,r,2}$. .... If $t_1<t_2$, with $\{t_1,t_2\}\subset\mathcal{R}$, and $\{V_{t_1},V_{t_2}\}$ denote the compact supports of $\{\rho_{t_1},\rho_{t_2}\}$, then as the supports vary continuously, and $\overline{J}_t$ and ρ_t are compactly supported for each $t\in[t_1,t_2]$, $\overline{J}_t$ and ρ_t are uniformly compacted supported for $t\in[t_1,t_2]$ in a ball $B(\overline{0},p)$, for some $p\in\mathcal{R}_{>0}$. In particularly;

$\int_{V_{t_1}}\rho_{t_1}dV=\int_{B(\overline{0},p)}\rho_{t_1}dV$

$\int_{V_{t_2}}\rho_{t_2}dV=\int_{B(\overline{0},p)}\rho_{t_2}dV$

For $t\in[t_1,t_2]$, using the continuity equation, the divergence theorem and the fact $\overline{J}_t$ is uniformly compacted supported for $t\in[t_1,t_2]$ in $B(\overline{0},p)$, we have that;

$\frac{d}{dt}(\int_{B(\overline{0},p)}\rho_t dV)=\int_{B(\overline{0},p)}\frac{\partial\rho}{\partial t}dV$

$=\int_{B(\overline{0},p)}div(\overline{J})_t dV$

$= \int_{\delta B(\bar{0},p)} \overline{J}_t \centerdot d\overline{S} dV$

$= 0$

so that;

$\int_{B(\bar{0},p)} \rho_{t_1} dV = \int_{B(\bar{0},p)} \rho_{t_2} dV$

$\int_{V_{t_1}} \rho_{t_1} dV = \int_{V_{t_2}} \rho_{t_2} dV$

In particularly, $\frac{d}{dt}(\int_{V_t}\rho_t dV) = 0$, ([3]). The same argument applies for $\frac{\partial\rho}{\partial t}$, with associated current $\overline{J}_1 = -c^2\bigtriangledown(\rho)$ and compact supports W_t, $t \in \mathcal{R}$, obeying the wave equation $\square^2(\overline{J}_1) = \overline{0}$. It follows from the Reynold's transport theorem, ([4]), the divergence theorem and the fact that $\overline{J}_1$ vanishes outside W_t and V_t, that;

3

In fact, the result is true for $(\rho, \overline{J})$ satisfying the continuity equation, when $\overline{J}$ fails to have compact support, and the components j_i, for $1 \leq i \leq 3$, are uniformly of rapid decay, in the sense, that for any finite interval $[t_1, t_2]$, there exists constants $C_{1,2,i,n} \in \mathcal{R}_{>0}$ such that $|j_i(\overline{x}, t)| \leq \frac{C_{1,i,2,n}}{|\overline{x}|^n}$ for $t \in [t_1, t_2]$ and $|\overline{x}| > 1$. In order to see this, suppose that on a finite interval (t_1, t_2), ρ is supported uniformly on $B(\overline{0}, p)$. and $\frac{d}{dt}\int_{V_t}\rho dV \neq 0$, for some $t \in [t_1, t_2]$. Then there exists an interval $(t_0 - \epsilon, t_0 + \epsilon) \subset (t_1, t_2)$, such that, without loss of generality, $\frac{d}{dt}\int_{V_t}\rho dV|_{(t_0-\epsilon,t_0+\epsilon)} > 0$, and, by the intermediate value theorem, we can assume that $\int_{V_t}\rho dV|_{(t_0-\epsilon,t_0+\epsilon)}$ is strictly increasing, with $\int_{V_{t_0+\epsilon}}\rho_{t_0+\epsilon}dV - \int_{V_{t_0}}\rho_{t_0}dV > \delta > 0$, $(*)$. Using the hypotheses on $\overline{J}$, we can choose $r > p$ sufficiently large such that for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $|\int_{\delta B(\overline{0},r)}\overline{J}_t \centerdot d\overline{S}| < \delta_1$, and by the continuity equation, for $t \in (t_0 - \epsilon, t_0 + \epsilon)$;

$$|\frac{d}{dt}\int_{B(\overline{0},r)} pdV| = |\int_{B(\overline{0},r)}\frac{\partial\rho}{\partial t}dV|$$

$$= |-\int_{B(\overline{0},r)} div(\overline{J})dV|$$

$$= |\int_{\delta B(\overline{0},r)}\overline{J}\centerdot d\overline{S}|$$

$$< \delta_1$$

and the intermediate value theorem;

$$|\int_{B(\overline{0},r)} p_{t_0+\epsilon}dV - \int_{B(\overline{0},r)} p_{t_0}dV| < \delta_1\epsilon$$

so choosing $\delta_1 = \frac{\delta}{2\epsilon}$, we obtain that;

$$|\int_{B(\overline{0},r)} p_{t_0+\epsilon}dV - \int_{B(\overline{0},r)} p_{t_0}dV| = |\int_{V_{t_0+\epsilon}} p_{t_0+\epsilon}dV - \int_{V_{t_0}} p_{t_0}dV|$$

$$< \frac{\delta}{2}$$

which contradicts $(*)$.

[4] The Reynolds transport theorem is true in this case, but is not the usual form, as, due to the failure of analyticity, there can be jumps in the support. There is also an issue with using the formula $\rho\overline{v} = \overline{J}$, when substituting for the velocity of the area element. This could be resolved in [**?**].

$\int_{V_t} \bigtriangledown^2(\rho) dV = \frac{1}{c^2}\int_{V_t} \frac{\partial^2 \rho}{\partial t^2} dV$

$= \frac{1}{c^2}(\frac{d}{dt}(\int_{V_t} \frac{\partial \rho}{\partial t} dV) - \int_{V_t} div(\overline{J}_1))$

$= -\frac{1}{c^2}\int_{V_t} div(\overline{J}_1) dV$

$= -\frac{1}{c^2}\int_{\delta V_t} \overline{J}_1 \cdot d\overline{S}$

$= 0$

In particular, at $t = 0$, we can assume that;

$\int_{V_0} \bigtriangledown^2(\rho_0) dV = \int_{V_0}(\sum_{i=1}^{3}(\frac{\partial^2 \rho}{\partial x_i^2})_0) dV = 0$ (O), ([5]).

We can define antiderivatives, by letting;

$p^a(\overline{x}, t) = \int_{-\infty}^{t} p(\overline{x}, s) ds$

$\overline{J}^a(\overline{x}, t) = \int_{-\infty}^{t} \overline{J}(\overline{x}, s) ds$ (if the integral exists)

As is easily checked, if $p \in C^\infty(\mathcal{R}^4)$ and the components $j_i \in C^\infty(\mathcal{R}^4)$, $1 \leq i \leq 3$, then $\rho^a \in C^\infty(\mathcal{R}^4)$ and the components $j_i^a \in C^\infty(\mathcal{R}^4)$, for $1 \leq i \leq 3$. The wave equation holds for ρ^a and $\overline{J}^a$, as, using the fundamental theorem of calculus, differentiating under the integral sign, the result about he left hand limit in [**?**], and using the fact that ρ satisfies the wave equation;

$\Box^2(\rho^a) = \int_{-\infty}^{t} \bigtriangledown^2(\rho) ds - \frac{1}{c^2}\frac{\partial \rho}{\partial t}$

$= \int_{-\infty}^{t} \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} ds - \frac{1}{c^2}\frac{\partial \rho}{\partial t}$

---

[5] Note that you can also deduce this, using the divergence theorem, and the fact that $\bigtriangledown(\rho_0)$ vanishes on δV_0;

$\int_{V_0} \bigtriangledown^2(\rho_0) dV = \int_{\delta V_0} \bigtriangledown \cdot (\bigtriangledown(\rho_0)) dV$

$= \int_{\delta V_0} \bigtriangledown(\rho_0) \cdot d\overline{S}$

$= 0$

$= \frac{1}{c^2}\frac{\partial\rho}{\partial t} - \frac{1}{c^2}\frac{\partial\rho}{\partial t}$

$= 0$

and;

$\Box^2(\overline{J}^a) = \int_{-\infty}^t \bigtriangledown^2(\overline{J})ds - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \int_{-\infty}^t \frac{1}{c^2}\frac{\partial^2\overline{J}}{\partial t^2}ds - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t} - \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}$

$= \overline{0}$

Differentiating under the integral sign and using the fundamental theorem of calculus, the fact that the continuity equation holds for $(\rho, \overline{J})$, the continuity equation holds as;

$\frac{\partial\rho^a}{\partial t} + \bigtriangledown \cdot \overline{J}^a$

$= \rho + \int_{-\infty}^t \bigtriangledown \cdot \overline{J}ds$

$= \rho + \int_{-\infty}^t + \int_{-\infty}^t -\frac{\partial\rho}{\partial s}ds$

$= \rho - \rho = 0$

and, differentiating under the integral sign, using the fundamental calculus of calculus and the connecting relation for $(\rho, \overline{J})$, the connecting relation holds;

$\bigtriangledown(\rho^a) + \frac{1}{c^2}\frac{\partial\overline{J}^a}{\partial t}$

$= \int_{-\infty}^t \bigtriangledown(\rho)ds + \frac{1}{c^2}\overline{J}$

$= \int_{-\infty}^t -\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}ds + \frac{1}{c^2}\overline{J}$

$= -\frac{1}{c^2}\overline{J} + \frac{1}{c^2}\overline{J}$

$= \overline{0}$,[6]

.... Then the fields $\{\overline{E}, \overline{B}\}$ are well defined by Jefimenko's equations and the components are of uniform very moderate decrease.

□

Lemma 0.2. *Cancellation Lemma*

Let $g \in C^\infty(\mathcal{R}^3)$ with compact support $V \subset \mathcal{R}^3$, then for a hyperplane $H \subset \mathcal{R}^3$, we have that;

$\int_{V \cap H} \bigtriangledown(g) d\mu = \overline{0}$

where μ is Lebesgue measure on $V \cap H$.

Proof. With out loss of generality, we can assume that $V = B(\overline{0}, r)$, for some $r \in \mathcal{R}_{>0}$ and H is a hyperplane passing through $\overline{0}$, with the equation $\alpha x + \beta y + \gamma z = 0$. Assume first that $\{\alpha, \beta, \gamma\} \subset \mathcal{R}$ are distinct and non zero. Let $pr_{12}, pr_{13}, pr_{23}$ be the projections onto the coordinates $(x, y), (x, z), (y, z)$. Let;

[6] We don't necessarily have that $(\rho^a, \overline{J}^a)$ has compact supports. On a finite interval $[t_1, t_2]$, for sufficiently large $\overline{x}$, we have $\frac{\partial \rho^a}{\partial t} = \rho = 0$, and;

$\bigtriangledown^2(\rho_a) = \frac{1}{c^2} \frac{\partial^2 \rho^a}{\partial^2 t}$

$= 0$

Let $h(\overline{x})$ define ρ^a for sufficiently large $\overline{x}$, then, as $\mathcal{R}^3 = \bigcup_{t \in \mathcal{R}} Supp(\rho_t)^c$;

$\bigtriangledown^2(h(\overline{x})) = \Box^2(h(\overline{x})) = 0$

everywhere. We can repeat the argument for the antiderivative $\overline{J}^a$ to obtain $\overline{c}(\overline{x})$ defining $\overline{J}^a$ for sufficiently large $\overline{x}$. so, as $\mathcal{R}^3 = \bigcup_{t \in \mathcal{R}} Supp(\overline{J}_t)^c$, we have that $\bigtriangledown^2(\overline{c}(\overline{x})) = \Box^2(\overline{c}(\overline{x})) = \overline{0}$, and, clearly, for the pair $(h(\overline{x}), \overline{c}(\overline{x}))$, we have that;

$div(\overline{c}(\overline{x})) = -\frac{\partial h}{\partial t} = 0$

$\bigtriangledown(h)(\overline{x}) = -c^2 \frac{\partial \overline{c}(\overline{x})}{\partial t}$

$= \overline{0}$

and $(\rho^a - h(\overline{x}), \overline{J}^a - \overline{c}(\overline{x}))$ has compact supports and inherits all the properties above for $(\rho^a, \overline{J}^a)$.

$g_{12}(x,y) = g(x,y,z(x,y)) = g(x,y,-\frac{\alpha x}{\gamma} - \frac{\beta y}{\gamma})$

$g_{13}(x,z) = g(x,y(x,z),z) = g(x,-\frac{\alpha x}{\beta} - \frac{\gamma z}{\beta},z)$

$g_{23}(y,z) = g(x(y,z),y,z) = g(-\frac{\beta y}{\alpha} - \frac{\gamma z}{\alpha},y,z)$

Then, by the chain rule;

$\frac{\partial g_{12}}{\partial x}|_{(x,y)} = (\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} - \frac{\alpha}{\gamma}\frac{\partial g}{\partial z})_{(x,y,z(x,y))}$

$\frac{\partial g_{12}}{\partial y}|_{(x,y)} = (\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} - \frac{\beta}{\gamma}\frac{\partial g}{\partial z})_{(x,y,z(x,y))}$

so that;

$\frac{\beta}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)} - \frac{\alpha}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)} = \frac{(\beta-\alpha)}{\gamma}(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y})|_{(x,y,z(x,y))}$

and;

$(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y})|_{(x,y,z(x,y))} = \frac{\gamma}{(\beta-\alpha)}(\frac{\beta}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)} - \frac{\alpha}{\gamma}\frac{\partial g_{12}}{\partial x}|_{(x,y)})$

and, a similar calculation holds for $\{g_{13}, g_{23}\}$. It follows that, using Fubini's theorem, the fundamental theorem of calculus and the fact that g_{12} vanishes on $\delta(pr_{12}(V \cap H))$;

$\int_{V\cap H}(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y})d\mu = \int_{pr_{12}(V\cap H)}(\frac{\beta}{\beta-\alpha}\frac{\partial g_{12}}{\partial x} - \frac{\alpha}{\beta-\alpha}\frac{\partial g_{12}}{\partial x})|_{(x,y)}c_{12}(\alpha,\beta,\gamma)dxdy$

$= 0$

where $c_{12}(\alpha,\beta,\gamma) \in \mathcal{R}$ is non-zero. Similarly, using $\{pr_{13}, pr_{23}\}$;

$\int_{V\cap H}(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z})d\mu = \int_{V\cap H}(\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z})d\mu$

so that;

$\int_{V\cap H}\frac{\partial g}{\partial x}d\mu = -\int_{V\cap H}\frac{\partial g}{\partial y}d\mu = \int_{V\cap H}\frac{\partial g}{\partial z} = -\int_{V\cap H}\frac{\partial g}{\partial x}d\mu$

and;

$\int_{V\cap H}\frac{\partial g}{\partial x}d\mu = 0$

Similarly;

$$\int_{V\cap H}\frac{\partial g}{\partial y}d\mu=\int_{V\cap H}\frac{\partial g}{\partial z}d\mu=0$$

and;

$$\int_{V\cap H}\bigtriangledown(g)d\mu=\overline{0}$$

By continuity, the result holds for any hyperplane H as the initial assumption was generic.

$\square$

Lemma 0.3. *Uniqueness of Representation of Arcs*

Suppose that $\overline{x}\in\mathcal{R}^3\setminus B(\overline{0},s)$ such that $\delta B(\overline{x},r)\cap B(\overline{0},s)\neq\emptyset$, then there exists a unique $0\leq w\leq s$ such that $B(\overline{0},w)$ intersects $B(\overline{x},r)$ at a single point $\overline{p}_{\overline{x},r}$, with the property that the spheres $\delta B(\overline{x},r)$ and $\delta B(\overline{0},w)$ share a tangent plane at $\overline{p}_{\overline{x},r}$.

Proof. Suppose that $\overline{0}\notin B(\overline{x},r)$. Let l be the line connecting the points $\{\overline{0},\overline{x}\}$, intersecting the sphere $\delta B(\overline{0},s)$ at $\overline{q}$. Then $\overline{q}\in B(\overline{x},r)$, otherwise $\delta B(\overline{x},r)\cap B(\overline{0},s)=\emptyset$. We have that $\delta B(\overline{x},r)\cap B(\overline{0},s)$ partitions $B(\overline{0},s)$ into 2 disjoint, connected regions, and the regions containing $\overline{0}$ and $\overline{q}$ are distinct. It follows that the line l between $\overline{0}$ and $\overline{q}$ intersects $B(\overline{x},r)$ at the point $\overline{p}_{\overline{x},r}\in B(\overline{0},s)$. Choose $0\leq w\leq s$ such that $\delta B(\overline{0},w)$ passes through $\overline{p}_{\overline{x},r}$. Then, as the tangent planes to the spheres $\delta B(\overline{0},w)$ and $\delta B(\overline{x},r)$ at $\overline{p}_{\overline{x},r}$ are both perpendicular to l and pass through $\overline{p}_{\overline{x},r}$, they must coincide. Suppose that the spheres $\delta B(\overline{0},w)$ and $\delta B(\overline{x},r)$ share a further intersection point $\overline{p}'$ with the properties that the tangent planes at $\overline{p}'$ coincide, then the lines l and l', where l' connects the points $\{\overline{0},\overline{p}'\}$, both pass through $\overline{0}$ and $\overline{x}$, so must coincide and $\overline{p}'\in l$. Then, as $\overline{p}_{\overline{x},r}$ and $\overline{p}'$ are distinct, it follows that $\overline{p}'\notin\delta B(\overline{x},r)$.

