

SOME RESULTS IN BIOCHEMISTRY AND BIOPHYSICS

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ABSTRACT. We consider some results in biochemistry and provide a dosage level of sodium bicarbonate to correct a ph imbalance from acidosis, noting the imbalance can be calculated from an arterial blood gas pressure test and an electrolyte panel. We consider some results in biophysics, and provide a dosage level of sodium bicarbonate to correct an increased membrane potential, which may result in psychosis, noting the potential can be calculated from an ECG. The level of acidosis is directly linked to an increase in membrane potential and sodium bicarbonate can correct both factors.

1. BIOCHEMISTRY

Lemma 1.1. *Henderson-Hasselbalch Equation for Carbonic Acid*

We have that;

$$ph \simeq -\log_{10}(K_a) + \log_{10}\left(\frac{n_s}{n_a}\right) \simeq -\log_{10}(K_a) + \log_{10}\left(\frac{m_s}{m_a}\right)$$

where K_a is the equilibrium constant for the disassociation of carbonic acid in water $H_2CO_3 \leftrightarrow H^+ + HCO_3^-$ where n_a is the number of moles of acid, n_s is the number of moles of the conjugate base HCO_3^- , and $\{m_a, m_s\}$ are the corresponding molalities, calculated with water as the solvent.

and;

$$ph \simeq -\log_{10}(K_b) + 4 + \log_{10}\left(\frac{[HCO_3^-]}{P_{CO_2}}\right)$$

where K_b is the equilibrium constant for the combination of CO_2 and H_2O into carbonic acid followed by its disassociation in water, $CO_2 + H_2O \leftrightarrow H_2CO_3 \leftrightarrow H^+ + HCO_3^-$, where $[HCO_3^-]$ is the concentration of the bicarbonate ion and P_{CO_2} is the partial pressure of carbon dioxide.

Proof. We use the methods in [7]. For the first claim, with reactions in which the solvent H_2O is not involved, we have that the equilibrium

constant is given by;

$$K_a = \frac{a(H_2O)a(H^+)a(HCO_3^-)}{a(H_2CO_3)}$$

We have that $a(H_2O) = \gamma_0 x_0 = \frac{\gamma_0 n_0}{n} \simeq 1$ where $\gamma_0 \simeq 1$ is the activity coefficient and $x_0 = \frac{n_0}{n} \simeq 1$ as we can assume we are working with a dilute solution, considering H_2O as substance 0, with $n = n_0 + n_a + 2n_s$. It follows that, using the definition of the molalities $m_a = \frac{\gamma_a n_a}{w}$, $m_s = \frac{\gamma_s n_s}{w}$, where $\{\gamma_a, \gamma_s\}$ are the activity coefficients and w is the weight of the solvent, the fact that $\gamma_s \simeq \gamma_a \simeq 1$, that;

$$\begin{aligned} K_a &\simeq \frac{a(H^+)a(HCO_3^-)}{a(H_2CO_3)} = \frac{a(H^+) \frac{\gamma_s n_s}{w}}{\frac{\gamma_a n_a}{w}} \\ &= \frac{a(H^+) \gamma_s n_s}{\gamma_a n_a} \\ &\simeq \frac{a(H^+) n_s}{n_a} \end{aligned}$$

so that, using the definition in [12] and the fact that $\frac{n_s}{n_a} \simeq \frac{m_s}{m_a}$;

$$\begin{aligned} ph &= -\log_{10}(a(H^+)) \\ &\simeq -\log_{10}\left(\frac{K_a n_a}{n_s}\right) \\ &= -\log_{10}(K_a) + \log_{10}\left(\frac{n_s}{n_a}\right) \\ &\simeq -\log_{10}(K_a) + \log_{10}\left(\frac{m_s}{m_a}\right) \end{aligned}$$

For the second claim, considering the reaction $CO_2 + H_2O \leftrightarrow H^+ + HCO_3^-$, we have that;

$$K_b = \frac{a(H^+)a(HCO_3^-)}{a(H_2O)a(CO_2)}$$

Again, we have that $a(H_2O) = \gamma_0 x_0 = \frac{\gamma_0 n_0}{n} \simeq 1$ where $\gamma_0 \simeq 1$ is the activity coefficient and $x_0 = \frac{n_0}{n} \simeq 1$ as we can assume we are working with a dilute solution, considering H_2O as substance 0, and CO_2 as substance 1, with $n = n_0 + n_1 + 2n_s$. It follows that, using the definition of the molality $m_s = \frac{\gamma_s n_s}{w}$, where γ_s is the activity coefficients and w is the weight of the solvent, the fact that $\gamma_s \simeq 1$, that;

$$K_b \simeq \frac{a(H^+)a(HCO_3^-)}{a(CO_2)}$$

$$\begin{aligned}
&= \frac{a(H^+)^{\frac{\gamma_s n_s}{w}}}{a(CO_2)} \\
&\simeq \frac{a(H^+) n_s}{w a(CO_2)} \\
&= \frac{a(H^+) n_s}{\rho g v a(CO_2)}
\end{aligned}$$

where $w = \rho g v$, ρ and v are the density and volume of the solvent, g is the gravitational constant.

so that, using the Definition in [12];

$$\begin{aligned}
ph &= -\log_{10}(a(H^+)) \\
&\simeq -\log_{10}\left(\frac{K_b \rho g v a(CO_2)}{n_s}\right) \\
&= -\log_{10}(K_b) - \log_{10}(\rho g) + \log_{10}\left(\frac{n_s}{v}\right) + \log_{10}\left(\frac{1}{a(CO_2)}\right) \\
&= -\log_{10}(K_b) - \log_{10}(\rho g) + \log_{10}\left(\frac{[HCO_3^-]}{a(CO_2)}\right) \\
&\simeq -\log_{10}(K_b) - 1 + \log_{10}\left(\frac{[HCO_3^-]}{a(CO_2)}\right)
\end{aligned}$$

We have that $a(CO_2) \simeq \frac{f_{PCO_2}}{P^\circ}$, where f is the fugacity of CO_2 calculated in a vapour mixture of H_2CO_3 , CO_2 and H_2O . This gives;

$$\begin{aligned}
ph &\simeq -\log_{10}(K_b) - 1 + \log_{10}\left(\frac{101325[HCO_3^-]}{f_{PCO_2}}\right) \\
&\simeq -\log_{10}(K_b) + 4 + \log_{10}\left(\frac{[HCO_3^-]}{f_{PCO_2}}\right)
\end{aligned}$$

□

Remarks 1.2. *From the equation;*

$$\frac{\partial G}{\partial \xi}|_{(T,P)} = \Delta G^\circ(T) + RT \ln(K_b)$$

and the fact that at chemical equilibrium (T, P) , $\frac{\partial G}{\partial \xi}|_{(T,P)} = 0$, we obtain that;

$$K_b = e^{-\frac{\Delta G^\circ(T)}{RT}}$$

In [2], values of ΔG° are given for various disassociations of acids, but I couldn't find the result for $H_2CO_3 \leftrightarrow H^+ + HCO_3^-$. However, they are in the order of 5 to 12 kcal/mol, which is about 20 to 50 J. I

found the value of $\Delta G^\circ(T)$ for $CO_2 + H_2O \leftrightarrow H_2CO_3$ to be 103.67×10^3 J/mol at 298.15K, about 25°C, significantly larger than the value for the disassociation, so we can use this value for $\Delta G^\circ(T)$ at approximately room temperature 298.15K. It follows that;

$$K_b = e^{\frac{-\Delta G^\circ(T)}{RT}}$$

$$\simeq e^{\frac{-10^5}{8.3 \times 300}} \simeq e^{-40}$$

$$\text{so that } -\log_{10}(K_b) \simeq -\log_{10}(10^{-20}) = 20$$

I found a theoretical calculation of f for a pure and an air mixture at 298K to be about 10^5 and 35 Pa, see [1], the last being about the same as the result for a seawater mixture, see [8]. As a seawater mixture probably contains some carbonic acid, we can estimate;

$$-\log_{10}(f) = -\log_{10}(35) \simeq -1.5$$

$$ph \simeq -\log_{10}(K_b) + 4 + \log_{10}\left(\frac{[HCO_3^-]}{fP_{CO_2}}\right)$$

$$\simeq 20 + 4 - 1.5 + \log_{10}\left(\frac{[HCO_3^-]}{P_{CO_2}}\right)$$

$$= 23 + \log_{10}\left(\frac{[HCO_3^-]}{P_{CO_2}}\right) (*)$$

The only reference I could find for this result was on Wikipedia, which gives;

$$ph = 6.1 + \log_{10}\left(\frac{[HCO_3^-]}{0.0307P_{CO_2}}\right)$$

$$= 6.1 - \log_{10}(0.0307) + \log_{10}\left(\frac{[HCO_3^-]}{P_{CO_2}}\right)$$

$$\simeq 7.6 + \log_{10}\left(\frac{[HCO_3^-]}{P_{CO_2}}\right)$$

so there is a discrepancy of 15.4 between the two results, provided the equilibrium reaction is obtained at about room temperature.

Lemma 1.3. *Nernst Equation in Ionic Homeostasis and Nervous Conduction*

In the case of a sodium channel, we have that;

$$\Delta\Psi = \frac{-RT}{F} \ln \frac{[Na^+]_{out}}{[Na^+]_{in}}$$

see [6], where $\Delta\Psi$ is the membrane potential, $[Na^+]_{out}$ and $[Na^+]_{in}$ are the concentration of sodium ions at (T, P) across the junction.

