MICROWAVE ENGINEERING

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Abstract. We give an explanation of charge and current driven radiation inside waveguides and magnetrons, using the equations found in [2], and by verifying compatibility with the TM and TE modes used in microwave engineering.

Lemma 0.1. *There exist* $(\rho, \overline{J}, \overline{E}, \overline{B})$ *satisfying;*

(i). $\square^2(\rho) = 0$.

(ii). $\square^2(\overline{J}) = \overline{0}$.

(iii). $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$.

(iv). $\frac{\partial \rho}{\partial t} = -\bigtriangledown \cdot \overline{J} = 0$.

(v). $\square^2(\overline{E}) = \bigtriangledown \times \overline{E} = \overline{0}$

(vi). $\overline{B} = \overline{0}$

(vii). $\bigtriangledown \cdot \overline{E} = \frac{\rho}{\epsilon_0}$

(viii) $\frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} + \mu_0 \overline{J} = \overline{0}$

such that;

$\rho(x, y, z, t) = p(x, y)e^{i(kz-\omega t)}$

$\overline{J} = \overline{j}(x, y)e^{i(kz-\omega t)}$, $\overline{j} = (j_1, j_2, j_3)$.

$\overline{E} = \overline{e}(x, y)e^{i(kz-\omega t)}$, $\overline{e} = (e_1, e_2, e_3)$.

$\overline{B} = \overline{b}(x, y)e^{i(kz-\omega t)}$, $\overline{b} = (b_1, b_2, b_3)$.

In particularly, Maxwell's equations are satisfied for $(\rho, \overline{J}, \overline{E}, \overline{B})$.

We have that there exists a potential V *with* $\bigtriangledown(V) = -\overline{E}$, *such that* $\Box^2(V) = 0$ *and* $V = \frac{c^2\rho}{\omega^2\epsilon_0}$, *with* $V = V' + d(t)$, *where* V' *is a usual electric potential. For a given reference point* (x_0, y_0, z_0), *we have that;*

$$V'(x,y,z,t) = \frac{c^2}{\omega^2\epsilon_0}[p(x,y)e^{ikz} - p(x_0,y_0)e^{ikz_0}]e^{-i\omega t}$$

There exist $(0, \overline{0}, \overline{E}', \overline{B}')$ *satisfying Maxwell's equations in vacuum;*

(i). $\bigtriangledown \cdot \overline{E}' = 0$

(ii). $\bigtriangledown \times \overline{E}' = -\frac{\partial \overline{B}'}{\partial t}$

(iii). $\bigtriangledown \cdot \overline{B}' = \overline{0}$

(iv) $\bigtriangledown \times \overline{B}' = \frac{1}{c^2}\frac{\partial \overline{E}'}{\partial t}$

$\overline{E}' = \overline{e}'(x,y)e^{i(kz-\omega t)}$, $\overline{e}' = (e_1', e_2', e_3')$.

$\overline{B}' = \overline{b}'(x,y)e^{i(kz-\omega t)}$, $\overline{b}' = (b_1', b_2', b_3')$.

with $\overline{B}' \neq \overline{0}$

Proof. For (i), we have, substituting $p(x,y)e^{i(kz-\omega t)}$ for ρ, that;

$$[p_{xx} + p_{yy} - k^2 p]e^{i(kz-\omega t)} = \frac{1}{c^2}p(-\omega^2)e^{i(kz-\omega t)}$$

so we require that $p_{xx} + p_{yy} + (\frac{\omega^2}{c^2} - k^2)p = 0$, $(*)$.

The proof that this can be solved in $\mathcal{R}^2$ will be shown as a special case of the next lemma. For (iii), we have, substituting $p(x,y)e^{i(kz-\omega t)}$ for ρ, and $\overline{j}(x,y)e^{i(kz-\omega t)}$ for $\overline{J}$, that;

$$(p_x, p_y, ikp)e^{i(kz-\omega t)} = -\frac{1}{c^2}(j_1, j_2, j_3)(-i\omega)e^{i(kz-\omega t)}$$

so that;

$$j_1 = \frac{c^2}{i\omega}p_x = -\frac{ic^2}{\omega}p_x$$

$j_2 = \frac{c^2}{i\omega}p_y = -\frac{ic^2}{\omega}p_y$

$j_3 = \frac{c^2 ik}{i\omega}p = \frac{c^2 k}{\omega}p$ $(**)$

If p satisfies $(*)$, differentiating, so do p_x and p_y, then, from $(**)$, the components $\{j_1, j_2, j_3\}$ satisfy $(*)$ and (ii) is satisfied. For (iv), we have, substituting again, and using $(**)$, that;