$\square$

Lemma 0.4. *Fix $0<w\leq s$ and with $\overline{r}\notin B(\overline{0},s)$, $t<0$, let $V_w(\overline{x})$ be the locus defined by;*

$B(\overline{0},w)$ intersects $B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ at a single point $\overline{p}_{\overline{x}}$, with the property that the spheres $\delta B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ and $\delta B(\overline{0},w)$ share a tangent plane at $\overline{p}_{\overline{x},r}$.

Then, $V_w(\overline{x}) \subset V_w^1(\overline{x}$*, where;*

$$V_w^1(\overline{x}) \equiv \exists\lambda\exists\overline{y}[(|\overline{y}| = w) \wedge ((|\overline{x}-\overline{y}| = -ct + |\overline{x}-\overline{r}|)$$

$$\vee(|\overline{x}-\overline{y}| = ct + |\overline{x}-\overline{r}|) \vee (|\overline{x}-\overline{y}| + |\overline{x}-\overline{r}| = -ct)) \wedge \overline{x} = \lambda\overline{y}]$$

and $V_w^1(\overline{x})$ *is generically a double cover of* $\delta B(\overline{0}, w)$*, and there exists parallel planes* $\{P_1, P_2\} \subset \mathcal{R}^3$*, such that, either;*

$V_w^1(\overline{x})$ *is bounded*

when $(P_1 \cap \delta B(\overline{0}, w)) = (P_2 \cap \delta B(\overline{0}, w)) = \emptyset$*, or;*

$V_w^1(\overline{x})$ *blows up at an exceptional locus* $Z_a \subset \delta B(\overline{0}, w)$

where $Z_a = (P_1 \cap \delta B(\overline{0}, w)) \cup (P_2 \cap \delta B(\overline{0}, w))$ *is the union of* 2 *circles on* $\delta B(\overline{0}, w)$*. For specific, non-generic* w*, these circles can coincide, but, in the generic case, when* Z_a *has two components,* $V_w(\overline{x})$ *basically has two asymptotic cones among* $Cone_1(\overline{0}, P_1 \cap \delta B(\overline{0}, w))$ *and* $Cone_2(\overline{0}, P_2 \cap \delta B(\overline{0}, w))$ *corresponding to distinct* $\{P_1, P_2\}$*, with a single pair of infinite opposite branches along asymptotes, which are bounded translations of the lines of the cones. The cover splits into a bounded and unbounded component centred along the asymptotes. In a special case of this generic behaviour, again corresponding to specific* w*,* $V_w^1(\overline{x})$ *can blow up along one component of* Z_a *and remain bounded over the other component. There is another special case, due to a specific link between* t *and* $\overline{r}$*, which can occur for non generic* w*, but it exhibits similar behaviour to the generic case.*

Proof. By the proof of Lemma 0.3, we have that;

$$V_w(\overline{x}) \equiv \exists\lambda_{\neq 0}\exists\overline{y}[(|\overline{y}| = w) \wedge (|\overline{x}-\overline{y}| = -ct + |\overline{x}-\overline{r}|) \wedge \overline{x} = \lambda\overline{y}]$$

Making the substitutions $\overline{x} = \lambda\overline{y}$ and $|\overline{y}| = w$, we have that;

$$|\overline{x}-\overline{y}| = -ct + |\overline{x}-\overline{r}| \iff |\lambda\overline{y}-\overline{y}| = -ct + |\lambda\overline{y}-\overline{r}|$$

$$\iff |\lambda - 1||\overline{y}| = -ct + |\lambda\overline{y}-\overline{r}|$$

$$\iff w|\lambda - 1| = -ct + |\lambda\overline{y}-\overline{r}|$$

$\implies w^2(\lambda-1)^2 = c^2t^2 + (\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2$

$-2ct|\lambda\overline{y} - \overline{r}|$

$\implies 4c^2t^2[(\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2]$

$= [w^2(\lambda-1)^2 - c^2t^2 - (\lambda y_1 - r_1)^2 - (\lambda y_2 - r_2)^2 - (\lambda y_3 - r_3)^2]^2$

$\iff 4c^2t^2[\lambda^2 w^2 - 2\lambda\overline{y}\centerdot\overline{r} + |\overline{r}|^2] = [-2\lambda w^2 + w^2 - c^2t^2 + 2\lambda\overline{y}\centerdot\overline{r} - |\overline{r}|^2]^2$

$\iff \lambda^2(4c^2t^2w^2 - (2\overline{y}\centerdot\overline{r} - 2w^2)^2) + \lambda(-8c^2t^2\overline{y}\centerdot\overline{r} - 2(2\overline{y}\centerdot\overline{r} - 2w^2)$

$(w^2 - c^2t^2 - |\overline{r}|^2)) + (4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2) = 0 \ (AA)$

If we reverse the two $\implies$ steps, we obtain the alternatives;

$w^2(\lambda-1)^2 = c^2t^2 + (\lambda y_1 - r_1)^2 + (\lambda y_2 - r_2)^2 + (\lambda y_3 - r_3)^2$

$+2ct|\lambda\overline{y} - \overline{r}|$

and $w|\lambda-1| = ct + |\lambda\overline{y} - \overline{r}|$ or $w|\lambda-1| = -ct - |\lambda\overline{y} - \overline{r}|$

which gives;

$|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|$ or $|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct$

so that the condition (AA) defines the admissible λ in the formula;

$V_w^1(\overline{x}) \equiv \exists\lambda_{\neq 0}\exists\overline{y}[(|\overline{y}| = w) \wedge ((|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}|)$

$\vee(|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|) \vee (|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct)) \wedge \overline{x} = \lambda\overline{y}]$

with $V_w(\overline{x}) \subset V_w^1(\overline{x})$. By the quadratic formula, we have that, if $a \neq 0$;

$\lambda = \frac{-b+/-\sqrt{b^2-4ac}}{2a} = \frac{\gamma_1}{a}$ or $\frac{\gamma_2}{a}$

where;

$a = 4c^2t^2w^2 - (2\overline{y}\centerdot\overline{r} - 2w^2)^2$

$b = -8c^2t^2\overline{y} \centerdot \overline{r} - 2(2\overline{y} \centerdot \overline{r} - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)$

$c = 4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2$ (QQ), ([7]).

Let;

$a(z) = 4c^2t^2w^2 - (2z - 2w^2)^2$

$b(z) = -8c^2t^2z - 2(2z - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)$

$c(z) = 4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2$

Then $a(z) \in \mathcal{R}[z]$ is a polynomial of degree 2, $b(z) \in \mathcal{R}[z]$ is a polynomial of degree 1 iff;

$-8c^2t^2 - 4(w^2 - c^2t^2 - |\overline{r}|^2) \neq 0$

iff $4|\overline{r}|^2 - 4c^2t^2 - 4w^2 \neq 0$

iff $|\overline{r}| \neq \sqrt{\frac{4w^2+4c^2t^2}{4}}$

iff $|\overline{r}| \neq \sqrt{w^2 + c^2t^2}$

and $c(z)$ is a constant. We have that $c(z) = 0$

iff $4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2 = 0$

---

[7]Generically the two roots corresponding to λ must provide one of the three alternatives;

(i). $|\overline{x} - \overline{y}| = -ct + |\overline{x} - \overline{r}|$

(ii). $|\overline{x} - \overline{y}| = ct + |\overline{x} - \overline{r}|$

(iii). $|\overline{x} - \overline{y}| + |\overline{x} - \overline{r}| = -ct$

for the corresponding $\overline{x} = \lambda y$. Clearly the points on $V^1_w(\overline{x})$ corresponding to case (iii) are bounded, so if we obtain any infinite points, they must correspond to cases (i) or (ii). By Lemma 0.8, the infinite points on opposite sides of the asymptotic line which we find below, must correspond to both cases (i) and (ii). To obtain cancellation, we therefore need to include the opposite time $-t$ in the calculation, which we can do by considering $\dot{\rho} + \dot{\rho}^{-2t}$, where $\dot{\rho}^s(\overline{x}, t) = \dot{\rho}(\overline{x}, t - s)$.

iff $|\overline{r}|^2=\frac{2w^2+2c^2t^2+/-\sqrt{(2c^2t^2+2w^2)^2-4(w^2-c^2t^2)^2}}{2}$

iff $|\overline{r}|^2=w^2+c^2t^2+/-\sqrt{(c^2t^2+w^2)^2-(w^2-c^2t^2)^2}$

$|\overline{r}|^2=w^2+c^2t^2+/-\sqrt{4w^2c^2t^2}$

iff $|\overline{r}|^2=w^2+c^2t^2+2wct=(w+ct)^2$ or $|\overline{r}|^2=w^2+c^2t^2-2wct=(w-ct)^2$

iff $|\overline{r}|=|w+ct|$ or $|\overline{r}|=|w-ct|=w-ct$

which can happen, with roots at 0 and $-\frac{b}{a}$, the finite point, calculated in $(*)$ below being 0. However, we consider the generic case when $c(z)\neq 0$, leaving further consideration of the other case to the reader.

Let;

$Z_a=\{\overline{x}\in\delta B(\overline{0},w):a(\overline{y}\centerdot\overline{r})=0\}$

$Z_b=\{\overline{x}\in\delta B(\overline{0},w):b(\overline{y}\centerdot\overline{r})=0\}$

As $a(z)$ has degree 2, we have, by the quadratic formula, that;

$$a=4c^2t^2w^2-(2\overline{y}\centerdot\overline{r}-2w^2)^2=0 \text{ iff } \overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2} \quad (PP)$$

which has at most 2 real solutions, corresponding to at most 2 (possibly empty) parallel intersection circles of the sphere $\delta B(\overline{0},w)$ with parallel planes $\{P_{1,a},P_{2,a}\}$. We will consider the generic case with two nonempty parallel circles, $\{C_{1,a},C_{2,a}\}$, which are not points, leaving the other cases to the reader, so that $Z_a=C_{1,a}\cup C_{2,a}$, ([8]). We have that $b(z)$ has degree at most 1, with at most 1 real solution, corresponding to at most 1 (possibly empty) intersection circle C_b of the sphere with a plane P_b, parallel to $P_{1,a}$ and $P_{2,a}$. Again, we will consider the generic

[8] The case when a has repeated roots, by the formula (PP) occurs when $1-\frac{4(1-c^2t^2)}{w^2}=0$, iff $w^2=4(1-c^2t^2)$, we can exclude this case by assuming $t^2>\frac{1}{c}$ by moving the initial conditions sufficiently far enough in advance of t and changing coordinates. Alternatively, we can obtain at most 2 possible solutions for w, which will account for a set of measure zero in the final integration, see footnote refcoincides. Observe that when a has two real roots, they cannot be maxima or minima, so a will change sign on opposite sides of the intersection circles $C_{1,a}$ and $C_{2,a}$.

case when C_b is nonempty and not a point, leaving the other cases to the reader. We have that C_b coincides with one of the circles $C_{1,a}$ or $C_{2,a}$ iff $a = b = 0$;

iff $(w^2 - c^2t^2 - r^2)(4w^2 - 4\overline{y} \centerdot \overline{r}) = 8c^2t^2\overline{y} \centerdot \overline{r}$ and (PP) holds

iff $(w^2-c^2t^2-r^2)(4w^2-4(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})) = 8c^2t^2(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})$

which can happen, in which case $V_w(\overline{x}$ does not blow up along C_b. Again, we leave this case to the interested reader.

For $\overline{y} \in \delta B(\overline{0}, w) \setminus Z_a$, we have that $p(\lambda, \overline{y}, \overline{r}) = 0$, where $p(z, \overline{y}, \overline{r}) \in \mathcal{R}[z]$ is a polynomial of degree 2, with coefficients in $\{\overline{y}, \overline{r}\}$, having at most 2 real roots.

Using the fact that;

$|\overline{y} \centerdot \overline{r}| \leq |\overline{y}||\overline{r}| = wr$

$|a| \leq (4c^2t^2w^2 + (2wr + 2w^2)^2) = C_1$

$|b| \leq 8c^2t^2wr + 2(2wr + 2w^2)(w^2 + c^2t^2 + r^2) = C_2$

$|c| \leq 4c^2t^2r^2 + (w^2 + c^2t^2 + r^2)^2 = C_3$

where $\{C_1, C_2, C_3\} \subset \mathcal{R}_{>0}$. Denoting the possible real roots of $p(\lambda, \overline{y}, \overline{r})$ by $\{\frac{\gamma_1}{a}, \frac{\gamma_2}{a}\}$, we have;

$max(|\gamma_1|, |\gamma_2|) \leq \frac{|b|+|\sqrt{b^2-4ac}|}{2} \leq \frac{C_2+\sqrt{C_2^2+4C_1C_3}}{2} = C_4$

where $C_4 \subset \mathcal{R}_{>0}$. Then, if;

$|a| = |4c^2t^2w^2 - (2\overline{y} \centerdot \overline{r} - 2w^2)^2| > \epsilon > 0$

it follows;

$max(|\frac{\gamma_1}{a}|, |\frac{\gamma_2}{a}|) \leq \frac{C_4}{\epsilon}$

In particular $V_w(\overline{x})$ can only blow up along the exceptional locus Z_a, ([9]).

In the generic case, with $C_{1,a} \neq \emptyset$, $C_{2,a} \neq \emptyset$, $C_{1,a} \neq C_{2,a}$, not points, we define the 2 asymptotic cones of $V_w(\overline{x})$ by;

$Cone(C_{1,a}) = \bigcup_{\overline{y} \in C_{1,a}} l_{\overline{0},\overline{y}}$

$Cone(C_{2,a}) = \bigcup_{\overline{y} \in C_{2,a}} l_{\overline{0},\overline{y}}$

where $l_{\overline{0},\overline{y}}$ is the line joining $\overline{0}$ and $\overline{y} \in C_{i,a}$, for $i \in \{1,2\}$.

We have that $Cone(C_{1,a}) \cap Cone(C_{2,a}) = \emptyset$ unless $pr^*(C_{1,a}) = C_{2,a}$, where pr^* is the orthogonal projection defined by the perpendicular line l passing through $\overline{0}$, perpendicular to the parallel planes $P_{1,a}$ and $P_{2,a}$, onto $P_{2,a}$, in which case $Cone(C_{1,a}) = Cone(C_{2,a})$. Again, we consider this generic case, leaving the case $Cone(C_{1,a}) = Cone(C_{2,a})$ to the reader.

We obtain no real roots, iff $b^2 - 4ac < 0$

iff $[-8c^2t^2\overline{y} \centerdot \overline{r} - 2(2\overline{y} \centerdot \overline{r} - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)]^2$

$-4[4c^2t^2w^2 - (2\overline{y} \centerdot \overline{r} - 2w^2)^2][4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2] < 0$

iff $q(\overline{y} \centerdot \overline{r}) < 0$, where $q \in \mathcal{R}[x]$ is a polynomial of degree at most 2, which by continuity determines an open set $Y_w \subset \mathcal{R}^3$, so that $X_w = Y_w \cap \delta B(\overline{0}, w)$ is open. We can exclude X_w from our calculations as the fibre is empty, and assume $b^2 - 4ac \geq 0$.

We obtain a repeated real root at $\frac{-b}{2a}$ iff;

$b^2 - 4ac = 0$

iff $[-8c^2t^2\overline{y} \centerdot \overline{r} - 2(2\overline{y} \centerdot \overline{r} - 2w^2)(w^2 - c^2t^2 - |\overline{r}|^2)]^2$

$-4[4c^2t^2w^2 - (2\overline{y} \centerdot \overline{r} - 2w^2)^2][4c^2t^2|\overline{r}|^2 - (w^2 - c^2t^2 - |\overline{r}|^2)^2] = 0$

[9] We can also note that if $C_{1,a} = C_{2,a} = \emptyset$ then $|a| > \epsilon_0$ on $\delta B(\overline{0}, w)$, and $max(|\frac{\gamma_1}{a}|, |\frac{\gamma_2}{a}|) \leq \frac{C_4}{\epsilon_0} = C_5$, where $C_5 \in \mathcal{R}_{>0}$, $V_w(\overline{x}) \subset B(\overline{0}, C_5 w)$ and $V_w(\overline{x})$ is bounded.

which again determines 2 intersection circles $Z_{rep} \subset B(\overline{0}, w)$, parallel to the circles $Z_a \cup Z_b$. Again, we consider the generic case that Z_{rep} is distinct from $Z_a \cup Z_b$, leaving the other cases to the reader, ([10]).