Proof. The proof is that we can consider the two concentrations as two species with molar amounts $\{n_1, n_2\}$. We have that the chemical Gibbs free energy of the two species is given by $G(T, P, n_1)$ and $G(T, P, n_2)$, with total chemical free energy given by $G_{chem}(T, P, n_1, n_2) = G(T, P, n_1) + G(T, P, n_2)$. The electrical free energy G_{el} of the two species is given by $-zFV_{out}n_1$ and $-zFV_{in}n_2$, where z is the valence of the ion, F is Faraday's constant and $\{V_{out}, V_{in}\}$ are the potentials. It follows that the electrical and chemical free energy G_{tot} is given by;

$$\begin{aligned} G_{tot}(T, P, n_1, n_2) &= G_{chem}(T, P, n_1, n_2) + G_{el}(n_1, n_2) \\ &= G_{chem}(T, P, n_1, n_2) - zFV_{out}n_1 - zFV_{in}n_2 \\ &= G(T, P, n_1) + G(T, P, n_2) - zFV_{out}n_1 - zFV_{in}n_2 \quad (*) \end{aligned}$$

At electrical and chemical equilibrium, we have that;

$$dG_{tot} = \frac{\partial G_{tot}}{\partial n_1} = \frac{\partial G_{tot}}{\partial n_2} = 0$$

so that, by (*);

$$\frac{\partial G}{\partial n_1} - zFV_{out} = \frac{\partial G}{\partial n_2} - zFV_{in} = 0, (**)$$

By the definition of chemical potential, $\mu(T, P)$, we have that;

$$\frac{\partial G}{\partial n_1}(T, P) = \mu(T, P) = \mu^\circ(T) + RT \ln(a_1)$$

$$\frac{\partial G}{\partial n_2}(T, P) = \mu(T, P) = \mu^\circ(T) + RT \ln(a_2), (***)$$

so that, by (**), (***);

$$\mu^\circ(T) + RT \ln(a_1) - zFV_{out} = \mu^\circ(T) + RT \ln(a_2) - zFV_{in} = 0, (****),$$

(1).

¹ According to Lemmas 1.4 and 1.5 we cannot strictly cancel the $\mu^\circ(T)$ term on both sides of the equation, as it depends on the number of ions on both sides of the channel. As the difference is quite small, we can make a good approximation

so that;

$$\Delta\Psi \simeq V_{out} - V_{in} = \frac{RT}{z} \ln\left(\frac{a_1}{a_2}\right)$$

By the definition of activities, we have that $a_1 = \frac{\gamma_1 n_1}{w}$, $a_2 = \frac{\gamma_2 n_2}{w}$, so that;

$$\frac{a_1}{a_2} = \frac{\gamma_1 n_1}{\gamma_2 n_2} \simeq \frac{\frac{n_1}{V}}{\frac{n_2}{V}} = \frac{[X]_{out}}{[X]_{in}}$$

assuming the activity coefficients $\gamma_1 \simeq \gamma_2 \simeq 1$ and V is the volume of the solvent. It follows;

$$\Delta\Psi \simeq V_{out} - V_{in} = \frac{RT}{z} \ln\left(\frac{[X]_{out}}{[X]_{in}}\right)$$

This result differs by a factor of -1 .

We also have from (***) , the fact that the density of water ρ is 1, and Lemma 1.5 that;

$$\begin{aligned} RT \ln(a_1) &\simeq RT \ln\left(\frac{n_1}{w}\right) \\ &= RT \ln\left(\frac{[X]_{out} W}{w}\right) \\ &= RT \ln\left(\frac{[X]_{out} W}{g \rho W}\right) \\ &= RT \ln\left(\frac{[X]_{out}}{g}\right) \\ &= z F V_{out} - \mu^\circ \end{aligned}$$

here. Otherwise, we can make a more accurate calculation;

$$\begin{aligned} \Delta\Psi &= \frac{RT}{z} \ln\left(\frac{a_1}{a_2}\right) + (kT - P^*V) + RT \ln(x_{out}) - (kT - P^*V) - RT \ln(x_{in}) \\ &\simeq \frac{RT}{z} \ln\left(\frac{[X]_{out}}{[X]_{in}}\right) + RT \ln\left(\frac{[X]_{out}}{[X]_{in}}\right) \\ &= RT \frac{z+1}{z} \ln\left(\frac{[X]_{out}}{[X]_{in}}\right) \end{aligned}$$

so that $[X]_{out} = ge^{\frac{zFV_{out} - \mu^\circ}{RT}}$ (\dagger), (2).

□

Lemma 1.4. *For an ideal gas, obeying the gas law $PV = knT$, we have that;*

$$\frac{\partial G}{\partial n} \Big|_{(T,P,n)} = kT(1 - n) = kT - PV$$

Proof. By the first law of thermodynamics and the definition of entropy, we have that;

$$dQ = dU + PdV = TdS$$

so that, using the definition of Gibbs energy;

$$dG = dU + PdV + VdP - TdS - SdT$$

$$= dQ + VdP - dQ - SdT$$

$$= VdP - SdT$$

Using the ideal gas law $PV = knT$, and assuming $dT = 0$, we have that;

² If we use Lemma 1.5, we obtain that;

$$\begin{aligned} [X]_{out} &= ge^{\frac{zFV_{out} - \mu^\circ}{RT}} \\ &= ge^{\frac{zFV_{out} - (kT - P^*V^* - RT \ln(x_{out}))}{RT}} \\ &= ge^{\frac{zFV_{out} - kT - P^*V^*}{RT}} e^{-\ln(x_{out})} \\ &= \frac{g}{x_{out}} e^{\frac{zFV_{out} - kT - P^*V^*}{RT}} \\ &= \frac{g\rho}{18 \times 10^{-3} [X]_{out}} e^{\frac{zFV_{out} - kT - P^*V^*}{RT}} \end{aligned}$$

so that;

$$\begin{aligned} [X]_{out}^2 &= \frac{g}{18 \times 10^{-3}} e^{\frac{zFV_{out} - kT - P^*V^*}{RT}} \\ [X]_{out} &= \frac{\sqrt{g} 10^{\frac{3}{2}}}{3\sqrt{2}} e^{\frac{zFV_{out} - kT - P^*V^*}{2RT}} \end{aligned}$$

$$dG = knT_{fix} \frac{dP}{P}$$

so that if we add ϵ moles of substance to an ideal gas, keeping the temperature at T_{fix} and record the difference in pressure while fixing the volume V , before changing the volume to $V_2 = \frac{(n+\epsilon)V}{n}$, we obtain that;

$$dG = V(P_2 - P_1) = V\left(\frac{k(n+\epsilon)T_{fix}}{V} - \frac{knT_{fix}}{V}\right) = k\epsilon T_{fix}$$

$$= G(T_{fix}, P_2, n + \epsilon) - G(T_{fix}, P_1, n)$$

$$G(T_{fix}, P_2, n + \epsilon) - G(T_{fix}, P_1, n + \epsilon) = kT_{fix}(n + \epsilon) \int_{P_1}^{P_2} \frac{dP}{P}$$

$$= kT_{fix}(n + \epsilon)(\ln(P_2) - \ln(P_1))$$

so that;

$$G(T_{fix}, P_1, n + \epsilon) - G(T_{fix}, P_1, n)$$

$$= k\epsilon T_{fix} - kT_{fix}(n + \epsilon)(\ln(P_2) - \ln(P_1))$$

$$= kT_{fix}(\epsilon - (n + \epsilon)\ln\left(\frac{P_2}{P_1}\right))$$

We have that $P_2 = \frac{k(n+\epsilon)T_{fix}}{V}$, $P_1 = \frac{knT_{fix}}{V}$, so that $\frac{P_2}{P_1} = \frac{(n+\epsilon)}{n}$

$$G(T_{fix}, P_1, n + \epsilon) - G(T_{fix}, P_1, n) = kT_{fix}(\epsilon - (n + \epsilon)\ln\left(\frac{n+\epsilon}{n}\right))$$

The base pressure is given by $P_1 = \frac{knT_{fix}}{V}$ and the return volume is given by;

$$P_1 = \frac{k(n+\epsilon)T_{fix}}{V_2} = \frac{knT_{fix}}{V}$$

so that $V_2 = \frac{(n+\epsilon)V}{n}$.

We have that;

$$\left[x - (n + x)\ln\left(\frac{n+x}{n}\right)\right]'_{x=0} = \left[1 - \ln\left(\frac{n+x}{n}\right) - \frac{(n+x)n}{(n+x)^2}\right]_{x=0}$$

$$= \left[1 - n - \ln\left(\frac{n+x}{n}\right)\right]_{x=0}$$

$$= 1 - n$$

It follows, using L'Hopital's rule, that;

$$\begin{aligned} \frac{\partial G}{\partial n} \Big|_{(T_{fix}, P_1, n)} &= \lim_{\epsilon \rightarrow 0} \frac{G(T_{fix}, P_1, n+\epsilon) - G(T_{fix}, P_1, n)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{kT_{fix}(\epsilon - (n+\epsilon)\ln(\frac{n+\epsilon}{n}))}{\epsilon} \\ &= kT_{fix}(1 - n) \\ &= kT_{fix} - P_1V \end{aligned}$$

□

Lemma 1.5. *Finding the chemical potential*

In a dilute solution of positive ions X in water, assumed to be an ideal solution, we have that the chemical potential;

$$\mu(T, P) = kT(1 - n) + RT\ln(x_1) = kT - P_1^*V + RT\ln(x_1)$$

where n_1 is the number of moles of ions X , n_2 is the number of moles of water, n is the combined number of moles of water and ions X , $x_1 = \frac{n_1}{n}$, $PV = knT$ for the vapour phase of the ions X , considered as an ideal gas, in equilibrium with X as a liquid and V the equilibrium volume.

Proof. As the solution is considered ideal, we have that;

$$\mu_1(T, P) = \mu_1^*(T, P) + RT\ln(x_1)$$

where the ions X are considered as substance 1. By the phase rule, if the solution is in equilibrium with its vapour phase, we have that $\mu_1^{*,(l)}(T, P) = \mu_1^{*,(g)}(T, P)$. Considering the vapour phase as an ideal gas, we can use Lemma 1.4 to give;

$$\mu_1^{*,(g)}(T, P) = kT - P_i^*V^*$$

where $P_i^*(T, P)$ is the equilibrium vapour pressure and $V^*(T, P)$ is the equilibrium vapour volume. The volume $V^*(T, P)$ is determined by temperature and pressure, as if not, with the temperature T and equilibrium vapour pressure P_i^* fixed, by the ideal gas law, we would have

a variation in the number of moles n' present in the vapour phase. Denoting these variations by $\{n'_1, n'_2\}$, we could force a reaction through the liquid phase forming a temperature and pressure loop. This contradicts the fact established in [11] that no substance can be formed in a loop. In practice, we would have to use a strong solution of the ions X in water to make the final calculation of V^* .