---

[10] If Z_a and Z_b are distinct, with $Z_{rep} = Z_a$, then $b^2 - 4ac = 0$ and $a = 0$, so $b = 0$, so that Z_a and Z_b have an intersection, which is a contradiction. Similarily, if Z_a and Z_b are distinct, with $Z_{rep} = Z_b$, then $b^2 - 4ac = 0$ and $b = 0$, so $ac = 0$, and $c = 0$, the blow up behaviour along Z_a being similar to the generic case. If $Z_b \subset Z_a$ with $Z_b \subset Z_{rep}$, then, we must have that $a = b = 0$, and;

$$\frac{-4w^2(w^2-c^2t^2-|\overline{r}|^2)}{-8c^2t^2-4(w^2-c^2t^2-|\overline{r}|^2)} = \frac{w^2+/w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$$

which, for fixed $\{t, |\overline{r}|\}$ has at most 8 solutions for w, $(*)$. Suppose that the spheres $\delta B(\overline{x}, -ct + |\overline{x} - \overline{r}|)$ and $\delta B(\overline{0}, w)$ share a tangent plane at $\overline{p}_{\overline{x},r}$, for some $0 < w < s$, so that the line $l_{\overline{0},\overline{p}_{\overline{x},r}}$ passes through $\overline{x}$. Without loss of generality, suppose that $|\overline{p}_{\overline{x},r}| < |\overline{x}|$ $\overline{p}_{\overline{x},r} = \frac{|\overline{p}_{\overline{x},r}|}{|\overline{x}|}\overline{x}$. Assume $\overline{x} \neq \overline{r}$ and consider the function $f_{\overline{x}}$ defined, for small λ by;

$$f_{\overline{x}}(\lambda) = -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - |\overline{x} + \lambda\overline{x} - \overline{p}_{\overline{x},r}|$$

$$= -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - |(1+\lambda)\overline{x} - \tfrac{|\overline{p}_{\overline{x},r}|}{|\overline{x}|}\overline{x}|$$

$$= -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - (1+\lambda - \tfrac{|\overline{p}_{\overline{x},r}|}{|\overline{x}|})|\overline{x}|$$

$$= -ct + |\overline{x} + \lambda\overline{x} - \overline{r}| - (1+\lambda)|\overline{x}| + |\overline{p}_{\overline{x},r}|$$

$$= -ct + [((1+\lambda)x_1 - r_1)^2 + ((1+\lambda)x_2 - r_2)^2 + ((1+\lambda)x_3 - r_3)^2]^{\frac{1}{2}} - (1+\lambda)|\overline{x}| + |\overline{p}_{\overline{x},r}|$$

$$= -ct + g_{\overline{x}}(\lambda) - (1+\lambda)|\overline{x}| + |\overline{p}_{\overline{x},r}|$$

in coordinates $\overline{x} = (x_1, x_2, x_3)$, $\overline{r} = (r_1, r_2, r_3)$, with $f_{\overline{x}}(0) = -ct + |\overline{x} - \overline{r}| - |\overline{x} - \overline{p}_{\overline{x},r}| = 0$, $g_{\overline{x}}(0) = |\overline{x} - \overline{r}|$. Then;

$$\frac{df}{d\lambda} = \frac{1}{2g_{\overline{x}}(\lambda)}(2((1+\lambda)x_1 - r_1)x_1 + 2((1+\lambda)x_2 - r_2)x_2 + 2((1+\lambda)x_3 - r_3)x_3) - |\overline{x}|$$

$$= \frac{1}{g_{\overline{x}}(\lambda)} < (1+\lambda)\overline{x} - \overline{r}, \overline{x} > -|\overline{x}|$$

$$= \frac{1}{g_{\overline{x}}(\lambda)}[(1+\lambda)|\overline{x}|^2 - < \overline{r}, \overline{x} >] - |\overline{x}|$$

so that $\frac{df}{d\lambda}(0) = 0$

iff

$$|\overline{x}|^2 - < \overline{r}, \overline{x} > = < \overline{x}, \overline{x} - \overline{r} > = |\overline{x}||\overline{x} - \overline{r}|$$

which implies that $\overline{r} \in l_{\overline{0},\overline{x}}$. Excluding this solution, as $f_{\overline{x}}$ is analytic, by O-minimality, for $\epsilon > 0$, we can assume that $f_{\overline{x}} = 0 \cap [-\epsilon, \epsilon]$ is a finite union of points and intervals. No interval can contain 0, as then $\frac{df}{d\lambda}(0)=0$, so that $f_{\overline{x}}\neq 0$ on some set of the form $(-\epsilon,\epsilon)\setminus\{0\}$. In particularly, this implies that we can obtain tangency of $\delta B(\overline{x}_1,-ct+|\overline{x}_1-\overline{r}|)$ with $\delta B(\overline{0},\overline{p}_{\overline{x},r,1})$ for mobile points $\overline{x}_1$ and $\overline{p}_{\overline{x},r,1}$ along the line $l_{\overline{0},\overline{p}_{\overline{x},r}}$. If $\overline{x}=\overline{r}$ or $\overline{r}\in l_{\overline{0},\overline{x}}$, we either have $|\overline{x}|<|\overline{r}|$, in which case, it is clear we can move $\overline{x}$ along $l_{\overline{0},\overline{x}}$ and obtain mobile points, or $|\overline{x}|>|\overline{r}|$, in which case we can move $\overline{x}$ through $\overline{r}$ towards $\overline{0}$, and eventually obtain mobile points, $(**)$. From $(**)$, the possible $0<w<s$ can represent arcs with the property that;

$B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ intersects $B(\overline{0},s)$

and such that the spheres $\delta B(\overline{x},-ct+|\overline{x}-\overline{r}|)$ and $\delta B(\overline{0},w)$ share a tangent plane at $\overline{p}_{\overline{x},r}$, see Lemma 0.4, is not discrete. It follows that the case $(*)$ accounts for a set of measure zero in the final parametrisation and doesn't effect the finiteness of the integral. When $w-ct=|\overline{r}|$, $\overline{y}=w$, $\overline{r}=(w-ct)\frac{\overline{y}}{|\overline{y}}=(1-\frac{ct}{w})\overline{y}$, $(***)$, we obtain, as above, that there exist solutions to $V_w(\overline{x}$ for $|\overline{x}|\geq|\overline{r}|$, $\overline{x}\in l_{\overline{0},\overline{y}}$. This corresponds to the case $a(\overline{y}\centerdot\overline{r})=b(\overline{y}\centerdot\overline{r})=c(\overline{y}\centerdot\overline{r})=0$, where;

$a(z)=4c^2t^2w^2-(2z-2w^2)^2$

$b(z)=-8c^2t^2z-2(2z-2w^2)(w^2-c^2t^2-|\overline{r}|^2)$

$c(z)=4c^2t^2|\overline{r}|^2-(w^2-c^2t^2-|\overline{r}|^2)^2$

We have from $(***)$ that;

$\overline{y}\centerdot\overline{r}=\overline{y}\centerdot(1-\frac{ct}{w})\overline{y}$

$=(1-\frac{ct}{w})|\overline{y}|^2$

$=(1-\frac{ct}{w})w^2$

$=w(w-ct)$

so that;

$a(\overline{y}\centerdot\overline{r})=a(w(w-ct))$

$=4c^2t^2w^2-(2w(w-ct)-2w^2)^2$

$=0$

$b(\overline{y}\centerdot\overline{r})=b(w(w-ct))$

$=-8c^2t^2w(w-ct)-2(2w(w-ct)-2w^2)(w^2-c^2t^2-|\overline{r}|^2)$

Assuming $b^2-4ac\geq 0$, we obtain that $\gamma_1=0$ or $\gamma_2=0$ iff;

$\sqrt{b^2-4ac}=b$ or $\sqrt{b^2-4ac}=-b$ iff $(b^2-4ac)=b^2$

iff $4ac=0$

iff $a=0$ or $c=0$

iff $4c^2t^2w^2-(2\overline{y}\centerdot\overline{r}-2w^2)^2=0$ or $4c^2t^2|\overline{r}|^2-(w^2-c^2t^2-|\overline{r}|^2)^2=0$

iff $\overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$ or $4c^2t^2r^2-(w^2-c^2t^2-r^2)^2=0$

iff $\overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$ or $r=|w+ct|$ or $r=w-ct$

iff Case 1. $\overline{y}\centerdot\overline{r}=\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2}$

or Case 2. $r=|w+ct|$ or $r=w-ct$

In Case 2, for $a\neq 0$, we obtain exactly 2 real roots $\frac{-b}{a}$ and 0, uniformly in $\overline{y}$.

In Case 1, with $b\neq 0$, we have, using Newton's expansion of $(1+y)^{\frac{1}{2}}$, for $|y|<1$, that;

$lim_{a\rightarrow 0}\frac{-b+\sqrt{b^2-4ac}}{2a}=lim_{a\rightarrow 0}\frac{-b+b(1-\frac{4ac}{b^2})^{\frac{1}{2}}}{2a}$

$=lim_{a\rightarrow 0}\frac{-b+b(1+\frac{y}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}y^n)}{2a}|_{y=-\frac{4ac}{b^2}}$

$=-8c^2t^2w(w-ct)-2(2w(w-ct)-2w^2)(w^2-c^2t^2-(w-ct)^2)$

$=0$

$c(\overline{y}\centerdot\overline{r})=c(w(w-ct))$

$=4c^2t^2(w-ct)^2-(w^2-c^2t^2-(w-ct)^2)^2$

$=0$

$= lim_{a\rightarrow 0}\frac{\frac{b(-\frac{4ac}{b^2})}{2}+b\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}(-\frac{4ac}{b^2})^n}{2a}$

$= lim_{a\rightarrow 0}\frac{1}{2}[-\frac{2c}{b}+\sum_{n=2}^{\infty}\frac{(-1)^{2n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}\frac{(4c)^n a^{n-1}}{b^{2n-1}}]$

$= \frac{-c}{b}$ $(*)$

and, with $b\neq 0$;

$lim_{a\rightarrow 0}\frac{-b-\sqrt{b^2-4ac}}{2a} = lim_{a\rightarrow 0}\frac{-b-b(1-\frac{4ac}{b^2})^{\frac{1}{2}}}{2a}$

$= lim_{a\rightarrow 0}\frac{-b-b(1+\frac{y}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}y^n)}{2a}|_{y=-\frac{4ac}{b^2}}$

$= lim_{a\rightarrow 0}\frac{-2b-\frac{b(-\frac{4ac}{b^2})}{2}-b\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}(-\frac{4ac}{b^2})^n}{2a}$

$= lim_{a\rightarrow 0}[-\frac{b}{a}+\frac{c}{b}-\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{2n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}\frac{(4c)^n a^{n-1}}{b^{2n-1}}]$

$= lim_{a\rightarrow 0}(-\frac{b}{a}+\frac{c}{b})$ $(**)$, ([11])

Letting $\overline{p}_{i,a}$ denote the centres of the blow up circle $S_{i,a}$, $1\leq i\leq 2$, and $\overline{q}_{i,a}=l_{\overline{0},\overline{p}_{i,a}}\cap\delta B(\overline{0},w)$, if $\overline{y}\in S_{i,a}$, we let $S_{i,\overline{y},a}$ denote the great circle passing through $\overline{y}$ and $\overline{q}_{i,a}$. Then, without loss of generality, we have that the region;

[11] This is a first order approximation for $V_w(\overline{x})$. We introduce the angle θ below and consider the leading term $-\frac{b}{a}$ which blows up as $a\rightarrow 0$. Strictly speaking, letting $d=-\frac{1}{2}[\sum_{n=2}^{\infty}\frac{(-1)^{2n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}\frac{(4c)^n a^{n-1}}{b^{2n-1}}]$, we have that;

$-\frac{b}{a}+\frac{c}{b}+d=-\frac{b}{a}(1-\frac{ca}{b^2}-\frac{da}{b})$

If we define $b_{new}(\theta)=b(\theta)(1-\frac{ca}{b^2}-\frac{da}{b})(\theta)$, with $a(0)=0$, so that;

$b_{new}(0)=b(0)$

$b'_{new}(\theta)=b'(\theta)(1-\frac{ca}{b^2}-\frac{da}{b})(\theta)+b(\theta)(\frac{2ca}{b^3}-\frac{c'a}{b^2}-\frac{ca'}{b^2}+\frac{da}{b^2}-\frac{d'a}{b}-\frac{da'}{b})$

$b'_{new}(0)=b'(0)+b(0)(-\frac{c(0)a'(0)}{b^2(0)}-\frac{d(0)a'(0)}{b(0)})$

$=b'(0)-\frac{c(0)a'(0)}{b(0)}-d(0)a'(0)$

the proof goes through replacing the instances of $\{b(0),b'(0)\}$ with $\{b_{new}(0),b'_{new}(0)\}$, which are all finite.

$a > 0 \cap B(\overline{0}, w) = \bigcup_{\overline{y} \in S_{1,a}} (S_{1,\overline{y},a} \cap a > 0)$

$a < 0 \cap B(\overline{0}, w) = \bigcup_{\overline{y} \in S_{1,a}} (S_{1,\overline{y},a} \cap a < 0)$

with $a < 0$ situated between the intersections $S_{1,a}$ and $S_{2,a}$, $a > 0$ situated above and below the intersections $S_{1,a}$ and $S_{2,a}$ on $\delta B(\overline{0}, w)$, and blow ups of opposite signs, see footnote 10, along $S_{1,\overline{y},a}$ at $\overline{y}$ and the corresponding opposite point $\overline{y}' \in S_{1,\overline{y},a} \cap S_{1,a}$ and points

$\overline{y}'', \overline{y}''' \subset S_{1,\overline{y},a} \cap S_{2,a}$, ([12]).

---

[12] In this case, $Cone_{1,a}$ and $Cone_{2,a}$ have the following asymptotic property. Fix $\overline{y} \in S_{1,a}$, and form the plane $Q_{1,\overline{y},a}$ determined by $l_{\overline{0},\overline{y}}$ and the tangent to the great circle $S_{1,\overline{y},a}$ at $\overline{y}$, so that $S_{1,\overline{y},a} \subset Q_{1,\overline{y},a}$. For a fixed $\overline{y}' \in S_{1,\overline{y},a}, \setminus \overline{y}$, let θ denote the angle between $l_{\overline{0},\overline{y}}$ and $l_{\overline{0},\overline{y}'}$ in the plane $Q_{1,\overline{y},a}$ and let $a(\theta) = a(\overline{y}')$, $b(\theta) = b(\overline{y}')$. Considering the first order approximation $-\frac{b}{a}$ for $V_w(\overline{x})$ along $S_{1,\overline{y},a}$, defining $V_{w,1}(\overline{x})$, we have that $|\overline{x}| = -\frac{b(\theta)w}{a(\theta)}$. Let pr^* be the orthogonal projection from $Q_{1,\overline{y},a}$ onto $l_{\overline{0},\overline{y}}$, and let $pr^*(\overline{x}) \in l_{\overline{0},\overline{y}}$ be the corresponding point, so that $pr^*(\overline{x})$ is the nearest point to $\overline{x}$ on $l_{\overline{0},\overline{y}}$, with $x = |\overline{x} - pr^*(\overline{x})|$ and $R = |pr^*(\overline{x})|$. By elementary trigonometry, assuming $\theta > 0$, we have that;

$x = |-\frac{b(\theta)wsin(\theta)}{a(\theta)}|$, $R = |-\frac{b(\theta)wcos(\theta)}{a(\theta)}|$, $\frac{x}{R} = tan(\theta)$ $(*)$

As the circles $S_{1,a}$ and $S_{2,a}$ are distinct and non-empty, we can factor a as $(\overline{y} \centerdot \overline{r} - \alpha)(\overline{y} \centerdot \overline{r} - \beta)$, where $S_{1,a}$ is defined by $(\overline{y} \centerdot \overline{r} = \alpha) \cap \delta B(\overline{0}, w)$, $S_{2,a}$ is defined by $(\overline{y} \centerdot \overline{r} = \beta) \cap \delta B(\overline{0}, w)$. Rotating coordinates so that $\overline{y}$ is situated at $(w, 0, 0)$, $\overline{y}'$ at $(wcos(\theta), wsin(\theta), 0)$, we have that;

$wr_1 - \alpha = 0$

where $\overline{r} = (r_1, r_2, r_3)$, and, without loss of generality, we can assume that $r_2 \neq 0$,. This follows as if we rotate $\overline{y}$ to $(w, 0, 0)$, $\overline{y}'$ to $(wcos(\theta), wsin(\theta), 0)$, with $r_2 = 0$, rotate $\overline{y}$ to $(0, w, 0)$ and $\overline{y}'$ to $(0, wcos(\theta), wsin(\theta))$ with $r_3 = 0$, and rotate $\overline{y}$ to $(0, 0, w)$ and $\overline{y}'$ to $(wsin(\theta), 0, wcos(\theta))$ with $r_1 = 0$, then $\overline{r} \in l_{\overline{0},\overline{y}}$, which we can exclude, as it accounts for a set of measure zero in the final integration. It follows that;

$a(\theta) = (wcos(\theta)r_1 + wsin(\theta)r_2 - \alpha)\gamma(\theta) = (\frac{w\alpha cos(\theta)}{w} + wsin(\theta)r_2 - \alpha)\gamma(\theta)$

$= (\alpha(cos(\theta) - 1) + wsin(\theta)r_2)\gamma(\theta)$ $(**)$

with $\gamma(0) \neq 0$, so that, from $(*)$,$(**)$;

$cos(\theta) = |-\frac{a(\theta)R}{b(\theta)w}|$

$= |-\frac{R}{b(\theta)w}||[(\alpha(cos(\theta) - 1) + wsin(\theta)r_2)\gamma(\theta)]|$ (L)

so that, using the power series expansions $cos(\theta) = 1 + O(\theta^2)$, $sin(\theta) = \theta + O(\theta^3)$;

$1 + O(\theta^2) = -\frac{R}{b(\theta)w}(\alpha O(\theta^2) + wr_2\theta + O(\theta^3)\gamma(\theta)$

and, rearranging;

$\theta = |\frac{-b(0)}{Rr_2}|(\frac{1}{|\gamma(0)|} + O(\theta))$ (D)

so $\theta = O(\frac{1}{R})$ $(***)$

so that, from $(*), (***)$;

$tan(\theta) = O(\frac{1}{R})$

$x = Rtan(\theta) = O(1)$

and, as $|\overline{x}|cos(\theta) = R = |pr^*(\overline{x})|$, we have that;

$|\overline{x}| - |pr^*(\overline{x})| = \frac{R}{cos(\theta)} - R$

$= R(1 + \frac{\theta^2}{2} + O(\theta^4)) - R$

$= O(\theta^2)$

$= O(\frac{1}{R^2})$

From (D);

$\theta = |\frac{-b(0)}{Rr_2}|(\frac{1}{|\gamma(0)|} + O(\theta))$ (D)

$= |\frac{-b(0)}{\gamma(0)Rr_2}|(1 + O(\theta))$

so that;

$\theta(1 + O(\theta))^{-1} = \theta(1 + O(\theta)) = |\frac{-b(0)}{\gamma(0)Rr_2}|$

$||\frac{-b(0)}{\gamma(0)Rr_2}| - \theta| \leq |\theta(1 + O(\theta)) - \theta| = O(\theta^2) = O(\frac{1}{R^2})$

so that;

$|| - \frac{b(0)}{\theta\gamma(0)r_2}| - R| \leq O(\frac{1}{R^2})O(\frac{R}{\theta}) = O(\frac{1}{R^2})O(R^2) = O(1)$

We have that, using (L);

$x = Rtan(\theta) = | - \frac{wb(\theta)sin(\theta)}{[(\alpha(cos(\theta)-1)+wsin(\theta)r_2)\gamma(\theta)]}|$

and using L'Hopital's rule;

$lim_{\theta\rightarrow 0}x = | - \frac{lim_{\theta\rightarrow 0}(wb(\theta)sin(\theta))'}{lim_{\theta\rightarrow 0}[(\alpha(cos(\theta)-1)+wsin(\theta)r_2)\gamma(\theta)]'}|$

$= | - \frac{lim_{\theta\rightarrow 0}(wb'(\theta)sin(\theta)+wb(\theta)cos(\theta))}{lim_{\theta\rightarrow 0}[(-\alpha sin(\theta)+wcos(\theta)r_2)\gamma(\theta)+((\alpha(cos(\theta)-1)+wsin(\theta)r_2))\gamma'(\theta)]}|$

$= | - \frac{wb(0)}{wr_2\gamma(0)}|$

$= | - \frac{b(0)}{r_2\gamma(0)}|$

so that the line formed by the translation of $l_{\overline{0},\overline{y}}$ by a perpendicular distance of $| - \frac{b(0)}{r_2\gamma(0)}|$ in the plane $Q_{1,\overline{y},a}$ is actually an asymptote. Moreover, as x is analytic

If $\overline{y} \in S_{1,a}$ is fixed, with corresponding $\{S_{1,\overline{y},a}, Q_{1,\overline{y},a}\}$, then as $b \neq 0$ along $C_{1,a}$, we can assume that for small enough $|\theta| < \delta$, see footnote 12, $|b(\theta)| > \epsilon$, uniformly in $\overline{y} \in S_{1,a}$, so that $|\frac{-c}{b(\theta)}| \leq \frac{|c|}{\epsilon}$, and the root found in $(*)$ has a maximum value M, varying $|\theta| < \delta$ and $\overline{y} \in S_{1,a}$. For the root $\overline{x}(\theta) = -\frac{b(\theta)}{a(\theta)} + \frac{c}{b(\theta)}\overline{w}$, defined by $(**)$, we can assume that

---

in θ, $x - |\frac{b(0)}{r_2\gamma(0)}| = O(\theta) = O(\frac{1}{R})$.

We also have, using (L), and L'Hopital's rule twice, that;

$|lim_{\theta\rightarrow 0}(-\frac{b(0)}{\theta r_2\gamma(0)} - R)|$

$= |lim_{\theta\rightarrow 0}|(-\frac{b(0)}{r_2\theta\gamma(0)}| + |\frac{b(\theta)wcos(\theta)}{[\alpha(cos(\theta)-1)+wsin(\theta)r_2]\gamma(\theta)}|)|$

$= lim_{\theta\rightarrow 0}|\frac{-b(0)\gamma(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2]+b(\theta)wr_2cos(\theta)\theta\gamma(0)}{r_2\gamma(0)\gamma(\theta)\theta[\alpha(cos(\theta)-1)+wsin(\theta)r_2]}|$

$= lim_{\theta\rightarrow 0}|\frac{-b(0)\gamma'(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2]-b(0)\gamma(\theta)[-\alpha sin(\theta)+wcos(\theta)r_2]+b'(\theta)wr_2\gamma(0)cos(\theta)\theta+b(\theta)wr_2\gamma(0)[cos(\theta)-\theta sin(\theta)]}{\gamma(0)\gamma'(\theta)\theta[\alpha(cos(\theta)-1)+wsin(\theta)r_2]+\gamma(0)\gamma(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2-\alpha\theta sin(\theta)+wcos(\theta)\theta r_2]}|$

$= lim_{\theta\rightarrow 0}|\frac{E(\theta)}{F(\theta)}|$

where;

$E(\theta) = -b(0)\gamma''(\theta)[\alpha(cos(\theta)-1)+wsin(\theta)r_2] - 2b(0)\gamma'(\theta)[-\alpha sin(\theta)$

$+wcos(\theta)r_2] - b(0)\gamma(\theta)[-\alpha cos(\theta) - wsin(\theta)r_2] + b''(\theta)wr_2\gamma(0)\theta cos(\theta)$

$+2b'(\theta)wr_2\gamma(0)[cos(\theta) - \theta sin(\theta)] + b(\theta)wr_2\gamma(0)[-2sin(\theta) - \theta cos(\theta)]$

$F(\theta) = \gamma(0)\gamma''(\theta)\theta[\alpha(cos(\theta)-1) + wsin(\theta)r_2] + \gamma(0)\gamma'(\theta)[\alpha(cos(\theta)-1)$

$+wsin(\theta)r_2] + \gamma(0)\gamma'(\theta)\theta[-\alpha sin(\theta) + wcos(\theta)r_2] + \gamma(0)\gamma'(\theta)[\alpha(cos(\theta)-1)$

$+wsin(\theta)r_2 - \alpha\theta sin(\theta) + wcos(\theta)\theta r_2] + \gamma(0)\gamma(\theta)[-2\alpha sin(\theta) + 2wcos(\theta)r_2$

$-\alpha\theta cos(\theta) - wsin(\theta)\theta r_2]$

so that;

$|lim_{\theta\rightarrow 0}(-\frac{b(0)}{r_2\theta\gamma(0)} - R)| = |\frac{-2b(0)\gamma'(0)wr_2+b(0)\gamma(0)\alpha+2b'(0)wr_2\gamma(0)}{2\gamma(0)^2wr_2}|$

It follows, as $\frac{-b(0)}{r_2\theta\gamma(0)} - R$ is analytic in θ, that;

$||\frac{-b(0)}{r_2\theta\gamma(0)}| - R| - |\frac{-2b(0)\gamma'(0)wr_2+b(0)\gamma(0)\alpha+2b'(0)wr_2\gamma(0)}{2\gamma(0)^2wr_2}| = O(\theta) = O(\frac{1}{R})$

for sufficiently small δ, $|\overline{x}| > Mw$, and we can unambiguously define $\overline{x}(\theta)_{opp} = \overline{x}(-\theta)$, ([13]).

Returning to the case notation above, we have that, Case 3, $a = b = 0$ iff $(1 - c^2t^2 - r^2) = \frac{8c^2t^2\overline{y}.\overline{r}}{4w^2-4\overline{y}.\overline{r}}$

$$\text{iff } (1 - c^2t^2 - r^2) = \frac{8c^2t^2(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})}{4w^2-4(\frac{w^2+/-w^2\sqrt{1-\frac{4(1-c^2t^2)}{w^2}}}{2})}$$

corresponding to specific values of w, a situation considered in the footnote above. We have that $V_w(\overline{x})$ is bounded over one of the components of Z_a and exhibits a blow up behaviour over the other component.

In Case 1, not Case 3, as we have seen in the above footnotes, we obtain two components, with one component having a pair of infinite opposite branches parallel to the lines in the asymptotic cones, and a bounded component corresponding to $\frac{-c}{b}$ over the singular locus Z_a.

In Case 2, we again, by a similar calculation to $(*)$, obtain two components, with one component having a pair of infinite opposite branches parallel to the lines in the asymptotic cones, and a bounded component corresponding to the root 0.

$\square$

Lemma 0.5. *Cancellation along asymptotes*

We have that, along the line $l_{\overline{0},\overline{y},sh}$, the integrals;

$$(i)\ \frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}$$

---

[13] By the calculation in footnote 12, we have that $\{\overline{x},\overline{x}_{opp}\}$ vary as $O(\frac{1}{\theta})$ with the angle θ. Moreover, by Lemma 0.8, for sufficiently small θ, if $\overline{x}$ corresponds to $-ct$, then $\overline{x}_{opp}$ corresponds to ct. By the definition of $V_w^1(\overline{x})$, $B(\overline{x},-ct+|\overline{x}-\overline{r})$ and $B(\overline{x}_{opp},ct+|\overline{x}_{opp}-\overline{r})$ pass through $\{\overline{y}',\overline{y}''\}\subset B(\overline{0},w)$, touching $\delta B(\overline{0},w)$, with $|\overline{y}-\overline{y}'| = 2w|\theta| = O(\frac{1}{R})$ and centred on "oppposite" sides of $B(\overline{0},w)$. As the boundaries $\delta B(\overline{x},-ct+|\overline{x}-\overline{r})$ and $\delta B(\overline{x}_{opp},-ct+|\overline{x}_{opp}-\overline{r})$ limit to the tangent planes of $\overline{y}'$ and $\overline{y}''$ for sufficiently large $\{\overline{x},\overline{x}_{opp}\}$ and the points $\overline{y}'$ and $\overline{y}''$ approach each other as we increase R, this will be enough to obtain cancellation in the indefinite integral, following the method above. Moreover, by the calculation in footnote 12, we can assume that $\overline{x}$ and $\overline{x}_{opp}$ in the limit as $\theta\to 0$ approach the same line consisting of a bounded translate of the line $l_{\overline{0},\overline{y}}$ in the plane $Q_{1,\overline{y},a}$

(ii) $\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}$

(iii) $\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}$

are $O(\frac{1}{R^3})$*, with* $R=|\overline{r}'|$

Proof. Using the notation in Lemma 0.4, we consider the restriction of $V_w(\overline{x})$ to a cover of $S_{1,\overline{y},a}$, for $\overline{y}\in S_{1,a}$. For $\overline{r}''(\theta)\in V_w(\overline{x})|_{S_{1,\overline{y},a}}$, let $\overline{r}'$ be the nearest point to $\overline{r}''$ on the asymptote $l_{\overline{0},\overline{y},sh}$, where $l_{\overline{0},\overline{y},sh}$ is a shift of $l_{\overline{0},\overline{y}}$ by the perpendicular distance $c_{\overline{y}}=|\frac{-b(0)}{r_2\gamma(0)}|$ in the plane $S_{1,\overline{y},a}$. Then, by the result of Lemma 0.4, we have, for any $0<\epsilon<1$, that;

$$|\overline{r}''(\theta)-\overline{r}'(\theta)|<\epsilon=O(\frac{1}{R})$$

for sufficiently small θ, with $|\overline{r}'(\theta)-\overline{v}_{\overline{y}}|=R$ and $|\overline{v}_{\overline{y}}|=|c_{\overline{y}}|$ and $|\overline{r}''(\theta)|-|pr^*(\overline{r}''(\theta))|<\frac{E}{R^2}$, where $|pr^*(\overline{r}''(\theta))|=|\overline{r}'(\theta)-\overline{v}_{\overline{y}}|=R$, so that, for sufficiently small $\theta(R)$ or large $R(\theta)$;

$$R-1<R-\frac{E}{R^2}<|\overline{r}''(\theta)|<R+\frac{E}{R^2}<R+1$$

$$R-2<R-1-\epsilon<|\overline{r}'(\theta)|<R+1+\epsilon<R+2$$

We also have that, by the result of Lemma 0.4, that, for sufficiently small $\theta(R)$, $0<\epsilon'<1$;

$$|\overline{r}''(\theta)|-|(\frac{-b(0)}{\theta r_2\gamma(0)})|=|\overline{r}''(\theta)|-|pr^*(\overline{r}''(\theta))|+|pr^*(\overline{r}''(\theta))|-|(\frac{-b(0)}{\theta r_2\gamma(0)})|$$

$$<\frac{E}{R^2}+\epsilon'+\epsilon_{\overline{y}}$$

$$=\frac{E}{R^2}+O(\frac{1}{R})+\epsilon_{\overline{y}}$$

$$<2\epsilon'+\epsilon_{\overline{y}}$$

where $\epsilon_{\overline{y}}=|\frac{-2b(0)\gamma'(0)wr_2+b(0)\gamma(0)\alpha+2b'(0)wr_2\gamma(0)}{2\gamma(0)^2wr_2}|$

and similarly;

$|\overline{r}''(\theta)_{opp}| - |(\frac{-b(0)}{-\theta r_2\gamma(0)})|$

$= |\overline{r}''(\theta)| - |pr^*(\overline{r}''(\theta)_{opp})| + |pr^*(\overline{r}''(\theta)_{opp})| - |(\frac{b(0)}{\theta r_2\gamma(0)})|$

$< \frac{E}{R^2} + \epsilon' + \epsilon_{\overline{y}}$

$< 2\epsilon' + \epsilon_{\overline{y}}$

so that;

$|\overline{r}''(\theta)| - |\overline{r}''(\theta)_{opp}| = |\overline{r}''(\theta)| - |(\frac{b(0)}{\theta r_2\gamma(0)})| + |(\frac{b(0)}{\theta r_2\gamma(0)})| - |\overline{r}''(\theta)_{opp}|$

$\leq 4\epsilon' + 2\epsilon_{\overline{y}}$

$= O(\frac{1}{R}) + 2\epsilon_{\overline{y}}$

and;

$|\overline{r}'(\theta)| - |\overline{r}'(\theta)_{opp}| = (|\overline{r}'(\theta)| - |\overline{r}''(\theta)|) + (|\overline{r}''(\theta)| - |\overline{r}''(\theta)_{opp}|) + (|\overline{r}''(\theta)_{opp}| -$

$|\overline{r}'(\theta)_{opp}|)$

$\leq (|\overline{r}'(\theta) - \overline{r}''(\theta)|) + (|\overline{r}''(\theta)| - |\overline{r}''(\theta)_{opp}|) + (|\overline{r}''(\theta)_{opp} - \overline{r}'(\theta)_{opp}|)$

$\leq 4\epsilon' + 2\epsilon_{\overline{y}} + 2\epsilon$

$= O(\frac{1}{R}) + 2\epsilon_{\overline{y}}$

In particularly, as, by Pythagoras' Theorem;

$|\overline{r}'(\theta)|^2 + |c_{\overline{y}}|^2 = |pr^*(\overline{r}'(\theta))|^2$

$|\overline{r}'(\theta)_{opp}|^2 + |c_{\overline{y}}|^2 = |pr^*(\overline{r}'(\theta)_{opp})|^2$

we have;

$= (|pr^*(\overline{r}'(\theta))|^2 - |c_{\overline{y}}|^2)^{\frac{1}{2}} - (|pr^*(\overline{r}'(\theta)_{opp})|^2 - |c_{\overline{y}}|^2)^{\frac{1}{2}}$

$= |\overline{r}'(\theta)| - |\overline{r}'(\theta)_{opp}|$

$\leq 4\epsilon' + 2\epsilon + 2\epsilon_{\overline{y}}$

so that, using Newton's expansion;

$|pr^*(\overline{r}'(\theta))|(1 - \frac{|c_{\overline{y}}|^2}{|pr^*(\overline{r}'(\theta))|^2})^{\frac{1}{2}} - |pr^*(\overline{r}'(\theta)_{opp})|(1 - \frac{|c_{\overline{y}}|^2}{|pr^*(\overline{r}'(\theta)_{opp})|^2})^{\frac{1}{2}}$

$\leq 4\epsilon' + 2\epsilon + 2\epsilon_{\overline{y}}$

$|pr^*(\overline{r}'(\theta))| - |pr^*(\overline{r}'(\theta)_{opp})| \leq 4\epsilon' + 2\epsilon + O(\frac{1}{R^2}) + 2\epsilon_{\overline{y}}$

and we can assume that for sufficiently small θ;