□

Lemma 1.6. *Use of Sodium Bicarbonate 1*

$$[NaHCO_3] = 10^{-D} - 10^{-7}$$

returns an acidosis of $7 - D$ to a neutral ph of 7, so that a dose of 5.5×10^{-D} g of sodium bicarbonate corrects an acidosis ph imbalance of $7 - D$, assuming the complete disassociation of sodium bicarbonate in water, and D can be calculated using an arterial blood gas pressure test and an electrolyte panel.

Proof. We have the disassociation of carbonic acid, $H_2CO_3 \leftrightarrow H^+ + HCO_3^-$, (*) and the disassociation of sodium bicarbonate $NaHCO_3 \leftrightarrow Na^+ + HCO_3^-$, (**). From remark 1.2, we have that the concentration $[H^+]$ in (*) is given by;

$$23 + \log_{10}\left(\frac{[HCO_3^-]}{P_{CO_2}}\right) = -\log_{10}([H^+])$$

The value D of the left hand side of the equation can be calculated with an arterial blood gas pressure test to measure P_{CO_2} and an electrolyte panel to measure the bicarbonate concentration $[HCO_3^-]$. Rearranging, we then obtain that;

$$[H^+] = 10^{-D} \ (\dagger)$$

so that we obtain $10^{-D}V$ moles of H^+ ions in a given volume. Assuming a complete disassociation in (**) and that the Na^+ ions in (**) react with Cl^- ions to form $NaCl$, while the remaining HCO_3^- ions react with H^+ in (**) to form CO_2 and H_2O , we obtain a reduction of moles of H^+ ions in a given volume V of $V[NaHCO_3]$, so by (\dagger), $H_{new}^+ = (10^{-D} - [NaHCO_3])V$ (moles), and the concentration is given by $[H^+]_{new} = (10^{-D} - [NaHCO_3])$ and;

$$ph_{new} = -\log_{10}([H^+]_{new})$$

$$\begin{aligned}
&= -\log_{10}(10^{-D} - [NaHCO_3]) \\
&= -\log_{10}(10^{-D}(1 - \frac{[NaHCO_3]}{10^{-D}})) \\
&= D - \log_{10}(1 - [NaHCO_3]10^D)
\end{aligned}$$

so that $ph_{new} = 7$ if;

$$D - \log_{10}(1 - [NaHCO_3]10^D) = 7$$

$$\text{iff } 1 - [NaHCO_3]10^D = 10^{D-7}$$

$$\text{iff } [NaHCO_3] = 10^{-D} - 10^{-D}10^{D-7}$$

$$\text{iff } [NaHCO_3] = 10^{-D} - 10^{-7}$$

Assuming the volume V of the human body is 0.0664, we require $0.0664(10^{-D} - 10^{-7}) \simeq 6.64 \times 10^{-D-2}$ moles of $NaHCO_3$ to correct an acidosis inbalance of $7 - D$ ph points. This amounts to $6.64 \times 10^{-D-2} \times (23 + 1 + 12 + 3.16) \simeq 5.5 \times 10^{-D}$ g.

□

2. BIOPHYSICS

Lemma 2.1. *Let (\bar{E}, \bar{B}) be an electromagnetic pair such that $\square^2 \bar{E} = \bar{0}$ and $\bar{B} = \bar{0}$, where \square is the d'Alembertian operator, then there exists a corresponding pair (V, \bar{A}) of potentials such that;*

$$(i). \bar{E} = -\nabla(V) - \frac{\partial \bar{A}}{\partial t}$$

$$(ii). \bar{B} = \nabla \times \bar{A}$$

$$(iii). \square^2 \bar{V} = 0$$

$$(iv). \square^2 \bar{A} = \bar{0}$$

There exists $(\rho, \bar{J}, \bar{E}, \bar{B})$ satisfying Maxwell's equation, with $\square^2 \rho = 0$, $\square^2 \bar{J} = \bar{0}$, $\nabla(\rho) + \frac{1}{c^2} \frac{\partial \bar{J}}{\partial t} = \bar{0}$, $\square^2 \bar{E} = \bar{0}$, $\bar{B} = \bar{0}$, and potentials (V, \bar{A}) , with $\bar{A} = \bar{0}$, $\square^2 V = 0$ such that V has the integral representation;

$$V(\bar{x}, t) = \int_{\mathcal{R}^3} (f(k)e^{ikct} + g(k)e^{-ikct})e^{i\bar{k} \cdot \bar{x}} d\bar{k}$$

where $k = |\bar{k}|$, $\bar{k} \neq \bar{0}$.

Proof. The existence of a pair (V_1, \bar{A}_1) satisfying (i), (ii) is well known, see [5]. We then have that;

$$\begin{aligned} \square^2 \bar{E} &= \square^2(-\nabla(V_1) - \frac{\partial \bar{A}_1}{\partial t}) \\ &= -\nabla(\square^2 V_1) - \frac{\partial \square^2 \bar{A}_1}{\partial t} \\ &= \bar{0} \\ \square^2 \bar{B} &= \square^2(\nabla \times \bar{A}_1) \\ \nabla \times \square^2 \bar{A}_1 \\ &= \bar{0} \end{aligned}$$

so that $(\square^2 V_1, \square^2 \bar{A}_1)$ are potentials for the electromagnetic pair $(\bar{0}, \bar{0})$, (†). If (V_2, \bar{A}_2) are a pair such that;

$$\begin{aligned} a. \quad \bar{0} &= -\nabla(V_2) - \frac{\partial \bar{A}_2}{\partial t} \\ b. \quad \bar{0} &= \nabla \times \bar{A}_2 \end{aligned}$$

Then, by b, there exists f such that $\bar{A}_2 = \nabla(f)$, (*) and by a;

$$\begin{aligned} -\nabla(V_2) - \frac{\partial \bar{A}_2}{\partial t} &= -\nabla(V_2) - \frac{\partial \nabla(f)}{\partial t} \\ &= \nabla(-V_2 - \frac{\partial f}{\partial t}) \\ &= \bar{0} \end{aligned}$$

so that by the fundamental theorem of calculus;

$$V_2 = -\frac{\partial f}{\partial t} + h(t) \quad (**)$$

and, conversely, given $\{f(\bar{x}, t), h(t)\}$, if we define (V_2, \bar{A}_2) by (*), (**), then a, b are satisfied. By (†), we then have that $\square^2(V_1) = -\frac{\partial k(\bar{x}, t)}{\partial t} + l(t)$, $\square^2 \bar{A}_1 = \nabla(k(\bar{x}, t))$. We claim that we can find a pair $(a(\bar{x}, t), b(t))$, such that;

$$\begin{aligned} \square^2\left(-\frac{\partial a}{\partial t} + b, \nabla(a)\right) &= -\left(-\frac{\partial k}{\partial t} + l, \nabla(k)\right) \\ \text{iff } \left(-\frac{\partial \square^2(a)}{\partial t} - c^2 b'', \nabla(\square^2(a))\right) &= \left(\frac{\partial k}{\partial t} - l, -\nabla(k)\right) \\ \text{iff } \square^2(a) = -k + f(t) \text{ and } c^2 f''(t) - c^2 b''(t) &= -l(t) \\ \text{if } \square^2(a) = -k \text{ and } b''(t) &= -l \end{aligned}$$

as we can take smooth antiderivatives of l and solve the non-homogeneous wave equation $\square^2(a) = -k$ using, for example, the method of Fourier transforms, see [4], and solving second order differential equations with a smooth forcing term, with Abel's method of characteristics, see [3]. If (V_3, \bar{A}_3) are defined by $(*)$, $(**)$ and $(a(\bar{x}, t), b(t))$, then it is clear that $V = V_1 + V_3$, $\bar{A} = \bar{A}_1 + \bar{A}_3$ defines a solution to (i) , (ii) , (iii) , (iv) .

For the second claim, we can use the method of [9], utilising the radial transform condition for \bar{J} , to find real (ρ, \bar{J}) such that the continuity equation is satisfied, $\square^2(\rho) = 0$, $\square^2(\bar{J}) = \bar{0}$, $\nabla(\rho) + \frac{1}{c^2} \frac{\partial \bar{J}}{\partial t} = \bar{0}$ and with \bar{J} having the integral representation;

$$\bar{J}(\bar{x}, t) = \int_{\mathcal{R}^3} (\alpha(k) \bar{k} e^{ikct} + \beta(k) \bar{k} e^{-ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k}$$

where $\{\alpha, \beta\} \subset S(\mathcal{R}^3)$, ⁽³⁾

³ When using the representations;

$$\rho(\bar{x}, t) = \int_{\mathcal{R}^3} (f(\bar{k}) e^{ikct} + g(\bar{k}) e^{-ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k}$$

$$\bar{J}(\bar{x}, t) = \int_{\mathcal{R}^3} (\bar{F}(\bar{k}) e^{ikct} + \bar{G}(\bar{k}) e^{-ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k}$$

from [9], we can always impose the conditions that $f(\bar{k}) = g(-\bar{k})^*$ and $\bar{F}(\bar{k}) = -\bar{G}(-\bar{k})^*$, to ensure the representation is real. This is because, we have that $\rho(\bar{x}, t)$ is real iff $\rho(\bar{x}, t) = \rho(\bar{x}, t)^*$ iff;

$$\begin{aligned} &\int_{\mathcal{R}^3} (f(\bar{k}) e^{ikct} + g(\bar{k}) e^{-ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k} \\ &= \left(\int_{\mathcal{R}^3} (f(\bar{k}) e^{ikct} + g(\bar{k}) e^{-ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k} \right)^* \\ &= \int_{\mathcal{R}^3} (f(\bar{k})^* e^{-ikct} + g(\bar{k})^* e^{ikct}) e^{-i\bar{k} \cdot \bar{x}} d\bar{k} \\ &= \int_{\mathcal{R}^3} (f(-\bar{k})^* e^{-ikct} + g(-\bar{k})^* e^{ikct}) e^{i\bar{k} \cdot \bar{x}} d\bar{k}, \quad ((\bar{k})' = -\bar{k}, d\bar{k}' = |-1| d\bar{k} = d\bar{k}) \end{aligned}$$