$pr^*(\overline{r}'(\theta)) = -pr^*(\overline{r}'(\theta)_{opp}) + \overline{\epsilon} + \overline{w}_{\overline{y}}$

with $|\epsilon| < 4\epsilon' + 3\epsilon$, $|\overline{w}_{\overline{y}}| = 2\epsilon_{\overline{y}}$, and;

$-\overline{r}'(\theta) = -(\overline{v}_{\overline{y}} + pr^*(\overline{r}'(\theta)))$

$= -\overline{v}_{\overline{y}} - pr^*(\overline{r}'(\theta))$

$= -\overline{v}_{\overline{y}} + pr^*(\overline{r}'(\theta)_{opp}) - \overline{\epsilon} - \overline{w}_{\overline{y}}$

$= -\overline{v}_{\overline{y}} + (\overline{r}'(\theta)_{opp} - \overline{v}_{\overline{y}}) - \overline{\epsilon} - \overline{w}_{\overline{y}}$

$= \overline{r}'(\theta)_{opp} - 2\overline{v}_{\overline{y}} - \overline{\epsilon} - \overline{w}_{\overline{y}}$

$= \overline{r}'(\theta)_{opp} - (2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}}) + O(\frac{1}{R})$

$\simeq \overline{r}'(\theta)_{opp} - (2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}})$

For the asymptote $l_{\overline{0},\overline{y},sh}$, with $\overline{r}' \in l_{\overline{0},\overline{y},sh}$, $|\overline{r}'| = R$, sufficiently large, there exists a unique $\overline{r}'' \in V_w(\overline{x})$, with $pr^*(\overline{r}'') = \overline{r}'$, where pr^1 is the orthogonal projection onto $l_{\overline{0},\overline{y},sh}$ in the plane $Q_{1,\overline{y},a}$. If $|pr^*(\overline{r}'')| = S$, then $|\overline{r}'' - \overline{r}'| = O(\frac{1}{S})$, $|\overline{r}'' - pr^*(\overline{r}'')| = O(\frac{1}{S^2})$, so that;

$S - \frac{1}{S} + O(\frac{1}{S^2}) \leq R \leq S + \frac{1}{S} + O(\frac{1}{S^2})$

so that $|\overline{r}'' - \overline{r}'| = O(\frac{1}{R}) = O(\frac{1}{S})$. We have that;

$\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|) \cap B(\overline{0}, s) = (\delta B(\overline{r}'', -ct + |\overline{r} - \overline{r}'|) + (\overline{r}' - \overline{r''})) \cap B(\overline{0}, s)$

$\simeq (\delta B(\overline{r}'', -ct + |\overline{r} - \overline{r}''|) + (\overline{r}' - \overline{r''})) \cap B(\overline{0}, s)$

with a radial adjustment of at most $|\overline{r}' - \overline{r}''|$, and $\delta B(\overline{r}'', -ct + |\overline{r} - \overline{r}''|)$ passes through $\overline{y}''$ with $|\overline{y}'' - \overline{y}| = w\theta = O(\frac{1}{S}) = O(\frac{1}{R})$. It follows that $\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|)$ passes through $\overline{y}'$ with $|\overline{y}'' - \overline{y}'| = O(\frac{1}{R})$, $|\overline{y} - \overline{y}'| = O(\frac{1}{R})$. Similarly, we have that for the pair $\{\overline{r}'_{opp}, \overline{r}''_{opp}\}$;

$\overline{r}'_{opp} = (2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}}) - \overline{r}' + O(\frac{1}{S})$

$= (2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}}) - \overline{r}' + O(\frac{1}{R})$

$= 2\overline{z}_{\overline{y}} - \overline{r}' + O(\frac{1}{R})$

where $\overline{z}_{\overline{y}} = \frac{1}{2}(2\overline{v}_{\overline{y}} + \overline{w}_{\overline{y}})$, so that $|\overline{r}''_{opp} - \overline{r}'_{opp}| = O(\frac{1}{R}) = O(\frac{1}{S})$. Moreover;

$\delta B(\overline{r}'_{opp}, ct + |\overline{r} - \overline{r}'_{opp}|) \cap B(\overline{0}, s) = (\delta B(\overline{r}''_{opp}, ct + |\overline{r} - \overline{r}'_{opp}|) + (\overline{r}'_{opp} - \overline{r''}_{opp})) \cap B(\overline{0}, s)$

$\simeq (\delta B(\overline{r}''_{opp}, ct + |\overline{r} - \overline{r}''_{opp}|) + (\overline{r}'_{opp} - \overline{r''}_{opp})) \cap B(\overline{0}, s)$

with a radial adjustment of at most $|\overline{r}'_{opp} - \overline{r}''_{opp}|$, and $\delta B(\overline{r}''_{opp}, ct + |\overline{r} - \overline{r}''_{opp}|)$ passes through $\overline{y}''_{opp}$ with $|\overline{y}''_{opp} - \overline{y}| = w\theta = O(\frac{1}{S}) = O(\frac{1}{R})$. It follows that $\delta B(\overline{r}'_{opp}, ct + |\overline{r} - \overline{r}'_{opp}|)$ passes through $\overline{y}'_{opp}$ with $|\overline{y}''_{opp} - \overline{y}'_{opp}| = O(\frac{1}{R})$, $|\overline{y} - \overline{y}'_{opp}| = O(\frac{1}{R})$.

We have that;

(i). Using the facts that $|\frac{\partial \rho}{\partial t}|_0| \leq M$ on $B(\overline{0}, s)$, the surface measure of $\delta B(\overline{r}', -ct + |\overline{r} - \overline{r}'|) \cap B(\overline{0}, s)$ is at most $2\pi s^2$, $\overline{r}'_{opp} = 2\overline{z}_{\overline{y}} - \overline{r}' + O(\frac{1}{R})$, we have, for sufficiently large $R = |\overline{r}'|$, that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t - \frac{|\overline{r}-\overline{r}'|}{c})^2} \int_{\delta B(\overline{r}', -ct + |\overline{r}-\overline{r}'|)} (t - \frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y}) \frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+ \frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t - \frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2} \int_{\delta B(\overline{r}'_{opp}, ct + |\overline{r}-\overline{r}'_{opp}|)} (-t - \frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y}) \frac{(r_1 - r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$= |\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t - \frac{|\overline{r}-\overline{r}'|}{c})} \frac{(r_1 - r_1')}{c|\overline{r}-\overline{r}'|^2} \int_{\delta B(\overline{r}', -ct + |\overline{r}-\overline{r}'|)} (\frac{\partial^2 \rho}{\partial t^2})(\overline{y}, 0)] dS(\overline{y})$

$$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})|$$

$$=|[\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})}\frac{(r_1-r'_1)}{c|\overline{r}-\overline{r}'|^2}+\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})$$

$$+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}](\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})$$

$$-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$=|\frac{1}{16\pi^2\epsilon_0 c^3}[\frac{(r_1-r'_1)((-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}+\frac{(r_1-r'_1)+(r_1-r'_{1,opp})}{(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})}\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$=|\frac{1}{16\pi^2\epsilon_0 c^3}[\frac{(r_1-r'_1)((-t-\frac{|\overline{r}+\overline{r}'-2p_{\overline{d}}|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}+\frac{2r_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R})}{(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}$$

$$\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})+[\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})}\frac{(r_1+r'_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R}))}{c|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}]$$

$$(\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y}))|$$

$$\leq\frac{Ms^2}{8\pi\epsilon_0 c^3}|\frac{(r_1-r'_1)((-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2-(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2)}{(t-\frac{|\overline{r}-\overline{r}'|}{c})|\overline{r}-\overline{r}'|^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}|$$

$$+\frac{Ms^2}{8\pi\epsilon_0 c^3}|\frac{2r_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R})}{(t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}|+|\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^2(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})}\frac{(r_1+r'_1-2\overline{z}_{\overline{y},1}+O(\frac{1}{R}))}{c|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|^2}|$$

$$|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

$$\leq\frac{Ms^2}{\pi\epsilon_0 c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0 c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0 c^3}\frac{1}{|(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})||\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}$$

$$|\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$$

$$(P)$$

where, we follow the method in (ii) below, noting the $O(|\overline{r}'|^3)$ term cancels in the first long term to obtain $\frac{O(|\overline{r}'|)O(|\overline{r}'|^2)}{O(|\overline{r}'|^6)}=\frac{1}{O(|\overline{r}'|^3)}$.

Change coordinates, so that the azimuth angle θ of the sphere $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$ is centred on the line passing through $\{\overline{r}',\overline{y}'\}$, giving coordinates;

$\overline{r}'+sin(\theta)cos(\phi)\overline{x}+sin(\theta)sin(\phi)\overline{y}+cos(\theta)(\overline{y}'-\overline{r}')$

$(0\leq\theta\leq\pi,-\pi\leq\phi\leq\pi)$

for a choice of orthogonal vectors $\{\overline{x},\overline{y},\overline{y}'-\overline{r}'\}$ with modulus $-ct+|\overline{r}-\overline{r}'|$. Similarly, choose the azimuth angle θ_{opp} of the sphere $\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)$ is centred on the line passing through $\{\overline{r}'_{opp},\overline{y}'_{opp}\}$, giving coordinates;

$\overline{r}'+sin(\theta_{opp})cos(\phi_{opp})\overline{x}_{opp}+sin(\theta_{opp})sin(\phi_{opp})\overline{y}_{opp}+cos(\theta_{opp})(\overline{y}'_{opp}-\overline{r}'_{opp})$

$(0\leq\theta_{opp}\leq\pi,-\pi\leq\phi_{opp}\leq\pi)$

for a choice of orthogonal vectors $\{\overline{x}_{opp},\overline{y}_{opp},\overline{y}'_{opp}-\overline{r}'_{opp}\}$ with modulus $ct+|\overline{r}-\overline{r}'_{opp}|$. We have, for points $\{\overline{q}',\overline{q}'_{opp}\}$ of intersection between $B(\overline{0},s)$ and $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, $B(\overline{0},s)$ and $\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)$ that;

$\theta(\overline{q}')\simeq sin(\theta(\overline{q}'))\leq\frac{2s}{-ct+|\overline{r}-\overline{r}'|}$

$\theta_{opp}(\overline{q}'_{opp})\simeq sin(\theta_{opp}(\overline{q}'_{opp}))\leq\frac{2s}{ct+|\overline{r}-\overline{r}'_{opp}|}$ (TT)

Let $\{m,m',m'_{opp}\}$ be perpendicular lines to the asymptotic line l containing $\{\overline{r}',\overline{r}'_{opp}\}$ with centre $\overline{z}_{\overline{y}}+O(\frac{1}{R})$, passing through the points $\{\overline{y},\overline{y}',\overline{y}'_{opp}\}$, with $\overline{p}=m\cap l$, $\overline{p}'=m'\cap l$, $\overline{p}'_{opp}=m'_{opp}\cap l$. Let $\{P,P',P'_{opp}\}$ be planes passing through $\{\overline{y},\overline{y}',\overline{y}'_{opp}\}$, perpendicular to the lines formed by translating l by the vectors $\{\overline{y}'-\overline{p}',\overline{y}'_{opp}-\overline{p}'_{opp}\}$ respectively. Let $v=|\overline{y}-\overline{p}|$, $v'=|\overline{y}'-\overline{p}'|$, $v'_{opp}=|\overline{y}'_{opp}-\overline{p}'_{opp}|$, $k'=|\overline{r}'-\overline{p}'|$, $k'_{opp}=|\overline{r}'_{opp}-\overline{p}'_{opp}$, then by elementary trigonometry, the angles $\{\alpha',\alpha'_{opp}\}$ between the lines $\{l,l_{\overline{y}',\overline{r}'}\}$ and $\{l,l_{\overline{y}'_{opp},\overline{r}'_{opp}}\}$ are given by;

$\alpha'\simeq tan(\alpha')=\frac{v'}{k'}=\frac{|\overline{y}'-\overline{p}'|}{|\overline{r}'-\overline{p}'|}$

$\alpha'_{opp}\simeq tan(\alpha'_{opp})=\frac{v'_{opp}}{k'_{opp}}=\frac{|\overline{y}'_{opp}-\overline{p}'_{opp}|}{|\overline{r}'_{opp}-\overline{p}'_{opp}|}=O(\frac{1}{R})$ (LM)

We have, for vectors $\{\overline{u},\overline{v},\overline{w}\}$, that;

$|\overline{u}-\overline{w}|\geq|\overline{u}-\overline{v}|-|\overline{w}-\overline{v}|$

so that;

$|\overline{r}'-\overline{p}'|\geq|\overline{r}'-\overline{z}_{\overline{y}}|-|\overline{p}'-\overline{z}_{\overline{y}}|$

$|\overline{r}'_{opp}-\overline{p}'_{opp}|\geq|\overline{r}'_{opp}-\overline{z}_{\overline{y}}|-|\overline{p}'_{opp}-\overline{z}_{\overline{y}}|$

$=|\overline{r}'-\overline{z}_{\overline{y}}+O(\frac{1}{R})|-|\overline{p}'_{opp}-\overline{z}_{\overline{y}}|$

and, moreover;

$|\overline{r}'|-|\overline{z}_{\overline{y}}|\leq|\overline{r}'-\overline{z}_{\overline{y}}|=|\overline{r}'_{opp}-\overline{z}_{\overline{y}}+O(\frac{1}{R})|\leq|\overline{r}'|+|\overline{z}_{\overline{y}}|$

so that;

$|\overline{r}'-\overline{z}_{\overline{y}}|=O(R)$, $|\overline{r}'_{opp}-\overline{z}_{\overline{y}}|=O(R)$

where $R=|\overline{r}'|$, and, using (LM), $\alpha=O(\frac{1}{R})$, $\alpha'=O(\frac{1}{R})$. Then, it is clear that that the maximal distance between points $\overline{q}'$ on the arc $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ and the orthogonal projections $pr^2(\overline{q}')$ onto the plane P' is at most $\alpha's=O(\frac{1}{R})$, and similarly, the maximal distance between points $\overline{q}'_{opp}$ on the arc $\delta B(\overline{r}',=ct+|\overline{r}-\overline{r}'_{opp}|)\cap B(\overline{0},s)$ and the orthogonal projections $pr^2(\overline{q}'_{opp})$ onto the plane P'_{opp} is at most $\alpha'_{opp}s=O(\frac{1}{R})$. Similarly, as the orthogonal distances between P' and P'_{opp} is $|\overline{y}'-\overline{y}'_{opp}|=O(\frac{1}{R})$, we can, for sufficiently large R, choose $\{\overline{x},\overline{y},\overline{x}_{opp},\overline{y}_{opp}\}$ compatibly, such that, uniformly;

$|\overline{q}'-\overline{q}'_{opp}|=O(\frac{1}{R})=\epsilon(R)$

for $\{\overline{q}',\overline{q}'_{opp}\}$ defined by coordinates $\theta=\theta_{opp}$, $\phi=\phi_{opp}$ with $0\leq\theta\leq max(\theta_{max},\theta_{max,opp})$, where;

$\theta_{max}=max_{0\leq\phi\leq 2\pi}\theta(\overline{q}')=O(\frac{1}{R})$

for $\overline{q}'$ in $B(\overline{0},s)\cap\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)$, with coordinates $\{\theta,\phi\}$, and;

$\theta_{max,opp}=max_{0\leq\phi\leq 2\pi}\theta_{opp}(\overline{q}'_{opp})=O(\frac{1}{R}$

for $\overline{q}'_{opp}$ in $B(\overline{0},s)\cap\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)$, with coordinates $\{\theta_{opp},\phi_{opp}\}$

It follows that, for sufficiently large R, using the surface measure $dS=r^2sin(\theta)$, the fact (TT) and $r^2(1-cos(\frac{1}{r}))=O(1)$, and footnote

2, for sufficiently large r;

$|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})-\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)dS(\overline{y})|$

$\leq 2\epsilon(R)|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(ct+|\overline{r}-\overline{r}'_{opp}|)^2\int_0^{max(\theta_{max},\theta_{max,opp})}sin(\theta)d\theta$

$=2\epsilon(R)|\bigtriangledown(((\frac{\partial^2\rho}{\partial t^2})_0))|_{B(\overline{0},s)}|2\pi^2(ct+|\overline{r}-\overline{r}'_{opp}|)^2(1-cos(max(\theta_{max},\theta_{max,opp})))$

$\leq C\epsilon(R)$

$\leq \frac{D}{|\overline{r}'|}$

where $\{C,D\}\subset\mathcal{R}_{>0}$.

It follows from (P), for sufficiently large $r(\epsilon)$, following the method of (ii), that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(t-\frac{|\overline{r}-\overline{r}'|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})(\frac{\partial^2\rho}{\partial t^2})(\overline{y},0)]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{Ms^2}{\pi\epsilon_0c^3|\overline{r}'|^3}+\frac{Ms^2}{2\pi\epsilon_0c^4|\overline{r}'|^3}+\frac{1}{16\pi^2\epsilon_0c^3}\frac{D}{|\overline{r}'|}\frac{1}{|(-t-\frac{|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}{c})||\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}|}$

$\leq \frac{E_1}{|\overline{r}'|^3}$

where $E_1\in\mathcal{R}_{>0}$.