Equating coefficients, we obtain that, for all $\bar{x} \in \mathcal{R}^3$;

If we define \overline{E}_1 by;

$$-\frac{1}{\epsilon_0} \int_{\mathcal{R}^3} \left(\frac{\alpha(k)}{ikc} \overline{k} e^{ikct} + \frac{\beta(k)}{ikc} \overline{k} e^{-ikct} \right) e^{i\overline{k} \cdot \overline{x}} d\overline{k} \quad (\dagger)$$

From the representation (\dagger) , and the fact that $\left\{ \frac{\alpha(k)\overline{k}}{k}, \frac{\beta(k)\overline{k}}{k} \right\}$ are bounded, with rapid decrease at infinity, we have that \overline{E}_1 exists, and then $\frac{\partial \overline{E}_1}{\partial t} = -\frac{1}{\epsilon_0} \overline{J}$. By the results of [9], there does exist \overline{E} with $\square^2 \overline{E} = \overline{0}$ $(\rho, \overline{J}, \overline{E}, \overline{0})$ satisfies Maxwell's equations, so that $\frac{\partial \overline{E}}{\partial t} = -\frac{1}{\epsilon_0} \overline{J}$. Differentiating under the integral sign, we can show that $\overline{E}_1 \in C^\infty(\mathcal{R}^4)$, and the form of the representation (\dagger) gives that $\square^2 \overline{E}_1 = \overline{0}$. Then, as $\frac{\partial(\overline{E} - \overline{E}_1)}{\partial t} = \overline{0}$ and $\square^2(\overline{E} - \overline{E}_1) = \overline{0}$, we obtain that $\nabla^2(\overline{E} - \overline{E}_1) = \overline{0}$, so that $\overline{E} - \overline{E}_1$ is harmonic. Using the fact that $\left\{ \frac{\alpha(k)\overline{k}}{k}, \frac{\beta(k)\overline{k}}{k} \right\} \subset L^1(\mathcal{R}^3)$, uniformly bounded in $L^1(\mathcal{R}^3)$ norm with respect to t , we have that \overline{E}_1 is bounded. Moreover, by the Riemann-Lebesgue lemma, we have that;

$$\lim_{|\overline{x}| \rightarrow \infty} \overline{E}_1(\overline{x}, t) = \overline{0}$$

uniformly in t . By the results of [10], \overline{E} is bounded, of moderate decrease, so that $\overline{E} - \overline{E}_1 = \overline{0}$ and $\overline{E} = \overline{E}_1$, so that \overline{E} has the representation (\dagger) . We have that $\nabla(e^{i\overline{k} \cdot \overline{x}}) = i\overline{k} e^{i\overline{k} \cdot \overline{x}}$, so that if we define V by;

$$V(\overline{x}, t) = \frac{-i}{\epsilon_0} \int_{\mathcal{R}^3} \left(\frac{\alpha(k)}{ikc} e^{ikct} + \frac{\beta(k)}{ikc} e^{-ikct} \right) e^{i\overline{k} \cdot \overline{x}} d\overline{k} \quad (\dagger\dagger)$$

$$\begin{aligned} & \int_{\mathcal{R}^3} f(\overline{k}) - g(-\overline{k})^* e^{i\overline{k} \cdot \overline{x}} d\overline{k} \\ &= \int_{\mathcal{R}^3} g(\overline{k}) - f(-\overline{k})^* e^{i\overline{k} \cdot \overline{x}} d\overline{k} \\ &= 0 \end{aligned}$$

so that, by the inversion theorem, $f(\overline{k}) - g(-\overline{k})^* = 0$. The vector version is similar with $F(\overline{k}) - G(-\overline{k})^* = \overline{0}$

When passing to the radial transform condition $F(\overline{k}) = \alpha(k)\overline{k}$, $G(\overline{k}) = \beta(k)\overline{k}$, we have that;

$$F(\overline{k}) = G(-\overline{k})^*$$

$$\text{iff } \alpha(k)\overline{k} = -\beta(k)^* \overline{k}$$

so this condition becomes $\alpha(k) = -\beta(k)^*$.

then, by the fact that $\{\frac{\alpha(k)}{ikc}, \frac{\beta(k)}{ikc}\} \subset L^1(\mathcal{R}^3)$, we have that V is well defined and $-\nabla(V) = \bar{E}$. As \bar{E} is real, we must have that $\nabla(Im(V)) = \bar{0}$, so that, by the FTC, $Im(V) = \bar{c}$. We have that $V \in C^\infty(\mathcal{R}^4)$ and by the form of $(\dagger\dagger)$, we have that $\square^2(V) = 0$. If we then define $\bar{A} = \bar{0}$, we have that $\square^2(\bar{A}) = \bar{0}$ and $(i) - (iv)$ in the statement of the Lemma are satisfied, so that (V, \bar{A}) are potentials. Again, we can use the fact that $\{\frac{\alpha(k)}{k}, \frac{\beta(k)}{k}\} \subset L^1(\mathcal{R}^3)$ and the Riemann-Lebesgue lemma, to show that $\bar{c} = 0$, so that V is real as well, ⁽⁴⁾. \square

Lemma 2.2. *Let V be a potential satisfying $\square^2 V = 0$, where \square is the d'Alembertian operator. Then, for $\bar{x} \in \mathcal{R}^3$, we can write;*

$$V_{\bar{x}}(t) = \int_{\mathcal{R}_{>0}} \omega_{k,\bar{x}}(t) dk \quad (*)$$

⁴ The second argument establishes the existence for the hypotheses, but we also want to impose the condition that $\bar{J}|_{S(\bar{0}, r_0)} = \bar{0}$. By the argument in [9], we choose;

$$\bar{J}(\bar{x}, t) = \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{S_l}{r_0}, k > 0} (\alpha(k)e^{ikct} + \beta(k)e^{-ikct}) \int_{S(\bar{0}, k)} \bar{k} e^{i\bar{k} \cdot \bar{x}} d\bar{k}$$

where $\{\alpha, \beta\} \subset S(\mathcal{R})$ but we are restricting to a discrete sum. In this case, we have that \bar{J} is bounded, but we don't necessarily have that $\lim_{|\bar{x}| \rightarrow \infty} \bar{J}(\bar{x}, t) = \bar{0}$. We then define \bar{E}_1 by;

$$\bar{E}_1 = -\frac{1}{\epsilon_0} \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{S_l}{r_0}, k > 0} (\frac{\alpha(k)}{ikc} e^{ikct} + \frac{\beta(k)}{ikc} e^{-ikct}) \int_{S(\bar{0}, k)} \bar{k} e^{i\bar{k} \cdot \bar{x}} d\bar{k} \quad (\dagger)'$$

From the representation $(\dagger)'$, the fact that $\{\alpha, \beta\} \subset S(\mathcal{R})$, we have that \bar{E}_1 exists and is bounded, and then $\frac{\partial \bar{E}_1}{\partial t} = -\frac{1}{\epsilon_0} \bar{J}$. We have to adapt the results of [9] and [10], to show there does exist \bar{E} with $\square^2 \bar{E} = \bar{0}$ $(\rho, \bar{J}, \bar{E}, \bar{0})$ satisfies Maxwell's equations, so that $\frac{\partial \bar{E}}{\partial t} = -\frac{1}{\epsilon_0} \bar{J}$. Again, in this case, we will only be able to show that \bar{E} is bounded. Differentiating under the integral sign, we can show that $\bar{E}_1 \in C^\infty(\mathcal{R}^4)$, and the form of the representation $(\dagger)'$ gives that $\square^2 \bar{E}_1 = \bar{0}$. Then, as $\frac{\partial(\bar{E} - \bar{E}_1)}{\partial t} = \bar{0}$ and $\square^2(\bar{E} - \bar{E}_1) = \bar{0}$, we obtain that $\nabla^2(\bar{E} - \bar{E}_1) = \bar{0}$, so that $\bar{E} - \bar{E}_1$ is harmonic. It follows that $\bar{E} - \bar{E}_1 = \bar{c}$ and $\bar{E} = \bar{E}_1 + \bar{c}$. However, as all of Maxwell's equations involve derivatives in the electrical and magnetic fields, and are real equations, we still have that $(\rho, \bar{J}, \bar{E}_1, \bar{0})$ and $(\rho, \bar{J}, Re(\bar{E}_1), \bar{0})$ solve Maxwell's equations. We have that $\nabla(e^{i\bar{k} \cdot \bar{x}}) = i\bar{k}e^{i\bar{k} \cdot \bar{x}}$, so that if we define V by;

$$V(\bar{x}, t) = \frac{-i}{\epsilon_0} \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{S_l}{r_0}, k > 0} (\frac{\alpha(k)}{ikc} e^{ikct} + \frac{\beta(k)}{ikc} e^{-ikct}) \int_{S(\bar{0}, k)} e^{i\bar{k} \cdot \bar{x}} dS_k(\bar{k}) \quad (\dagger'\dagger')$$

then, we have that V is well defined and $-\nabla(V) = \bar{E}_1$, $-\nabla(Re(V)) = Re(\bar{E}_1)$. We have that $Re(V) \in C^\infty(\mathcal{R}^4)$ and by the form of $(\dagger'\dagger')$, we have that $\square^2(Re(V)) = 0$. If we then define $\bar{A} = \bar{0}$, we have that $\square^2(\bar{A}) = \bar{0}$ and $(i) - (iv)$ in the statement of the Lemma are satisfied, so that $(Re(V), \bar{A})$ are potentials.

where $\omega_{k,\bar{x}}(t)$ solves the differential equation for simple harmonic motion with frequency $\frac{kc}{2\pi}$, $\omega''_{k,\bar{x}}(t) = -k^2c^2\omega_{k,\bar{x}}(t)$. If the potential V is real, we have that each $\omega_{k,\bar{x}}$ in the representation (*) is real as well.