(ii). Using the facts that $|\frac{\partial\rho}{\partial t}|_0|\leq M$ on $B(\overline{0},s)$, the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$ is at most $2\pi s^2$, $\overline{r}'_{opp}=2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})$, we have, for sufficiently large $R=|\overline{r}'$, that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}(\frac{\partial\rho}{\partial t}(\overline{y},0))]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{1}{4\pi\epsilon_0c}\frac{2\pi Ms^2}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}+\frac{1}{4\pi\epsilon_0c}\frac{2\pi Ms^2}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}$

$=\frac{Ms^2}{8\pi c\epsilon_0(ct-|\overline{r}-\overline{r}'|)^2|\overline{r}-\overline{r}'|}+\frac{Ms^2}{8\pi c\epsilon_0(-ct-|\overline{r}_1+\overline{r}'|)^2|\overline{r}_1+\overline{r}'|}$

$=\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}-\overline{r}'|^3|\frac{ct}{|\overline{r}-\overline{r}'|}+1|^2}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3|(\frac{-ct}{|\overline{r}_1+\overline{r}'|}-1)|^2}$

$\leq \frac{Ms^2}{4\pi c\epsilon_0|\overline{r}-\overline{r}'|^3}+\frac{Ms^2}{8\pi c\epsilon_0|\overline{r}_1+\overline{r}'|^3}$

$\leq \frac{3Ms^2}{8\pi c\epsilon_0|\overline{r}'|^3}$

$= \frac{E_2}{|\overline{r}'|^3}$

where $\overline{r}_1=\overline{r}-2\overline{z}_{\overline{y}}+O(\frac{1}{R})$, $E_2\in\mathcal{R}_{>0}$.

(iii). We have that;

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$=|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}(-ct+|\overline{r}-\overline{r}'|)\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y}))]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}(ct+|\overline{r}-\overline{r}'_{opp}|)\int_{\delta B(\overline{r}'_{opp},-ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot$
$(\overline{z}_{opp}(\overline{y})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq \frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}'|)}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}_{opp}(\overline{y})dS(\overline{y})|$
(NN)

Letting $\overline{z}_0=\frac{(\overline{y}'-\overline{r}')}{-ct+|\overline{r}-\overline{r}'|}$, so that $|\overline{z}_0|=1$, R the surface measure of $\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)\cap B(\overline{0},s)$, using Lemma 0.2, following the method of (i), we have that, for sufficiently large R;

$|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot\overline{z}(\overline{y})dS(\overline{y})|$

$=|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})+\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot$

$\overline{z_0}dS(\overline{y})|$

$\leq|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\centerdot(\overline{z}(\overline{y})-\overline{z_0})dS(\overline{y})|+|[\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)]\centerdot$

$\overline{z_0}dS(\overline{y})|$

$\leq Rmax_{\overline{y}\in B(\overline{0},s)}|D(\frac{\partial\rho}{\partial t})(\overline{y},0)||\overline{z}(\overline{y})-\overline{z_0}|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})\cdot \overline{z_0}|$

$\leq RMmax_{\overline{y}\in B(\overline{0},s)}|\overline{z}(\overline{y})-\overline{z_0}|+|\overline{z_0}||\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RM|(1-cos(\theta_{max}),sin(\theta_{max})|+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})$

$-\int_{P'}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|+|\int_{P'}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$=\sqrt{2}RM(1-cos(\theta_{max}))^{\frac{1}{2}}+|\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})-\int_{P_{\overline{d}}}D(\frac{\partial\rho}{\partial t})(\overline{y},0)dS(\overline{y})|$

$\leq RMF\theta_{max}+O(\frac{1}{R})$

$\leq \frac{2sH}{-ct+|\overline{r}-\overline{r}'|}+\frac{W}{|\overline{r}'|}$

$=\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|\overline{r}'|}$

where $\{F,G,W,H,A_1,B_1\}\subset\mathcal{R}_{>0}$. Similarly, using P'_{opp}, there exist $\{A_2,B_2\}\subset\mathcal{R}_{>0}$, such that

$|\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot\overline{z}(\overline{y})dS(\overline{y})|\leq\frac{A_2}{ct+|\overline{r}-\overline{r}'_{opp}|}+\frac{B_2}{|1+\overline{r}'_{opp}|}$

$=\frac{A_2}{ct+|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}+\frac{B_2}{|2\overline{z}_{\overline{y}}-\overline{r}'|+O(\frac{1}{R})}$

so that, from (NN), following the method of (ii)

$|\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2}\int_{\delta B(\overline{r}',-ct+|\overline{r}-\overline{r}'|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}')]dS(\overline{y})\frac{(r_1-r_1')}{c|\overline{r}-\overline{r}'|^2}$

$+\frac{1}{4\pi\epsilon_0}[\frac{1}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2}\int_{\delta B(\overline{r}'_{opp},ct+|\overline{r}-\overline{r}'_{opp}|)}D(\frac{\partial\rho}{\partial t})(\overline{y},0)\cdot(\overline{y}-\overline{r}'_{opp})]dS(\overline{y})\frac{(r_1-r'_{1,opp})}{c|\overline{r}-\overline{r}'_{opp}|^2}|$

$\leq\frac{1}{4\pi\epsilon_0 c}\frac{(-ct+|\overline{r}-\overline{r}')}{4\pi c^2(t-\frac{|\overline{r}-\overline{r}'|}{c})^2|\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$

$+\frac{1}{4\pi\epsilon_0 c}\frac{(ct+|\overline{r}-\overline{r}'_{opp})}{4\pi c^2(-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c})^2|\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{ct+|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}+\frac{B_2}{|1+2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})|})$

$=\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|(t-\frac{|\overline{r}-\overline{r}'|}{c})||\overline{r}-\overline{r}'|}(\frac{A_1}{-ct+|\overline{r}-\overline{r}'|}+\frac{B_1}{|1+\overline{r}'|})$

$+\frac{1}{16\pi^2\epsilon_0 c^2}\frac{1}{|-t-\frac{|\overline{r}-\overline{r}'_{opp}|}{c}||\overline{r}-\overline{r}'_{opp}|}(\frac{A_2}{ct+|\overline{r}+\overline{r}'-2\overline{z}_{\overline{y}}+O(\frac{1}{R})|}+\frac{B_2}{|1+2\overline{z}_{\overline{y}}-\overline{r}'+O(\frac{1}{R})|})$

$\leq\frac{E_3}{|\overline{r}'|^3}$

where $E_3 \in \mathcal{R}_{>0}$ $((i),(ii),(iii))$

$\square$

Definition 0.6. *For the blow up circles* $\{S_{1,a}, S_{2,a}\}$, *we define the corresponding shifted asymptotic cones* $\{SCone(S_{1,a}), SCone(S_{2,a})\}$ *by;*

$SCone(S_{1,a}) = \bigcup_{\overline{y}\in S_{1,a}} l_{\overline{0},\overline{y},sh}$

$SCone(S_{2,a}) = \bigcup_{\overline{y}\in S_{2,a}} l_{\overline{0},\overline{y},sh}$

Fix base points $\overline{y}_{1,a} \in S_{1,a}$ *and* $\overline{y}_{2,a} \in S_{2,a}$, *the circles having centres* $\{\overline{c}_{1,a}, \overline{c}_{2,a}\}$ *with radii* $\{r_{1,a}, r_{2,a}\}$ *and points on the circle* $\{\overline{z}_{1,a}, \overline{z}_{2,a}\}$, *such that* $l_{\overline{c}_{i,a},\overline{y}_{i,a}}$ *and* $l_{\overline{c}_{i,a},\overline{z}_{i,a}}$ *are perpendicular for* $1 \leq i \leq 2$ *then we can define parameterisations* $\beta_1 : [0, 2\pi) \rightarrow S_{1,a}$, $\beta_2 : [0, 2\pi) \rightarrow S_{2,a}$, *by;*

$\beta_i(\gamma) = \overline{c}_{i,a} + r_{i,a}(\overline{y}_{i,a} - \overline{c}_{i,a})cos(\gamma) + r_{i,a}(\overline{z}_{i,a} - \overline{c}_{i,a})sin(\gamma)$

We define the maps $\{\theta_1, \theta_2\}$, $\theta_i : \mathcal{R} \times (0, 2\pi) \rightarrow SCone(S_{i,a})$, $1 \leq i \leq 2$, *by;*

$\theta_i(r, \gamma) = \overline{z}_{\beta_i(\gamma)} + \frac{r}{w}\beta_i(\gamma)$

where, for $\overline{y} \in S_{i,a}$, $\overline{u}_{\overline{y}}$ *has modulus* $|\frac{-b(\overline{y})}{r_2\gamma(\overline{y})}|$ *with* $\overline{u}_{\overline{y}} \in S_{1,\overline{y},a}$ *perpendicular to* $\overline{y} \in l_{\overline{0},\overline{y}}$.

Lemma 0.7. *Cancellation along the shifted asymptotic cone and* $V_w(\overline{x})$

Proof. Using the notation above, we have that, for $i \in \{1, 2\}$

(i). $\theta_i(0, \gamma) = \overline{u}_{\beta_i(\gamma)}$

(ii). $\theta_i(r, \gamma)_{opp} = \theta_i(-r, \gamma) + O(\frac{1}{r})$, for sufficiently large $r > 0$, ([14]).

[14] As, by the above, if $\overline{r}' = \theta_i(r, \gamma)$, then;

$\overline{r}'_{opp} = -(\overline{r}' - \overline{v}_{\beta_i(\gamma)}) + \overline{w}_{\beta_i(\gamma)} + \overline{v}_{\beta_i(\gamma)} + O(\frac{1}{R});$

$= -\overline{r}' + (2\overline{v}_{\beta_i(\gamma)} + \overline{w}_{\beta_i(\gamma)}) + O(\frac{1}{r})$

so that $|\overline{r}' - \overline{z}_{\beta_i(\gamma)}| = |\overline{r}'_{opp} - \overline{z}_{\beta_i(\gamma)}| + O(\frac{1}{r})$

where $\overline{z}_{\beta_i(\gamma)} = \frac{1}{2}(2\overline{v}_{\beta_i(\gamma)} + \overline{w}_{\beta_i(\gamma)})$ and $\overline{z}_{\beta_i(\gamma)} \in l_{\overline{0},\beta_i(\gamma),sh}$.

(iii). There exist $R_i \subset \mathcal{R}_{>0}$ with θ_i diffeomorphisms outside $[-R_i, R_i] \times [0, 2\pi)$, with the partial derivatives uniformly bounded.

(iv). $Im(\theta_1|_{\mathcal{R}\setminus[-R_1,R_1]\times[0,2\pi)} \cap Im(\theta_2|_{\mathcal{R}\setminus[-R_2,R_2]\times[0,2\pi)}) = \emptyset$

(v). For $r_2 > r_1 > R_i$, $|\theta_i(r_2, \gamma) - \theta_i(r_1, \gamma)| = r_2 - r_1$, and for $r_2 < r_1 < -R_i$, $|\theta_i(r_2, \gamma) - \theta_i(r_1, \gamma)| = r_1 - r_2$

It follows from $(iii), (v)$ that, for $1 \leq i \leq 2$, the pullbacks;

$$\theta_i|^*_{\mathcal{R}\setminus[-R_1,R_1]\times[0,2\pi)}(dLeb|_{SCone(S_{i,a})}) = |\frac{\partial \theta_1}{\partial r} \times \frac{\partial \theta_1}{\partial \gamma}|drd\gamma = f(r, \gamma)drd\gamma$$

has the property that $f(r, \gamma)$ has order $O(r)$, uniformly in γ and $f(r, \gamma) = f(-r, \gamma)$, for $r \in \mathcal{R}_{>0}$. For $R \in \mathcal{R}_{>0}$, with $R > R_i$, we can define the regions $S_{R,i} \subset \mathcal{R} \times [\alpha, \beta)$, by;

$$S_{R,i} = \{(r', \gamma) : R_i \leq |r'| \leq R, \gamma \in [\alpha, \beta)\}$$

with corresponding regions $\theta_i(S_{R,i}) \subset SCone(S_{i,a})$

Then, by the calculation above, using fact (ii), Lemma 0.5 and the mean value theorem, letting;

$$H^+(\overline{r}') = (\frac{1}{4\pi\epsilon_0} \frac{\dot{\rho}(\overline{r}', t_r)\hat{\mathfrak{r}}}{|\overline{r} - \overline{r}'|})_1$$

$$H^-(\overline{r}') = (\frac{1}{4\pi\epsilon_0} \frac{\dot{\rho}(\overline{r}', -t_r)\hat{\mathfrak{r}}}{|\overline{r} - \overline{r}'|})_1$$

where by t_r we mean $t - \frac{|\overline{r}-\overline{r}'|}{c}$ and by $-t_r$ we mean $-t - \frac{|\overline{r}-\overline{r}'|}{c}$, we have that, for $r > R_i$;

$$|\theta_1^* H^+(r, \gamma) + \theta_1^* H^-(-r, \gamma)|$$

$$= |H^+(\overline{r}') + H^-(\overline{r}'_{opp} + O(\tfrac{1}{r}))|$$

$$\leq |H^+(\overline{r}') + H^-(\overline{r}'_{opp})| + |H^-(\overline{r}'_{opp} + O(\tfrac{1}{r})) - H^-(\overline{r}'_{opp})|$$

$$\leq \frac{C}{r^3} + |H^-(\overline{r}'_{opp} + O(\tfrac{1}{r})) - H^-(\overline{r}'_{opp})|$$

$$= \frac{C}{r^3} + |DH^-(\overline{r}'_1) \centerdot O(\tfrac{1}{r})|$$

$\leq \frac{C}{r^3}+\frac{E}{r}|\bigtriangledown(H^{-})(\overline{r}'_1)|$ (YY)

where $|\overline{r}'_1-\overline{r}'_{opp}|=O(\frac{1}{r})$

We have that;

$|\bigtriangledown(H^{-})(\overline{r}'_1)|\leq\sqrt{3}max_{1\leq i\leq 3}(\frac{\partial H^{-}}{\partial r'_i})|_{\overline{r}'_1}$

and;

$\frac{\partial H^{-}}{\partial r'_i}|_{\overline{r}'_1}=\frac{1}{4\pi\epsilon_0}[[(\frac{\partial\dot{\rho}}{\partial r'_i}(\overline{r}',-t_r)+\frac{\partial^2\rho}{\partial^2 t}(\overline{r}',-t_r)\frac{(r_i-r'_i)}{c|\overline{r}-\overline{r}_i|})]\frac{r_1-r'_1}{|\overline{r}-\overline{r}'|^2}-\frac{\dot{\rho}(\overline{r}',-t_r)}{|\overline{r}-\overline{r}'|^2}+\frac{2\dot{\rho}(\overline{r}',-t_r)|r_1-r'_1|^2}{|\overline{r}-\overline{r}'|^3}]]|_{\overline{r}'_1}$

so that, using the fact that $|\dot{\rho}|\leq M$ for some $M\in\mathcal{R}_{>0}$;

$|\frac{\partial H}{\partial r'_i}|_{\overline{r}'_1}|\leq[\frac{|\frac{\partial\dot{\rho}}{\partial r'_i}(\overline{r}',-t_r)|+|\frac{1}{c}\frac{\partial^2\rho}{\partial^2 t}(\overline{r}',-t_r)|}{4\pi\epsilon_0|\overline{r}-\overline{r}'|}+\frac{M}{4\pi\epsilon_0|\overline{r}-\overline{r}'|^2}+\frac{\dot{\rho}(\overline{r}',-t_r)}{2\pi\epsilon_0|\overline{r}-\overline{r}'|}]]|_{\overline{r}'_1}$

We have that ρ obeys the wave equation $\bigtriangledown^2(\rho)+\frac{1}{c^2}\frac{\partial^2\rho}{\partial^2 t}=0$, determined by the initial conditions $\{\rho_0,(\frac{\partial\rho}{\partial t})_0\}$, so that $\dot{\rho}$ obeys the same wave equation determined by the initial conditions $\{(\frac{\partial\rho}{\partial t})_0,-c^2\bigtriangledown^2(\rho_0)\}$, $\frac{\partial^2\rho}{\partial t^2}$ obeys the wave equation determined by the initial conditions $\{-c^2\bigtriangledown^2(\rho_0),-c^2\bigtriangledown^2(\frac{\partial\rho_0}{\partial t})\}$, $\frac{\partial\dot{\rho}}{\partial r'_i}$, $1\leq i\leq 3$, obeys the wave equation determined by the initial conditions $\{\frac{\partial^2\rho_0}{\partial t\partial r'_i},-c^2\bigtriangledown^2(\frac{\partial\rho_0}{\partial r_i})\}$.

Using Kirchoff's formula, it follows that there exist $\{D_{1i},E_{1i},D_2,E_2,D_3,E_3\}\subset\mathcal{R}_{>0}$, for $1\leq i\leq 3$, such that, for sufficiently large $|\overline{r}'|$;

$|\frac{\partial\dot{\rho}}{\partial r'_i}|_{\overline{r}',-t_r}\leq\frac{D_{1i}}{-t-\frac{|\overline{r}-\overline{r}'|}{c}}\leq\frac{E_{1i}}{|\overline{r}'|}$

$|\frac{\partial^2\rho}{\partial t^2}|_{\overline{r}',-t_r}\leq\frac{D_2}{-t-\frac{|\overline{r}-\overline{r}'|}{c}}\leq\frac{E_2}{|\overline{r}'|}$

$|\dot{\rho}|_{\overline{r}',-t_r}\leq\frac{D_3}{-t-\frac{|\overline{r}-\overline{r}'|}{c}}\leq\frac{E_3}{|\overline{r}'|}$

so that, for sufficiently large $|\overline{r}'|$, there exists $\{G,H,K_i\}\subset\mathcal{R}_{>0}$, for $1\leq i\leq 3$;

$|\frac{\partial H}{\partial r'_i}|_{\overline{r}'_1}|\leq[\frac{1}{4\pi\epsilon_0|\overline{r}-\overline{r}'|}(\frac{E_{1i}}{|\overline{r}'|}+\frac{E_2}{|\overline{r}'|}+\frac{M}{|\overline{r}-\overline{r}'|^2}+\frac{2E_3}{|\overline{r}'|})]|_{\overline{r}'_1}$

$\leq [\frac{G}{4\pi\epsilon_0|\overline{r}'|}(\frac{E_{1i}}{|\overline{r}'|}+\frac{E_2}{|\overline{r}'|}+\frac{MH}{|\overline{r}'|^2}+\frac{2E_3}{|\overline{r}'|})]|_{\overline{r}'_1}$

$\leq \frac{K_i}{|\overline{r}'|^2}|_{\overline{r}'_1}$

$= \frac{K_i}{|\overline{r}'_1|^2}$

and, for some $\{X,Y,Z\}\subset\mathcal{R}_{>0}$;

$|\bigtriangledown(H)(\overline{r}'_1)|\leq\frac{\sqrt{3}max_{1\leq i\leq 3}K_i}{|\overline{r}'_1|^2}$

$=\frac{\sqrt{3}max_{1\leq i\leq 3}K_i}{|\overline{r}'_{opp}+O(\frac{1}{r})|^2}$

$\leq\frac{X}{|\overline{r}'_{opp}|^2}$

$\leq\frac{Y}{|\overline{r}'|^2}$

$\leq\frac{Z}{r^2}$

so that, from (YY)

$|\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma)|$

$\leq\frac{C}{r^3}+\frac{E}{r}\frac{Z}{r^2}$

$=\frac{C+EZ}{r^3}$ $(*)$

We also have;

$|f(r,\gamma)|\leq Dr$

$|(\theta_1^*H^+(r,\gamma)+\theta_1^*H^-(-r,\gamma))f(r,\gamma)|\leq\frac{(C+F)D}{r^2}$

where $\{D,F\}\subset\mathcal{R}_{>0}$, $F=EZ$, so that;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}(H^++H^-)(\overline{r}')d\overline{r}'$

$=lim_{R\rightarrow\infty,R>R_i}\int_{S_{R,i}}(\theta_1^*(H^++H^-))(r,\gamma)f(r,\gamma)drd\gamma$

$=lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}[\int_{R_i}^R\theta_1^*H^+(r,\gamma)f(r,\gamma)dr+\int_{-R}^{-R_i}\theta_1^*H^-(r,\gamma)f(r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}[\int_{R_i}^{R}\theta_1^* H^+(r,\gamma)f(r,\gamma)dr+\int_{-R}^{-R_i}\theta_1^* H^-(-r,\gamma)f(-r,\gamma)dr]d\gamma$

$= lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}\int_{R_i}^{R}(\theta_1^* H^+(r,\gamma)+\theta_1^* H^-(-r,\gamma))f(r,\gamma)drd\gamma$

$= \int_{[0,2\pi)}\int_{R_i}^{\infty}(\theta_1^* H^+(r,\gamma)+\theta_1^* H^-(-r,\gamma))f(r,\gamma)drd\gamma$

where, letting $G(\gamma)=\int_{R_i}^{\infty}(\theta_1^* H^+(r,\gamma)+\theta_1^* H^-(-r,\gamma))f(r,\gamma)dr$;

$|G(\gamma)|\leq\int_{R_i}^{\infty}\frac{CD}{r^2}dr=[\frac{-CD}{r}]_{R_i}^{\infty}$

$=\frac{CD}{R_i}$

so that;

$lim_{R\rightarrow\infty,R>R_i}\int_{[0,2\pi)}\int_{R_i}^{\infty}(\theta_1^* H^+(r,\gamma)+\theta_1^* H^-(-r,\gamma))f(r,\gamma)drd\gamma=\int_{[\alpha,\beta)}G(\gamma)d\gamma$

exists and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{[0,\pi)}\int_{R_i}^{\infty}(\theta_1^* H^+(r,\gamma)+\theta_1^* H^-(-r,\gamma))f(r,\gamma)drd\gamma|\leq\frac{CD(\beta-\alpha)}{R_i}$

It follows;

$lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}(H^+ + H^-)(\overline{r}')d\overline{r}'$

exists, and;

$|lim_{R\rightarrow\infty,R>R_i}\int_{\theta_i(S_{R,i})}(H^+ + H^-)(\overline{r}')d\overline{r}'|\leq\frac{CD(\beta-\alpha)}{R_i}$

as well. (UU)

Idea for $V_w(\overline{x})$, using calculation $(*)$ above;

With the same notation as above, for sufficiently large R, letting $\overline{r}''\in V_w(\overline{x})$, with $\overline{r}'=pr^*(\overline{r}'')$, with pr^* the orthogonal projection of $S_{1,\overline{y},a}$ onto the asymptotic line $l_{\overline{0},\overline{y},sh}$, $\overline{r}'_{opp}$ the opposite point to $\overline{r}'$ and $\overline{r}''_{opp}$ the nearest point to $\overline{r}'_{opp}$ on $V_w(\overline{x})\cap S_{1,\overline{y},a}$. Let $dV_{\overline{y}}$ be the restriction of Lebesgue measure to $V_w(\overline{x})\cap S_{1,\overline{y},a}$, $dZ_{\overline{y}}$ the restriction of Lebesgue measure to $l_{\overline{0},\overline{y},sh}=SCone_{1,a}\cap S_{1,\overline{y},a}$.

Using the notation above, we have that;

$||\frac{-b(0)}{\theta r_2\gamma(0)}|-R|=\epsilon_{\overline{y}}+O(\frac{1}{R})$

$\overline{r}''=\overline{r}'+O(\frac{1}{R})$

$\overline{r}'=\frac{R\overline{y}}{w}+\overline{v}_{\overline{y}}$

so that;

$\overline{r}''(\theta)=\frac{R\overline{y}}{w}+\overline{v}_{\overline{y}}+O(\frac{1}{R})$

$=(\frac{-b(0)}{\theta r_2\gamma(0)w}+\frac{\epsilon_{\overline{y}}}{w}+O(\frac{1}{R}))\overline{y}+\overline{v}_{\overline{y}}+O(\frac{1}{R})$

$=(\frac{-b(0)}{\theta r_2\gamma(0)w}+\frac{\epsilon_{\overline{y}}}{w})\overline{y}+\overline{v}_{\overline{y}}+O(\frac{1}{R})$

$=(\frac{-b(0)}{\theta r_2\gamma(0)w}+\frac{\epsilon_{\overline{y}}}{w})\overline{y}+\overline{v}_{\overline{y}}+\delta(\theta)$

where $\delta(\theta)=O(\frac{1}{R})$ is analytic in θ, so that $|\delta'(\theta)|\leq N$, for some $N\in\mathcal{R}_{>0}$. It follows that;

$\frac{d\overline{r}''}{d\theta}=\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\delta'(\theta)$

It follows that, using Newton's expansion;

$\frac{|\frac{d\overline{r}''(\theta)}{d\theta}|}{|pr^*(\frac{d\overline{r}''(\theta)}{d\theta})|}=\frac{|\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\delta'(\theta)|}{|pr^*(\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w})+pr^*(\delta'(\theta))|}$

$=\frac{|\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\delta'(\theta)|}{|\frac{b(0)\overline{y}}{\theta^2 r_2\gamma(0)w}+\overline{v}_{\overline{y}}+pr^*(\delta'(\theta))|}$

$=\frac{|\frac{b(0)\overline{y}}{r_2\gamma(0)w}+\theta^2\delta'(\theta)|}{|\frac{b(0)\overline{y}}{r_2\gamma(0)w}+\theta^2(\overline{v}_{\overline{y}}+pr^*(\delta'(\theta)))|}$

$=\frac{(1+\theta^2\frac{r_2^2\gamma(0)^2w^2|\delta'(\theta)|^2}{b(0)^2|\overline{y}|^2})^{\frac{1}{2}}}{(1+\theta^2\frac{r_2^2\gamma(0)^2w^2|\overline{v}_{\overline{y}}+pr^*(\delta'(\theta))|^2}{b(0)^2|\overline{y}|^2})^{\frac{1}{2}}}$

$=1+O(\theta^2)$

$=1+O(\frac{1}{R^2})\ (SS)$

so that;

$dV_{\overline{y}}(\overline{r}'')=dZ_{\overline{y}}(\overline{r}')+O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}')$

and, similarly;

$dV_{\overline{y}}(\overline{r}''_{opp})=dZ_{\overline{y}}(\overline{r}'_{opp})+O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}'_{opp})$

By the above, we have that;

(i). $\overline{r}''=\overline{r}'+O(\frac{1}{R})$

(ii). $\overline{r}''_{opp}=\overline{r}'_{opp}+O(\frac{1}{R})$

(iii). $H^+(\overline{r}'')=H^+(\overline{r}')+O(\frac{1}{R^3})$

(iv). $H^-(\overline{r}''_{opp})=H^-(\overline{r}'_{opp})+O(\frac{1}{R^3})$

(v). $dV_{\overline{y}}(\overline{r}'')=dZ_{\overline{y}}(\overline{r}')+O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}')$

(vi). $dV_{\overline{y}}(\overline{r}''_{opp})=dZ_{\overline{y}}(\overline{r}'_{opp})+O(\frac{1}{R^2})dZ_{\overline{y}}(\overline{r}'_{opp})$

(vii). $H^+(\overline{r}')+H^-(\overline{r}'_{opp})=O(\frac{1}{R^3})$

$(viii)$. $dZ_{\overline{y}}(\overline{r}')=dZ_{\overline{y}}(\overline{r}'_{opp})=O(R)$

Then, using $(i)-(viii)$;

$H^+(\overline{r}'')dV_{\overline{y}}(\overline{r}'')+H^-(\overline{r}''_{opp})dV_{\overline{y}}(\overline{r}''_{opp})$

$=[H^+(\overline{r}')+O(\frac{1}{R^3})]dV_{\overline{y}}(\overline{r}'')+[H^-(\overline{r}'_{opp})+O(\frac{1}{R^3})]dV_{\overline{y}}(\overline{r}''_{opp})$

$=[H^+(\overline{r}')+O(\frac{1}{R^3})][dZ_{\overline{y}}(\overline{r}')+O(\frac{1}{R^2})dZ(\overline{r}')]+[H^-(\overline{r}'_{opp})+O(\frac{1}{R^3})][dZ_{\overline{y}}(\overline{r}'_{opp})$

$+O(\frac{1}{R^2})dZ(\overline{r}'_{opp})]$

$=H^+(\overline{r}')dZ_{\overline{y}}(\overline{r}')+H^-(\overline{r}'_{opp})dZ_{\overline{y}}(\overline{r}'_{opp})+H^+(\overline{r}')O(\frac{1}{R^2})O(R)+H^-(\overline{r}'_{opp})O(\frac{1}{R^2})O(R)$

$+O(\frac{1}{R^3})O(R)+O(\frac{1}{R^3})O(\frac{1}{R^2})O(R)+O(\frac{1}{R^3})O(R)+O(\frac{1}{R^3})O(\frac{1}{R^2})O(R)$

$=H^+(\overline{r}')dZ_{\overline{y}}(\overline{r}')+H^-(\overline{r}'_{opp})dZ_{\overline{y}}(\overline{r}'_{opp})+O(\frac{1}{R^2})$

$=O(\frac{1}{R^3})O(R)+O(\frac{1}{R^2})$

$=O(\frac{1}{R^2})$

With the same notation as above, let dV be the restriction of Lebesgue measure to $V_w(\overline{x})$, dZ the restriction of Lebesgue measure to $SCone_{1,a}$.

Choose a parametrisation $\overline{\beta}:[0,2\pi)\rightarrow S_{1,a}$. Following the calculation (SS) above, we have that, for $t\in[0,2\pi)$;

$||\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))}|-R|=\epsilon_{\overline{\beta}(t)}+O(\frac{1}{R})$

$\overline{r}''=\overline{r}'+O(\frac{1}{R})$

$\overline{r}'=\frac{R\overline{\beta}(t)}{w}+\overline{v}_{\overline{\beta}(t)}$

so that;

$\overline{r}''(\theta)=\frac{R\overline{\beta}(t)}{w}+\overline{v}_{\overline{\beta}(t)}+O(\frac{1}{R})$

$=(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w}+O(\frac{1}{R}))\overline{\beta}(t)+\overline{v}_{\overline{\beta}(t)}+O(\frac{1}{R})$

$=(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w})\overline{\beta}(t)+\overline{v}_{\overline{\beta}(t)}+O(\frac{1}{R})$

$=(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w})\overline{\beta}(t)+\overline{v}_{\overline{\beta}(t)}+\delta(\theta,t)$

where $\delta(\theta,t)=O(\frac{1}{R})$, uniformly in t, and is analytic in θ and t, so that $max(|\frac{\partial\delta}{\partial\theta}|,|\frac{\partial\delta}{\partial t}|)\leq N$, for some $N\in\mathcal{R}_{>0}$. It follows that;

$\frac{\partial\overline{r}''}{\partial\theta}=\frac{b(\overline{\beta}(t))\overline{\beta}(t)}{\theta^2 r_2\gamma(\overline{\beta}(t))w}+\frac{\partial\delta(\theta,t)}{\partial\theta}$

$=\frac{A_1(t)}{\theta^2}\overline{\beta}(t)+\frac{\partial\delta'(\theta,t)}{\partial\theta}$

where $A_1(t)=\frac{b(\overline{\beta}(t))}{r_2\gamma(\overline{\beta}(t))w}$

$\frac{\partial\overline{r}''}{\partial t}=(\frac{(-b\circ\overline{\beta})'(t)}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{b(\overline{\beta}(t))(\gamma\circ\overline{\beta})'(t)}{\theta r_2(\gamma\circ\overline{\beta})^2(t)w}+\frac{(\epsilon\circ\overline{\beta})'(t)}{w})\overline{\beta}(t)+(\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}+\frac{\epsilon_{\overline{\beta}(t)}}{w})\overline{\beta}'(t)$

$+(\overline{v}\circ\overline{\beta})'(t)+\frac{\partial\delta(\theta,t)}{\partial t}$

$=(\frac{A_2(t)}{\theta}+A_3(t))\overline{\beta}(t)+(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}'(t)+(\overline{v}\circ\overline{\beta})'(t)+\frac{\partial\delta(\theta,t)}{\partial t}$

where;

$A_2(t)=\frac{(-b\circ\overline{\beta})'(t)}{r_2\gamma(\overline{\beta}(t))w}+\frac{b(\overline{\beta}(t))(\gamma\circ\overline{\beta})'(t)}{r_2(\gamma\circ\overline{\beta})^2(t)w}$

$A_3(t)=\frac{(\epsilon\circ\overline{\beta})'(t)}{w}$

$A_4(t)=\frac{-b(\overline{\beta}(t))}{\theta r_2\gamma(\overline{\beta}(t))w}$

$A_5(t)=\frac{\epsilon_{\overline{\beta}(t)}}{w}$

so that $\{A_1,A_2,A_3,A_4,A_5\}$ are analytic and bounded on the interval $[0,2\pi]$. We have, for $t\in[0,2\pi)$, that $(\overline{v}\circ\overline{\beta})(t)\centerdot\overline{b}(t)=0$, $pr^*((\overline{v}\circ\overline{\beta})(t))=\overline{0}$, so that $pr^*((\overline{v}\circ\overline{\beta})'(t))=\overline{0}$. Similarly, $pr^*(\overline{\beta}(t))=\overline{\beta}(t)$ so that $pr^*(\overline{\beta}'(t))=\overline{\beta}'(t)$.