Proof. As $\square^2V = 0$, taking Fourier transforms, we have that;

$$\begin{aligned} \mathcal{F}(\square^2V)(\bar{k}, t) &= \mathcal{F}\left(\frac{\partial^2V}{\partial x^2} + \frac{\partial^2V}{\partial y^2} + \frac{\partial^2V}{\partial z^2} - c^2\frac{\partial^2V}{\partial t^2}\right)(\bar{k}, t) \\ &= -k_1^2\mathcal{F}(V)(\bar{k}) - k_2^2\mathcal{F}(V)(\bar{k}) - k_3^2\mathcal{F}(V)(\bar{k}, t) - c^2\frac{\partial^2\mathcal{F}(V)}{\partial t^2}(\bar{k}, t) \\ &= -k^2\mathcal{F}(V)(\bar{k}, t) - c^2\frac{\partial^2\mathcal{F}(V)}{\partial t^2}(\bar{k}, t) \\ &= 0 \end{aligned}$$

where $k^2 = k_1^2 + k_2^2 + k_3^2 > 0$. For fixed $\bar{k} \in \mathcal{R}^3$, solving the corresponding second order ODE in t , we obtain that;

$$\mathcal{F}(V)(\bar{k}, t) = A(\bar{k})e^{ikct} + B(\bar{k})e^{-ikct}$$

and applying the inversion theorem, assuming sufficient decay in V , and using polar coordinates, (k, θ, ϕ) we obtain that;

$$\begin{aligned} V(\bar{x}, t) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} \mathcal{F}(V)(\bar{k}, t) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} (A(\bar{k})e^{ikct} + B(\bar{k})e^{-ikct}) e^{i\bar{k}\cdot\bar{x}} d\bar{k} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}>0} \int_0^\pi \int_{-\pi}^\pi (A(k, \theta, \phi) e^{ik(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)k^2} e^{ikct} \\ &\quad + (B(k, \theta, \phi) e^{ik(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)k^2}) e^{-ikct} dk d\theta d\phi \end{aligned}$$

For fixed $k > 0$, let;

$$\begin{aligned} \omega_{k,\bar{x}}(t) &= \int_0^\pi \int_{-\pi}^\pi (A(k, \theta, \phi) e^{ik(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)k^2} e^{ikct} \\ &\quad + (B(k, \theta, \phi) e^{ik(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)k^2}) e^{-ikct} d\theta d\phi \end{aligned}$$

Then, differentiating under the integral sign;

$$\begin{aligned} \omega''_{k,\bar{x}}(t) &= \int_0^\pi \int_{-\pi}^\pi (ikc)^2 (A(k, \theta, \phi) e^{ik(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)k^2} e^{ikct} \\ &\quad + (-ikc)^2 (B(k, \theta, \phi) e^{ik(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)k^2}) e^{-ikct} d\theta d\phi \end{aligned}$$

$$= -k^2 c^2 \omega_{k, \bar{x}}(t)$$

so that;

$$V_{\bar{x}}(t) = \int_{k>0} \omega_{k, \bar{x}}(t) dk$$

as required.

If V is a real solution to $\square^2 V = 0$, we have that, using polar coordinates again;

$$\begin{aligned} V(\bar{x}, t) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} ([Re(A)(\bar{k}) + iIm(A)(\bar{k})][\cos(kct) + i\sin(kct)] \\ &+ [Re(B)(\bar{k}) + iIm(B)(\bar{k})][\cos(kct) - i\sin(kct)]) \\ &[\cos(\bar{k} \cdot \bar{x}) + i\sin(\bar{k} \cdot \bar{x})] d\bar{k} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} ((Re(A + B)(\bar{k})\cos(kct) + Im(B - A)\sin(kct))\cos(\bar{k} \cdot \bar{x}) \\ &+ (Im(-A - B)\cos(kct) + Re(B - A)\sin(kct))\sin(\bar{k} \cdot \bar{x})) d\bar{k} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}^3} [f_1(\bar{k})\cos(\bar{k} \cdot \bar{x})\cos(kct) + f_2(\bar{k})\cos(\bar{k} \cdot \bar{x})\sin(kct) \\ &+ f_3(\bar{k})\sin(\bar{k} \cdot \bar{x})\cos(kct) + f_4(\bar{k})\sin(\bar{k} \cdot \bar{x})\sin(kct)] d\bar{k} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathcal{R}_{>0}} \int_0^\pi \int_{-\pi}^\pi [f_1(k, \theta, \phi)\cos(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 \\ &+ \cos(\theta)x_3))\cos(kct) \\ &+ f_2(k, \theta, \phi)\cos(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3))\sin(kct) \\ &+ f_3(k, \theta, \phi)\sin(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3))\cos(kct) \\ &+ f_4(k, \theta, \phi)\sin(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3))\sin(kct)] k^2 dk d\theta d\phi \end{aligned}$$

where $f_1 = Re(A + B)$, $f_2 = Im(B - A)$, $f_3 = -Im(A + B)$, $f_4 = Re(B - A)$.

For $k \in \mathcal{R}_{>0}$, let;

$$\omega_{\bar{x}, k}(t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\pi \int_{-\pi}^\pi [f_1(k, \theta, \phi)\cos(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2$$

$$\begin{aligned}
& + \cos(\theta)x_3)) \cos(kct) \\
& + f_2(k, \theta, \phi) \cos(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)) \sin(kct) \\
& + f_3(k, \theta, \phi) \sin(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)) \cos(kct) \\
& + f_4(k, \theta, \phi) \sin(\cos(\sin(\theta)\cos(\phi)x_1 + \sin(\theta)\sin(\phi)x_2 + \cos(\theta)x_3)) \sin(kct)] k^2 d\theta d\phi
\end{aligned}$$

Then, clearly $\omega_{\bar{x},k}$ is real, $\omega''_{\bar{x},k}(t) = -k^2 c^2 \omega_{\bar{x},k}$ and;

$$V_{\bar{x},t} = \int_{\mathcal{R}_{>0}} \omega_{\bar{x},k}(t) dk$$

□

Lemma 2.3. *We have that;*

$$\begin{aligned}
\int_{S(\bar{0},k)} e^{-i\bar{k}\cdot\bar{x}} dS(\bar{k}) &= 4\pi k \frac{\sin k|\bar{x}|}{|\bar{x}|}, \quad \bar{x} \neq \bar{0} \\
&= 4\pi k^2, \quad \bar{x} = \bar{0}
\end{aligned}$$

In the case when \bar{J} satisfies the radial transform condition of [9] and $\{\alpha, \beta\} \subset S(\mathcal{R})$, we have that;

$$V(\bar{x}, t) = -\frac{4\pi}{\epsilon_0} \int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k)\sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k)\sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right) dk$$

where $V(\bar{x}, t)$ is given by Lemma 2.1, and;

$$\bar{J} = -(4\pi c^4 \int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{ikct} + \frac{\beta(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{-ikct} \right) dk) \bar{x}$$

If $\bar{x}_1 \neq \bar{x}_2$, using the notation of Lemma 2.2, we have that, for $k \in \mathcal{R}_{>0}$;

$$\begin{aligned}
& \omega_{\bar{x}_1,k} - \omega_{\bar{x}_2,k} \\
& = \theta(\bar{x}_1, \bar{x}_2) (Re(\alpha + \beta)(k) \cos(kct) + Im(\beta - \alpha)(k) \sin(kct))
\end{aligned}$$

The amplitudes $\{H_{\bar{x}_1}, H_{\bar{x}_2}\}$ of the potentials $\{\omega_{\bar{x}_1,k}, \omega_{\bar{x}_2,k}\}$ are given by;

$$\begin{aligned}
H_{\bar{x}_1} &= \frac{G \sin(k|\bar{x}_1|)}{c|\bar{x}_1| \theta(\bar{x}_1, \bar{x}_2)} \\
H_{\bar{x}_2} &= \frac{G \sin(k|\bar{x}_2|)}{c|\bar{x}_1| \theta(\bar{x}_1, \bar{x}_2)}
\end{aligned}$$

where;

$$\theta(\bar{x}_1, \bar{x}_2) = \frac{[|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)]}{c|\bar{x}_1||\bar{x}_2|}$$

and G , together with the frequency $\frac{kc}{2\pi}$ can be measured from an ECG. The potentials $\{\omega_{\bar{x}_1, k}, \omega_{\bar{x}_2, k}, \omega_{\bar{x}_1, k} - \omega_{\bar{x}_2, k}\}$ are synchronous in time, but with different amplitudes.

Proof. The case when $\bar{x} = \bar{0}$ is clear. When $\bar{x} \neq 0$, first observe that the integral is invariant, in the sense that if $|\bar{x}_1| = |\bar{x}_2|$, then;

$$\int_{S(\bar{0}, k)} e^{-i\bar{k} \cdot \bar{x}_1} dS(\bar{k}) = \int_{S(\bar{0}, k)} e^{-i\bar{k} \cdot \bar{x}_2} dS(\bar{k})$$

Using polar coordinates, $k_1 = k \cos(\theta)$, $k_2 = k \sin(\theta) \cos(\phi)$, $k_3 = k \sin(\theta) \sin(\phi)$, $0 \leq \theta \leq \pi$, $-\pi \leq \phi < \pi$, we can, by rotating coordinates, assume that $\bar{x} = (|\bar{x}|, 0, 0)$, so that $\bar{k} \cdot \bar{x} = k_1 |\bar{x}| = k \cos(\theta) |\bar{x}|$. Using the measure $dS(\bar{k}) = k^2 \sin(\theta) d\theta d\phi$, we have that;

$$\begin{aligned} \int_{S(\bar{0}, k)} e^{-i\bar{k} \cdot \bar{x}_1} dS(\bar{k}) &= \int_{-\pi}^{\pi} \int_0^{\pi} e^{-ik \cos(\theta) |\bar{x}|} k^2 \sin(\theta) d\theta d\phi \\ &= 2\pi k^2 \int_0^{\pi} e^{-ik \cos(\theta) |\bar{x}|} \sin(\theta) d\theta \end{aligned}$$

Making the substitution, $\xi = \cos(\theta)$, with $d\xi = -\sin(\theta) d\theta$, we have that;

$$\begin{aligned} \int_0^{\pi} e^{-ik \cos(\theta) |\bar{x}|} \sin(\theta) d\theta &= \int_1^{-1} e^{-ik \xi |\bar{x}|} - d\xi \\ &= \int_{-1}^1 e^{-ik \xi |\bar{x}|} d\xi \\ &= \left[\frac{e^{-ik \xi |\bar{x}|}}{-ik |\bar{x}|} \right]_{-1}^1 \\ &= \frac{e^{-ik |\bar{x}|} - e^{ik |\bar{x}|}}{-ik |\bar{x}|} \\ &= \frac{2 \sin(k |\bar{x}|)}{k |\bar{x}|} \end{aligned}$$

so that;

$$\begin{aligned} \int_{S(\bar{0}, k)} e^{-i\bar{k} \cdot \bar{x}_1} dS(\bar{k}) &= 2\pi k^2 \frac{2 \sin(k |\bar{x}|)}{k |\bar{x}|} \\ &= 4\pi k \frac{\sin(k |\bar{x}|)}{|\bar{x}|} \end{aligned}$$

Using the result, we obtain that;

$$\begin{aligned}
V(\bar{x}, t) &= -\frac{i}{\epsilon_0} \int_{\mathcal{R}^3} \left(\frac{\alpha(k)}{ikc} e^{ikct} + \frac{\beta(k)}{ikc} e^{-ikct} \right) e^{i\bar{k} \cdot \bar{x}} d\bar{k} \\
&= -\frac{1}{\epsilon_0} \int_{\mathcal{R}^3} \left(\frac{\alpha(k)}{kc} e^{ikct} + \frac{\beta(k)}{kc} e^{-ikct} \right) e^{i\bar{k} \cdot \bar{x}} d\bar{k} \\
&= -\frac{1}{\epsilon_0} \int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k)}{kc} e^{ikct} + \frac{\beta(k)}{kc} e^{-ikct} \right) \int_{S(\bar{0}, k)} e^{i\bar{k} \cdot \bar{x}} dS_k(\bar{k}) dk \\
&= -\frac{1}{\epsilon_0} \int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k)}{kc} e^{ikct} + \frac{\beta(k)}{kc} e^{-ikct} \right) 4\pi k \frac{\sin(k|\bar{x}|)}{|\bar{x}|} dk \\
&= -\frac{4\pi}{\epsilon_0} \int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right) dk, \quad (\#) \quad (5).
\end{aligned}$$

By Maxwell's first equation, the definition of the potential V , the fact that V satisfies the wave equation $\square^2 V = 0$, the explicit form of V , we have that;

$$\begin{aligned}
\rho &= \epsilon_0 \operatorname{div}(\bar{E}) \\
&= -\epsilon_0 \operatorname{div}(\nabla(V)) \\
&= -\epsilon_0 \nabla^2(V) \\
&= -\epsilon_0 c^2 \frac{\partial^2 V}{\partial t^2} \\
&= -\frac{1}{\mu_0} \frac{\partial^2 V}{\partial t^2} \\
&= \frac{4\pi}{\epsilon_0 \mu_0} \frac{\partial^2}{\partial t^2} \left(\int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right) dk \right)
\end{aligned}$$

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When we impose the condition that $\bar{J}|_{S(\bar{0}, r_0)} = \bar{0}$, we obtained that, in the footnote to Lemma 2.1, that;

$$\begin{aligned}
\operatorname{Re}(V)(\bar{x}, t) &= \operatorname{Re} \left(\frac{-i}{\epsilon_0} \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{\mathcal{S}_l}{r_0}, k > 0} \left(\frac{\alpha(k)}{ikc} e^{ikct} + \frac{\beta(k)}{ikc} e^{-ikct} \right) \int_{S(\bar{0}, k)} e^{i\bar{k} \cdot \bar{x}} dS_k(\bar{k}) \right) \\
&= \operatorname{Re} \left(-\frac{4\pi}{\epsilon_0} \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{\mathcal{S}_l}{r_0}, k > 0} \left(\frac{\alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right) dk \right) \\
&= -\frac{4\pi}{\epsilon_0} \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{\mathcal{S}_l}{r_0}, k > 0} \left(\frac{\alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right) dk
\end{aligned}$$

by comparison with the continuous spectrum case.

$$\begin{aligned}
&= 4\pi c^2 \int_{\mathcal{R}_{>0}} \left(\frac{-k^2 c^2 \alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} - \frac{k^2 c^2 \beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right) dk \\
&= -4\pi c^3 \int_{\mathcal{R}_{>0}} \left(\frac{k^2 \alpha(k) \sin(k|\bar{x}|)}{|\bar{x}|} e^{ikct} + \frac{k^2 \beta(k) \sin(k|\bar{x}|)}{|\bar{x}|} e^{-ikct} \right) dk, \quad (6)
\end{aligned}$$

We have that;

$$\nabla \left(\frac{\sin(k|\bar{x}|)}{|\bar{x}|} \right) = \frac{(|\bar{x}|k \cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} \bar{x}$$

By the fact that $\nabla(\rho) + \frac{1}{c^2} \frac{\partial \bar{J}}{\partial t} = \bar{0}$, we have that;

$$\begin{aligned}
\frac{\partial \bar{J}}{\partial t} &= -c^2 \nabla(\rho) \\
&= (4\pi c^5 \int_{\mathcal{R}_{>0}} \left(\frac{k^2 \alpha(k) (k|\bar{x}| \cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{ikct} + \frac{k^2 \beta(k) (k|\bar{x}| \cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{-ikct} \right) dk) \bar{x}
\end{aligned}$$

As above, using the fact that $\square^2(\bar{J}) = \bar{0}$, $\square^2(-c^2 \nabla(\rho)) = \bar{0}$, the usual wave to harmonic argument, and the fact that $\lim_{|\bar{x}| \rightarrow \infty} \bar{J}(\bar{x}, t) = \bar{0}$, we have that;

$$\bar{J} = -(4\pi c^4 \int_{\mathcal{R}_{>0}} \left(\frac{\alpha(k) (k|\bar{x}| \cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{ikct} + \frac{\beta(k) (k|\bar{x}| \cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{-ikct} \right) dk) \bar{x}$$

It follows from (#) that;

$$\begin{aligned}
&\omega_{\bar{x}_1, k} - \omega_{\bar{x}_2, k} \\
&= \left(\frac{\alpha(k) \sin(k|\bar{x}_1|)}{c|\bar{x}_1|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}_1|)}{c|\bar{x}_1|} e^{-ikct} \right) - \left(\frac{\alpha(k) \sin(k|\bar{x}_2|)}{c|\bar{x}_2|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}_2|)}{c|\bar{x}_2|} e^{-ikct} \right) \\
&= \left(\frac{\alpha(k) \sin(k|\bar{x}_1|)}{c|\bar{x}_1|} - \frac{\alpha(k) \sin(k|\bar{x}_2|)}{c|\bar{x}_2|} \right) e^{ikct} + \left(\frac{\beta(k) \sin(k|\bar{x}_1|)}{c|\bar{x}_1|} - \frac{\beta(k) \sin(k|\bar{x}_2|)}{c|\bar{x}_2|} \right) e^{-ikct} \\
&= \left(\frac{\alpha(k) [|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)]}{c|\bar{x}_1||\bar{x}_2|} \right) e^{ikct} + \left(\frac{\beta(k) [|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)]}{c|\bar{x}_1||\bar{x}_2|} \right) e^{-ikct} \\
&= \left(\frac{Re(\alpha+\beta)(k) [|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)]}{c|\bar{x}_1||\bar{x}_2|} \right) \cos(kct) + \left(\frac{Im(\beta-\alpha)(k) [|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)]}{c|\bar{x}_1||\bar{x}_2|} \right) \sin(kct) \\
&= \frac{[|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)]}{c|\bar{x}_1||\bar{x}_2|} (Re(\alpha+\beta)(k) \cos(kct) + Im(\beta-\alpha)(k) \sin(kct))
\end{aligned}$$