It follows that;

$\frac{\partial\overline{r}''(\theta)}{\partial\theta}\times\frac{\partial\overline{r}''(\theta)}{\partial t}=\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O(\frac{1}{\theta^2},t)$

$pr^*(\frac{\partial\overline{r}''(\theta)}{\partial\theta})=\frac{A_1(t)}{\theta^2}\overline{\beta}(t)+pr^*(\frac{\partial\delta'(\theta,t)}{\partial\theta})$

$=\frac{A_1(t)}{\theta^2}\overline{\beta}(t)+O(1,t)$

$pr^*(\frac{\partial\overline{r}''(\theta)}{\partial t})=(\frac{A_2(t)}{\theta}+A_3(t))\overline{\beta}(t)+(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}'(t)+pr^*((\overline{v}\circ\overline{\beta})')(t)$

$+pr^*(\frac{\partial\delta(\theta,t)}{\partial t})$

$=(\frac{A_2(t)}{\theta}+A_3(t))\overline{\beta}(t)+(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}'(t)+O(1,t)$

$pr^*(\frac{\partial\overline{r}''(\theta)}{\partial\theta})\times pr^*(\frac{\partial\overline{r}''(\theta)}{\partial t})=\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\frac{1}{\theta^2},t)$

It follows that, using Newton's expansion;

$\frac{|\frac{\partial\overline{r}''(\theta)}{\partial\theta}\times\frac{\partial\overline{r}''(\theta)}{\partial t}|}{|pr^*(\frac{\partial\overline{r}''(\theta)}{d\theta})\times pr^*(\frac{\partial\overline{r}''(\theta)}{dt})|}=\frac{|\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O(\frac{1}{\theta^2},t)|}{|\frac{A_1(t)}{\theta^2}(\frac{A_4(t)}{\theta}+A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\frac{1}{\theta^2},t)|}$

$=\frac{|(A_1(t)A_4(t)+\theta A_1(t)A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O(\theta,t)|}{|(A_1(t)A_4(t)+\theta A_1(t)A_5(t))\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\theta,t)|}$

$=\frac{|A_1(t)A_4(t)\overline{\beta}(t)\times\overline{\beta}'(t)+O(\theta,t)|}{|A_1(t)A_4(t)\overline{\beta}(t)\times\overline{\beta}'(t)+O'(\theta,t)|}$

$=\frac{(1+O''(\theta,t))^{\frac{1}{2}}}{(1+O'''(\theta,t))^{\frac{1}{2}}}$

$=(1+\frac{1}{2}O''(\theta,t)+O''''(\theta^2,t))(1-\frac{1}{2}O'''(\theta,t)+O'''''(\theta^2,t))$

$= 1 + O(\theta, t)$

$= 1 + O(\frac{1}{R}, t)$ (SSS)

so that;

$dV(\overline{r}'') = dZ(\overline{r}') + O(\frac{1}{R}, t)dZ(\overline{r}')$

and, similarly;

$dV(\overline{r}''_{opp}) = dZ(\overline{r}'_{opp}) + O(\frac{1}{R}, t)dZ(\overline{r}'_{opp})$

As above, using (SSS) now for $(v), (vi)$, we have that;

(i). $\overline{r}'' = \overline{r}' + O(\frac{1}{R}, t)$

(ii). $\overline{r}''_{opp} = \overline{r}'_{opp} + O(\frac{1}{R}, t)$

(iii). $H^+(\overline{r}'') = H^+(\overline{r}') + O(\frac{1}{R^3}, t)$

(iv). $H^-(\overline{r}''_{opp}) = H^-(\overline{r}'_{opp}) + O(\frac{1}{R^3}, t)$

(v). $dV(\overline{r}'') = dZ(\overline{r}') + O(\frac{1}{R}, t)dZ(\overline{r}')$

(vi). $dV(\overline{r}''_{opp}) = dZ(\overline{r}'_{opp}) + O(\frac{1}{R}, t)dZ(\overline{r}'_{opp})$

(vii). $H^+(\overline{r}') + H^-(\overline{r}'_{opp}) = O(\frac{1}{R^3}, t)$

$(viii)$. $dZ(\overline{r}') = dZ(\overline{r}'_{opp}) = O(R, t)$

Then, using $(i) - (viii)$;

$H^+(\overline{r}'')dV(\overline{r}'') + H^-(\overline{r}''_{opp})dV(\overline{r}''_{opp})$

$= [H^+(\overline{r}') + O(\frac{1}{R^3}, t)]dV(\overline{r}'') + [H^-(\overline{r}'_{opp}) + O(\frac{1}{R^3}, t)]dV(\overline{r}''_{opp})$

$= [H^+(\overline{r}')+O(\frac{1}{R^3}, t)][dZ(\overline{r}')+O(\frac{1}{R}, t)dZ(\overline{r}')]+[H^-(\overline{r}'_{opp})+O(\frac{1}{R^3}, t)][dZ(\overline{r}'_{opp})$

$+O(\frac{1}{R}, t)dZ(\overline{r}'_{opp})]$

$= H^+(\overline{r}')dZ(\overline{r}')+H^-(\overline{r}'_{opp})dZ(\overline{r}'_{opp})+H^+(\overline{r}')O(\frac{1}{R}, t)O(R)+H^-(\overline{r}'_{opp})O(\frac{1}{R}, t)O(R, t)$

$+O(\frac{1}{R^3},t)O(R,t)+O(\frac{1}{R^3},t)O(\frac{1}{R},t)O(R,t)+O(\frac{1}{R^3},t)O(R,t)+O(\frac{1}{R^3},t)O(\frac{1}{R},t)O(R,t)$

$= H^+(\overline{r}')dZ(\overline{r}') + H^-(\overline{r}'_{opp})dZ(\overline{r}'_{opp}) + O(\frac{1}{R^2},t)$

$= O(\frac{1}{R^3},t)O(R,t) + O(\frac{1}{R^2},t)$

$= O(\frac{1}{R^2},t)$

.... Look at argument of (UU) again, $drd\gamma$, $(0 \leq \gamma < 2\pi)$. Final integration over $0 \leq w \leq s$, exclude discrete case, use Lemma 0.3.

□

Lemma 0.8. *Let* $\{\overline{r},\overline{y}\} subset \mathcal{R}^3$, *let* $l \subset \mathcal{R}^3$ *be a line, with* $\{\overline{p},\overline{p}'\} \subset l$ *and* $\overline{p} \neq \overline{p}'$. *Then if* $\overline{x}_\lambda = \overline{p} + \lambda(\overline{p}' - \overline{p})$, *we have that;*

$lim_{\lambda\rightarrow\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|) = -lim_{\lambda\rightarrow-\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|)$

Proof. By rotating and translating coordinates (x,y,z), which preserves distance, we may assume that l is the line $y = z = 0$, $\overline{p} = \overline{0}$, $\overline{p}' = (x_0,0,0)$, $\overline{y} = (y_1,y_2,0)$ and $\overline{r} = (r_1,r_2,r_3)$. Then, using Newton's expansion;

$|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|$

$= |(\lambda x_0,0,0) - (y_1,y_2,0)| - |(\lambda x_0,0,0) - (r_1,r_2,r_3)|$

$= [(\lambda x_0 - y_1)^2 + y_2^2]^{\frac{1}{2}} - [(\lambda x_0 - r_1)^2 + r_2^2 + r_3^2]^{\frac{1}{2}}$

$= [\lambda^2 x_0^2 - 2\lambda x_0 y_1 + y^2]^{\frac{1}{2}} - [\lambda^2 x_0^2 - 2\lambda x_0 r_1 + r^2]^{\frac{1}{2}}$

$= |\lambda x_0|[1 - \frac{2y_1}{\lambda x_0} + \frac{y^2}{\lambda^2 x_0^2}]^{\frac{1}{2}} - |\lambda x_0|[1 - \frac{2r_1}{\lambda x_0} + \frac{r^2}{\lambda^2 x_0^2}]^{\frac{1}{2}}$

$= |\lambda x_0|(1 - \frac{y_1}{\lambda x_0} + O(\frac{1}{\lambda^2})) - |\lambda x_0|(1 - \frac{r_1}{\lambda x_0} + O(\frac{1}{\lambda^2}))$

$= -\frac{sign(\lambda)y_1}{x_0} + \frac{sign(\lambda)r_1}{x_0} + O(\frac{1}{\lambda})$

where $y = |\overline{y}$ and $r = |\overline{r}|$, so that;

$lim_{\lambda\rightarrow\infty}(|\overline{x}_\lambda - \overline{y}| - |\overline{x}_\lambda - \overline{r}|) = -\frac{y_1}{x_0} + \frac{r_1}{x_0}$

$$lim_{\lambda\rightarrow -\infty}(|\overline{x}_\lambda-\overline{y}|-|\overline{x}_\lambda-\overline{r}|)=\frac{y_1}{x_0}-\frac{r_1}{x_0}$$

$$=-lim_{\lambda\rightarrow\infty}(|\overline{x}_\lambda-\overline{y}|-|\overline{x}_\lambda-\overline{r}|)$$

□

Definition 0.9. *For $f\in C^\infty(\mathcal{R}^4)$ and $h\in\mathcal{R}$, we define the time shift f^h by $f^h(\overline{x},t)=f(\overline{x},t+h)$. For a field $\overline{f}$, with $\overline{f}=(f_1,f_2,f_3)$ and $f_i\in C^\infty(\mathcal{R}^4)$, $1\leq i\leq 3$, we define $\overline{f}^h=(f_1^h,f_2^h,f_3^h)$.*

Lemma 0.10. *Let $(\rho,\overline{J})$ be a charge and current configuration with $\rho\in C^\infty(\mathcal{R}^4)$, $\overline{J}=(j_1,j_2,j_3)$, and $j_i\in C^\infty(\mathcal{R}^4)$, $1\leq i\leq 3$, such that $(\rho,\overline{J})$ satisfies the continuity equation. Then, for $h\in\mathcal{R}_{>0}$, the time shifts $(\rho^h,\overline{J}^h)$ satisfy the continuity equation and so do the sums $(\rho+\rho^h,\overline{J}+\overline{J}^h)$. If for $h\in\mathcal{R}_{>0}$, there exists electric and magnetic fields $(\overline{E}_h,\overline{B}_h)$ such that $(\rho+\rho^h,\overline{J}+\overline{J}^h,\overline{E}_h,\overline{B}_h)$ satisfy Maxwell's equations, then there exist fields $\overline{E}$ and $\overline{B}$ such that $(\rho,\overline{J},\overline{E},\overline{B})$ satisfy Maxwell's equations.*

Proof. By the hypotheses, we have for $\{h_1,h_2\}\subset\mathcal{R}_{>0}$, with $h_2>h_1$ that there exist pairs $(\overline{E}_{h_1},\overline{B}_{h_1})$ and $(\overline{E}_{h_2},\overline{B}_{h_2})$ such that $(\rho+\rho^{h_1},\overline{J}+\overline{J}^{h_1},\overline{E}_{h_1},\overline{B}_{h_1}$ and $(\rho+\rho^{h_2},\overline{J}+\overline{J}^{h_2},\overline{E}_{h_2},\overline{B}_{h_2}$ satisfy Maxwell equations, so that, taking the difference, $(\rho^{h_1}-\rho^{h_2},\overline{J}^{h_1}-\overline{J}^{h_2},\overline{E}_{h_1}-\overline{E}_{h_2},\overline{B}_{h_1}-\overline{B}_{h_2})$ satisfy Maxwell's equations, $(*)$. Then $h_2-h_1>0$, so that, by the hypotheses, there exist $(\overline{E}_{h_2-h_1},\overline{B}_{h_2-h_1})$ such that $(\rho+\rho^{h_2-h_1},\overline{J}+\overline{J}^{h_2-h_1},\overline{E}_{h_2-h_1},\overline{B}_{h_2-h_1})$ satisfy Maxwell's equations, $(**)$. As is easily checked, if $(\rho,\overline{J},\overline{E},\overline{B})$ satisfy Maxwell's equations, then, for $h\in\mathcal{R}$, $(\rho^h,\overline{J}^h,\overline{E}^h,\overline{B}^h)$ satisfy Maxwell's equations, so that, from $(**)$;

$$(\rho^{h_1}+\rho^{h_2-h_1+h_1},\overline{J}^{h_1}+\overline{J}^{h_2-h_1+h_1},\overline{E}^{h_1}_{h_2-h_1},\overline{B}^{h_1}_{h_2-h_1})$$

$$=(\rho^{h_1}+\rho^{h_2},\overline{J}^{h_1}+\overline{J}^{h_2},\overline{E}^{h_1}_{h_2-h_1},\overline{B}^{h_1}_{h_2-h_1})\ (***)$$

satisfies Maxwell's equations. Then adding the equations $(*)$, $(***)$, we obtain that;

$$(2\rho^{h_1},2\overline{J}^{h_1},\overline{E}_{h_1}-\overline{E}_{h_2}+\overline{E}^{h_1}_{h_2-h_1},\overline{B}_{h_1}-\overline{B}_{h_2}+\overline{B}^{h_1}_{h_2-h_1})$$

satisfies, Maxwell's equation and;

$$(\rho^{h_1},\overline{J}^{h_1},\tfrac{1}{2}(\overline{E}_{h_1}-\overline{E}_{h_2}+\overline{E}^{h_1}_{h_2-h_1}),\tfrac{1}{2}(\overline{B}_{h_1}-\overline{B}_{h_2}+\overline{B}^{h_1}_{h_2-h_1}))$$

satisfies Maxwell's equations. Again, by the observation above, it follows that;

$$(\rho^{h_1-h_1},\overline{J}^{h_1-h_1},\tfrac{1}{2}(\overline{E}_{h_1}^{-h_1}-\overline{E}_{h_2}^{-h_1}+\overline{E}_{h_2-h_1}^{h_1-h_1}),\tfrac{1}{2}(\overline{B}_{h_1}^{-h_1}-\overline{B}_{h_2}^{-h_1}+\overline{B}_{h_2-h_1}^{h_1-h_1}))$$

$$(\rho,\overline{J},\tfrac{1}{2}(\overline{E}_{h_1}^{-h_1}-\overline{E}_{h_2}^{-h_1}+\overline{E}_{h_2-h_1}),\tfrac{1}{2}(\overline{B}_{h_1}^{-h_1}-\overline{B}_{h_2}^{-h_1}+\overline{B}_{h_2-h_1}))$$

satisfies Maxwell's equations, as required. □

Lemma 0.11. *Let $(\rho_w,\overline{J}_w)$ for $w\neq c$, be the smooth charge and current configurations defined above, satisfying the continuity equation. Then the causal fields $(\overline{E}_w,\overline{B}_w)$ defined by Jefimenko's equations exist for $w\neq c$, with $(\rho_w,\overline{J}_w,\overline{E}_w,\overline{B}_w)$ satisfying Maxwell's equations. Moreover $lim_{w\rightarrow c}\overline{E}_w$ and $\lim_{w\rightarrow c}\overline{B}_w$ exist and define fields $(\overline{E}_c,\overline{B}_c$ such that $(\rho_c,\overline{J}_c,\overline{E}_c,\overline{B}_c)$ satisfy Maxwell's equations.*

Proof. The first claim will be proved later, the second claim follows from a result in [**?**]. For $h\in\mathcal{R}_{>0}$, we have that $(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h)$ satisfies the continuity equation, $w\neq c$. By the observation in the previous lemma, $(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h,\overline{E}_w+\overline{E}_w^h,\overline{B}_w+\overline{B}_w^h)$ satisfies Maxwell's equations, and is defined by Jefimenko's equations relative to $(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h)$. By the main proof, (choosing the initial conditions at $\frac{t+h}{2}$, between t and $t+h$) we have that $lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^h)=\overline{E}_{c,h}$ and $lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^h)=\overline{B}_{c,h}$ exist, so that (more proof required);

$$lim_{w\rightarrow c}(\rho_w+\rho_w^h,\overline{J}_w+\overline{J}_w^h,\overline{E}_w+\overline{E}_w^h,\overline{B}_w+\overline{B}_w^h)$$

$$=(\rho_c+\rho_c^h,\overline{J}_c+\overline{J}_c^h,\overline{E}_{c,h},\overline{B}_{c,h})$$

satisfies Maxwell's equations. By Lemma 0.10, for $\{h_1,h_2\}\subset\mathcal{R}_{>0}$, with $h_1<h_2$, we have that;

$$(\rho_c,\overline{J}_c,\tfrac{1}{2}(\overline{E}_{c,h_1}^{-h_1}-\overline{E}_{c,h_2}^{-h_1}+\overline{E}_{c,h_2-h_1}),\tfrac{1}{2}(\overline{B}_{c,h_1}^{-h_1}-\overline{B}_{c,h_2}^{-h_1}+\overline{B}_{c,h_2-h_1}))$$

$$=(\rho_c,\overline{J}_c,\tfrac{1}{2}(lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^{h_1})^{-h_1}-lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^{h_2})^{-h_1}$$

$$+lim_{w\rightarrow c}(\overline{E}_w+\overline{E}_w^{h_2-h_1})),\tfrac{1}{2}(lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^{h_1})^{-h_1}$$

$$-lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^{h_2})^{-h_1}+lim_{w\rightarrow c}(\overline{B}_w+\overline{B}_w^{h_2-h_1})))$$

$$=(\rho_c,\overline{J}_c,lim_{w\rightarrow c}\overline{E}_w,lim_{w\rightarrow c}\overline{B}_w)$$

satisfies Maxwell's equations, as required.

□

References

Flat 3, Redesdale House, 85 The Park, Cheltenham, GL50 2RP
E-mail address: t.depiro@curvalinea.net