Using the fact that;

$$A \cos(kct) + B \sin(kct) = G \cos(kct + w)$$

⁶ When $\bar{J}|_{S(\bar{0}, r_0)}$ we obtain;

$$\rho = -4\pi c^3 \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in \frac{\mathcal{S}_l}{r_0}, k > 0} \left(\frac{k^2 \alpha(k) \sin(k|\bar{x}|)}{|\bar{x}|} e^{ikct} + \frac{k^2 \beta(k) \sin(k|\bar{x}|)}{|\bar{x}|} e^{-ikct} \right)$$

$$\text{iff } A\cos(kct) + B\sin(kct) = G\cos(kct)\cos(w) - G\sin(kct)\sin(w)$$

$$\text{iff } A = G\cos(w) \text{ and } B = G\sin(w)$$

$$\text{iff } G^2 = A^2 + B^2 \text{ and } \tan(w) = \frac{B}{A}.$$

where G is the amplitude of the nerve signal, $\frac{kc}{2\pi}$ is the frequency, both of which are measurable using an ECG. For a typical sodium channel, we obtain that;

$$\frac{[|\bar{x}_2|\sin(k|\bar{x}_1)| - |\bar{x}_1|\sin(k|\bar{x}_2)|]}{c|\bar{x}_1||\bar{x}_2|} \text{Re}(\alpha + \beta)(k) = G\cos(w)$$

$$\frac{[|\bar{x}_2|\sin(k|\bar{x}_1)| - |\bar{x}_1|\sin(k|\bar{x}_2)|]}{c|\bar{x}_1||\bar{x}_2|} \text{Im}(\beta - \alpha)(k) = G\sin(w)$$

so that;

$$\text{Re}(\alpha + \beta)(k) = \frac{G\cos(w)}{\theta(\bar{x}_1, \bar{x}_2)} \quad \text{Im}(\beta - \alpha)(k) = \frac{G\sin(w)}{\theta(\bar{x}_1, \bar{x}_2)}$$

where;

$$\theta(\bar{x}_1, \bar{x}_2) = \frac{[|\bar{x}_2|\sin(k|\bar{x}_1)| - |\bar{x}_1|\sin(k|\bar{x}_2)|]}{c|\bar{x}_1||\bar{x}_2|}$$

is determined by the positions $\{\bar{x}_1, \bar{x}_2\}$ of the opposite ends of the channel. It follows that;

$$\begin{aligned} \omega_{\bar{x}_1, k} &= \frac{\sin(k|\bar{x}_1|)}{c|\bar{x}_1|} (\text{Re}(\alpha + \beta)(k)\cos(kct) + \text{Im}(\beta - \alpha)(k)\sin(kct)) \\ &= \frac{\sin(k|\bar{x}_1|)}{c|\bar{x}_1|} \left(\frac{G\cos(w)}{\theta(\bar{x}_1, \bar{x}_2)} \cos(kct) + \frac{G\sin(w)}{\theta(\bar{x}_1, \bar{x}_2)} \sin(kct) \right) \\ &= H_{\bar{x}_1} \cos(kct + v) \end{aligned}$$

where;

$$\begin{aligned} H_{\bar{x}_1} &= \frac{G\sin(k|\bar{x}_1|)}{c|\bar{x}_1|\theta(\bar{x}_1, \bar{x}_2)} \\ v &= \tan^{-1}\left(\frac{G\sin(w)}{G\cos(w)}\right) = w \end{aligned}$$

Similarly;

$$\omega_{\bar{x}_2, k} = \frac{\sin(k|\bar{x}_2|)}{c|\bar{x}_2|} (\text{Re}(\alpha + \beta)(k)\cos(kct) + \text{Im}(\beta - \alpha)(k)\sin(kct))$$

$$= \frac{\sin(k|\bar{x}_2|)}{c|\bar{x}_2|} \left(\frac{G\cos(w)}{\theta(\bar{x}_1, \bar{x}_2)} \cos(kct) + \frac{G\sin(w)}{\theta(\bar{x}_1, \bar{x}_2)} \sin(kct) \right)$$

$$= H_{\bar{x}_2} \cos(kct + u)$$

where;

$$H_{\bar{x}_2} = \frac{G\sin(k|\bar{x}_2|)}{c|\bar{x}_2|\theta(\bar{x}_1, \bar{x}_2)}$$

$$u = \tan^{-1} \left(\frac{G\sin(w)}{G\cos(w)} \right) = w$$

□

Lemma 2.4. *Use of Sodium Bicarbonate 2*

$$[NaHCO_3] = \frac{(c-1)ge^{\frac{zFE_{out} - \mu^\circ(T)}{RT}}}{1 - ce^{\frac{FzE}{RT}}}$$

$$\text{where } c = e^{\frac{Fz(70 \times 10^{-3} - E)}{RT}}$$

returns a membrane potential of E to 70×10^{-3} V, where $E = E_{out} - E_{in}$.

In particular, if the membrane potential is $70 \times 10^{-3} + A$, we obtain;

$$[NaHCO_3] = \frac{(c-1)ge^{\frac{zFE_{out} - \mu^\circ(T)}{RT}}}{1 - c(e^{\frac{Fz(70 \times 10^{-3} + A)}{RT}})}$$

$$\text{where } c = e^{\frac{-FzA}{RT}}$$

to restore the membrane potential to a resting potential. The membrane potential E can be calculated using an ECG to calculate the amplitude of the signal away from the average, the value of E_{out} at one side of the membrane can be calculated using Lemma 2.3, knowing E and the positions $\{\bar{x}_1, \bar{x}_2\}$ of the sides of the membrane.

Proof. A concentration of $[NaHCO_3]$ relative to V will result in an addition of Na^+ ions on both sides of a membrane channel by $[NaHCO_3]W$, with W the volume of the channel. The new concentrations will be given by;

$$\frac{[Na^+]_{out}W + [NaHCO_3]W}{W} = [Na^+]_{out} + [NaHCO_3]$$

$$\frac{[Na^+]_{in}W + [NaHCO_3]W}{W} = [Na^+]_{in} + [NaHCO_3]$$

Using the formula above, we obtain that;

$$\begin{aligned}\Delta\Psi_{new} &= \frac{RT}{Fz} \ln\left(\frac{[Na^+]_{out} + [NaHCO_3]}{[Na^+]_{in} + [NaHCO_3]}\right) \\ &= \frac{RT}{Fz} \ln\left(\frac{[Na^+]_{out}}{[Na^+]_{in}}\right) + \frac{RT}{Fz} \ln\left(\frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}}\right) \\ &= \Delta\Psi + \frac{RT}{Fz} \ln\left(\frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}}\right)\end{aligned}$$

Assuming $\Delta\Psi = E$, with $V_{out} = E_{out}$, $V_{in} = E_{in}$, with a normal resting potential 70×10^{-3} V, we have that $\Delta\Psi_{new} = 70 \times 10^{-3}$ if;

$$\begin{aligned}\frac{RT}{Fz} \ln\left(\frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}}\right) &= 70 \times 10^{-3} - E \\ \text{iff } \frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}} &= e^{\frac{Fz(70 \times 10^{-3} - E)}{RT}} \\ \text{iff } [NaHCO_3] &= \frac{(c-1)[Na^+]_{out}}{1 - c\left(\frac{[Na^+]_{out}}{[Na^+]_{in}}\right)} \\ \text{iff } [NaHCO_3] &= \frac{(c-1)[Na^+]_{out}}{1 - ce^{\frac{zFE}{RT}}} \\ \text{iff } [NaHCO_3] &= \frac{(c-1)ge^{\frac{zFE_{out} - \mu^\circ(T)}{RT}}}{1 - c\left(e^{\frac{zFE}{RT}}\right)} \\ \text{where } c &= e^{\frac{Fz(70 \times 10^{-3} - E)}{RT}}, \quad (7).\end{aligned}$$

⁷ If we carry out the above calculation using the greater accuracy of the footnotes above and the formula for $\mu^\circ(T)$, we obtain that;

$$\begin{aligned}\Delta\Psi_{new} &= \frac{RT(Fz+1)}{Fz} \ln\left(\frac{[Na^+]_{out} + [NaHCO_3]}{[Na^+]_{in} + [NaHCO_3]}\right) \\ &= \frac{RT(Fz+1)}{Fz} \ln\left(\frac{[Na^+]_{out}}{[Na^+]_{in}}\right) + \frac{RT(Fz+1)}{Fz} \ln\left(\frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}}\right) \\ &= \Delta\Psi + \frac{RT(Fz+1)}{Fz} \ln\left(\frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}}\right)\end{aligned}$$

Assuming $\Delta\Psi = E$, with $V_{out} = E_{out}$, $V_{in} = E_{in}$, with a normal resting potential -70×10^{-3} V, we have that $\Delta\Psi_{new} = -70 \times 10^{-3}$ if;

$$\begin{aligned}\frac{RT(Fz+1)}{Fz} \ln\left(\frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}}\right) &= 70 \times 10^{-3} - E \\ \text{iff } \frac{1 + \frac{[NaHCO_3]}{[Na^+]_{out}}}{1 + \frac{[NaHCO_3]}{[Na^+]_{in}}} &= e^{\frac{Fz(70 \times 10^{-3} - E)}{(Fz+1)RT}} \\ \text{iff } [NaHCO_3] &= \frac{(c-1)[Na^+]_{out}}{1 - c\left(\frac{[Na^+]_{out}}{[Na^+]_{in}}\right)}\end{aligned}$$

□

$$\text{iff } [NaHCO_3] = \frac{(c-1)[Na^+]_{out}}{1-c(e^{\frac{zFE}{(Fz+1)RT}})}$$

$$\text{iff } [NaHCO_3] = \frac{(c-1)\sqrt{g}10^{\frac{3}{2}}e^{\frac{zFV_{out}-kT-P^*V^*}{2RT}}}{3\sqrt{2}(1-c(e^{\frac{zFE}{(Fz+1)RT}}))}$$

where $c = e^{\frac{Fz(70 \times 10^{-3} - E)}{(Fz+1)RT}}$. For the second clause, with a membrane potential of $70 \times 10^{-3} + A$, we obtain that;

$$[NaHCO_3] = \frac{(c-1)\sqrt{g}10^{\frac{3}{2}}e^{\frac{zFV_{out}-kT-P^*V^*}{2RT}}}{3\sqrt{2}(1-c(e^{\frac{Fz(70 \times 10^{-3} + A)}{(Fz+1)RT}}))}$$

where $c = e^{\frac{-FzA}{(Fz+1)RT}}$.

From Lemma 2.3, we have that;

$$V_{out} = \frac{E \sin(k|\bar{x}_1|)}{c|\bar{x}_1|\theta(\bar{x}_1, \bar{x}_2)}$$

$$= \frac{E \sin(k|\bar{x}_1|)|\bar{x}_2|}{|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|)}$$

Assume $E = 90 \times 10^{-3}$

$$\frac{kw}{2\pi} = f$$

f based on 80 beats/minute $f = \frac{80}{60} = \frac{4}{3} Hz$, $k = \frac{2\pi f}{w} = \frac{8\pi}{3w}$, with $w < c$

$$|\bar{x}_1| = 0.5$$

$$|\bar{x}_2| = 0.5 + \frac{1}{w}$$

Using the small angle approximation for sine;

$$|\bar{x}_2| \sin(k|\bar{x}_1|) - |\bar{x}_1| \sin(k|\bar{x}_2|) \simeq \frac{k^3 |\bar{x}_1| |\bar{x}_2| (|\bar{x}_2|^2 - |\bar{x}_1|^2)}{6}$$

$$= \frac{k^3 |\bar{x}_1| |\bar{x}_2| (|\bar{x}_2| + |\bar{x}_1|) (|\bar{x}_2| - |\bar{x}_1|)}{6}$$

$$\simeq \left(\frac{8\pi}{3w}\right)^3 \cdot 0.5^2 \cdot 1 \cdot \frac{1}{w}$$

$$= \frac{512\pi^3}{27 \cdot 4 \cdot w^4}$$

$$= \frac{128\pi^3}{27w^4}$$

$$V_{out} \simeq \frac{9 \times 10^{-4} \times 0.5 \times \frac{8\pi}{3w}}{\frac{128\pi^3}{27w^4}}$$

$$= \frac{81w^3 \times 10^{-4}}{32\pi^2}$$

$$F = 10^5$$

$$A = 20 \times 10^{-3}$$

$$c = e^{\frac{-FzA}{(Fz+1)RT}}$$

$$z = 1$$

$$T = 310K$$

$$R = 8.3$$

$$c = e^{\frac{-FzA}{(Fz+1)RT}} = e^{\frac{-10^5 \times 20 \times 10^{-3}}{(10^5+1) \times 8.3 \times 310}}$$

$$\simeq e^{-0.8 \times 10^{-5}}$$

$$\simeq 1 - (0.8 \times 10^{-5})$$

$$k = R$$

$$P^* = 54025$$

$$V^* = 0.2$$

Using $e^x \simeq 1 + x$, for $x \simeq 0$, we have that;

$$\begin{aligned} [NaHCO_3] &= \frac{(c-1)\sqrt{g}10^{\frac{3}{2}}e^{\frac{zFV_{out}-kT-P^*V^*}{2RT}}}{3\sqrt{2}(1-c(e^{\frac{Fz(70 \times 10^{-3}+A)}{(Fz+1)RT}}))} \\ &\simeq \frac{(-0.8 \times 10^{-5})(3 \times 10^{\frac{3}{2}})e^{\frac{FV_{out}-kT-P^*V^*}{2RT}}}{3\sqrt{2} \times (1 - e^{\frac{10^5(90 \times 10^{-3})}{10^5 \times 8.3 \times 310}})} \\ &\simeq \frac{(-0.8 \times 10^{-5})(3 \times 10^{\frac{3}{2}})e^{\frac{FV_{out}-kT-P^*V^*}{2RT}}}{3\sqrt{2} \times (1 - (1 + 15 \times 10^{-5}))} \\ &= \frac{(0.8 \times 10^{-5})(3 \times 10^{\frac{3}{2}})e^{\frac{FV_{out}-kT-P^*V^*}{2RT}}}{3\sqrt{2} \times (15 \times 10^{-5})} \\ &\simeq 0.1 \times (3 \times 10^{\frac{3}{2}})e^{\frac{10^5 \times 81w^3 \times 10^{-4} - (8.3 \times 310)}{2.32.8.3.310\pi^2} - \frac{54025.0.2}{2.(8.3).310}} \\ &\simeq 10 \times e^{\frac{810w^3}{4 \times 10^6} - 2} \\ &\simeq 10 \times e^{(2w^3 \times 10^{-4}) - 2} \\ &\simeq (0.06 \times 84 \times 10 \times e^{2w^3 \times 10^{-4} - 2})g \\ &\simeq (50 \times e^{2w^3 \times 10^{-4} - 2})g \end{aligned}$$

For an acidosis imbalance of $D = 2$, we obtained $(5.5 \times 10^{-D})g = (5.5 \times 10^{-2})g$, so that;

Lemma 2.5. *If we impose the condition that $\bar{J} = \bar{0}$ on some sphere $S(\bar{0}, r_0)$, then, by the above, we can write;*

$$V(\bar{x}, t) = -\frac{4\pi}{\epsilon_0} \sum_{k \in W} \left(\frac{\alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right)$$

$$\bar{J} = -(4\pi c^4 \sum_{k \in W} \left(\frac{\alpha(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{ikct} + \frac{\beta(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{-ikct} \right)) \bar{x}$$

where $W = \{k > 0 : \tan(kr_0) = kr_0\}$.

We have approximately that;

$$V(\lambda, t) = -\frac{4\pi}{\epsilon_0} \left(\frac{\alpha(k_0) \sin(k_0\lambda)}{c\lambda} e^{ikct} + \frac{\beta(k_0) \sin(k_0\lambda)}{c\lambda} e^{-ikct} \right)$$

along the nerve fibre, where k_0 is measurable using an ECG, and use Lemma 2.3.

Proof. If we impose the condition that $\bar{J} = \bar{0}$ on some sphere $S(\bar{0}, r_0)$, then, by the above, we can write;

$$V(\bar{x}, t) = -\frac{4\pi}{\epsilon_0} \sum_{l \in \mathcal{Z}_{\geq 0}} \sum_{k \in S_l, k \geq 0} \left(\frac{\alpha(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k) \sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct} \right)$$

where S_l is the zero set of the Bessel function j_l . Repeating the steps of Lemma 2.1 with this condition, we can replace the integral formula for \bar{J} by a discrete sum;

$$\bar{J} = -(4\pi c^4 \sum_{k \in W} \left(\frac{\alpha(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{ikct} + \frac{\beta(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{-ikct} \right)) \bar{x} + \bar{c}$$

where;

$$\text{with } (50 \times (1 + (2w^3 \times 10^{-4}) - 2)) = (5.5 \times 10^{-2})$$

$$(1 + (2w^3 \times 10^{-4}) - 2) \simeq 10^{-3}$$

$$(2w^3 \times 10^{-4}) \simeq 1$$

$$w^3 \simeq 10^4$$

$$w \simeq 10$$

The speed 10 m/s seems reasonable for signals propagating in the human body.

$$\begin{aligned} W &= \{k > 0 : kr_0 \cos(kr_0) - \sin(kr_0) = 0\} \\ &= \{k > 0 : \tan(kr_0) = kr_0\} \end{aligned}$$

Clearly, the vanishing condition implies that $\bar{c} = \bar{0}$, so that;

$$\bar{J} = -(4\pi c^4 \sum_{k \in W} (\frac{\alpha(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{ikct} + \frac{\beta(k)(k|\bar{x}|\cos(k|\bar{x}|) - \sin(k|\bar{x}|))}{|\bar{x}|^3} e^{-ikct})) \bar{x}$$

It follows that, we can also write the potential V as;

$$V(\bar{x}, t) = -\frac{4\pi}{\epsilon_0} \sum_{k \in W} (\frac{\alpha(k)\sin(k|\bar{x}|)}{c|\bar{x}|} e^{ikct} + \frac{\beta(k)\sin(k|\bar{x}|)}{c|\bar{x}|} e^{-ikct}), \quad (8).$$

Translating coordinates if necessary, and approximating a nerve fibre as a line passing through the origin $\bar{0}$, $\bar{x} = \lambda\bar{\tau}$, with $|\tau| = 1$, we have that the restriction of V along the fibre is given by;

$$\begin{aligned} V(\lambda, t) &= -\frac{4\pi}{\epsilon_0} \sum_{k \in W} (\frac{\alpha(k)\sin(k|\lambda|)}{c|\lambda|} e^{ikct} + \frac{\beta(k)\sin(k|\lambda|)}{c|\lambda|} e^{-ikct}) \\ &= -\frac{4\pi}{\epsilon_0} \sum_{k \in W} (\frac{\alpha(k)\sin(k\lambda)}{c\lambda} e^{ikct} + \frac{\beta(k)\sin(k\lambda)}{c\lambda} e^{-ikct}) \\ &(\lambda \in \mathcal{R}) \end{aligned}$$

Assuming the length of the fibre is $2r_0$, we require that $V(r_0, t) = V(-r_0, t) = 0$, (\dagger), for all $t \in \mathcal{R}$. We have that, if $\sin(kr_0) = 0$ and $k \in W$, then $kr_0 \cos(kr_0) = 0$, so that as $\cos(kr_0) \neq 0$, $k = 0$, contradicting the fact that $k \in W$. It follows that (\dagger) cannot be satisfied with the choice of r_0 for the nerve fibre and for the vanishing of surface current. If we consider a small adjustment $r_1 = r_0 + \epsilon$, we have that;

$$\begin{aligned} \sin(kr_1) &= 0, \text{ iff } \sin(kr_0 + k\epsilon) = 0 \\ &\text{iff } \sin(kr_0)\cos(k\epsilon) + \cos(kr_0)\sin(k\epsilon) = 0 \\ &\text{iff } \tan(kr_0) = -\tan(k\epsilon) \end{aligned}$$

so that if $\tan(kr_0) = kr_0$, we have that;

$$kr_0 = -\tan(k\epsilon)$$

⁸ This shows that $\bigcup_{l \in \mathcal{Z}_{\geq 0}} \frac{S_l}{r_0} = W$

This has infinitely many solutions for $\epsilon \neq 0$, but the smallest solution k_0 will dominate the potential along the nerve fibre, so that;

$$V(\lambda, t) \simeq -\frac{4\pi}{\epsilon_0} \left(\frac{\alpha(k_0)\sin(k_0\lambda)}{c\lambda} e^{ikct} + \frac{\beta(k_0)\sin(k_0\lambda)}{c\lambda} e^{-ikct} \right)$$

□

Remarks 2.6. *In order to support the above theory further, one can perform a spectral analysis of the heart signal using a Fourier transform. There are various programs available for this which use the FFT (fast fourier transform). The author would conjecture that the spectrum exhibits sharp discrete peaks.*

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