$-i\omega p e^{i(kz-\omega t)} = -(j_{1x} + j_{2x} + j_{3x})$

$= -(\frac{c^2}{i\omega}p_{xx} + \frac{c^2}{i\omega}p_{yy} + \frac{k^2c^2}{i\omega}p)e^{i(kz-\omega t)}$

so that;

$-\frac{c^2}{i\omega}p_{xx} - \frac{c^2}{i\omega}p_{yy} + (\frac{k^2c^2}{i\omega} + i\omega)p = 0$

and multiplying by $-\frac{i\omega}{c^2}$;

$p_{xx} + p_{yy} + (\frac{\omega^2}{c^2} - k^2)p = 0$

which is $(*)$. As all the steps are reversible, we obtain (iv). For (v), substituting $\overline{e}(x, y)e^{i(kz-\omega t)}$ for $\overline{E}$, we require that;

$\frac{\partial e_3}{\partial y} - ike_2 = 0$

$\frac{\partial e_3}{\partial x} - ike_1 = 0$

$\frac{\partial e_2}{\partial x} - \frac{\partial e_1}{\partial y} = 0$ $(***)$

so that;

$e_1 = \frac{1}{ik}\frac{\partial e_3}{\partial x}$

$e_2 = \frac{1}{ik}\frac{\partial e_3}{\partial y}$ $(****)$

and we automatically obtain that $\frac{\partial e_2}{\partial x} - \frac{\partial e_1}{\partial y} = 0$, as the partial derivatives $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ commute. For (vii), we require that;

$(\frac{\partial e_1}{\partial x} + \frac{\partial e_2}{\partial y} + ike_3)e^{i(kz-\omega t)} = \frac{p}{\epsilon_0}e^{i(kz-\omega t)}$

so that;

$\frac{\partial e_1}{\partial x}+\frac{\partial e_2}{\partial y}+ike_3=\frac{p}{\epsilon_0}$

and, using $(****)$;

$-\frac{i}{k}\frac{\partial^2 e_3}{\partial x^2}-\frac{i}{k}\frac{\partial^2 e_3}{\partial y^2}+ike_3=\frac{p}{\epsilon_0}$

so that, multiplying by ik;

$\frac{\partial^2 e_3}{\partial x^2}+\frac{\partial^2 e_3}{\partial y^2}-k^2e_3=\frac{ikp}{\epsilon_0}$ $(\dagger)$

By (v) again, the component e_3 has to satisfy;

$\frac{\partial^2 e_3}{\partial x^2}+\frac{\partial^2 e_2}{\partial^2 y^2}+(\frac{\omega^2}{c^2}-k^2)e_3=0$ $(\dagger\dagger)$

and, combining $(\dagger),(\dagger\dagger)$, we obtain that;

$-\frac{\omega^2}{c^2}e_3=\frac{ikp}{\epsilon_0}$

$e_3=\frac{-ikc^2p}{\omega^2\epsilon_0}$ $(\dagger\dagger\dagger)$

By $(****)$, we then obtain $\square^2\overline{E}=\overline{0}$, so that $(v),(vii)$ are satisfied.

By $(****)$ and $(\dagger\dagger\dagger)$, we have that;

$e_1=\frac{1}{ik}\frac{\partial e_3}{\partial x}=\frac{1}{ik}\frac{-ikc^2}{\omega^2\epsilon_0}p_x=-\frac{c^2}{\omega^2\epsilon_0}p_x$

$e_2=\frac{1}{ik}\frac{\partial e_3}{\partial y}=\frac{1}{ik}\frac{-ikc^2}{\omega^2\epsilon_0}p_y=-\frac{c^2}{\omega^2\epsilon_0}p_y$ $(\sharp)$

For $(viii)$, we require that;

$-\frac{i\omega}{c^2}\overline{e}e^{i(kz-\omega t)}=-\mu_0\overline{j}e^{i(kz-\omega t)}$

so that;

$j_1=\frac{i\omega}{\mu_0c^2}e_1$

$j_2=\frac{i\omega}{\mu_0c^2}e_2$

$j_3=\frac{i\omega}{\mu_0c^2}e_3$ $(\sharp\sharp)$

Combining ($\sharp\sharp$) with ($\sharp$), ($\dagger\dagger\dagger$), we obtain that;

$j_1 = \frac{i\omega}{\mu_0 c^2} e_1 = \frac{i\omega}{\mu_0 c^2} \frac{-c^2 p_x}{\omega^2 \epsilon_0} = -\frac{ic^2 p_x}{\omega}$

$j_2 = \frac{i\omega}{\mu_0 c^2} e_2 = \frac{i\omega}{\mu_0 c^2} \frac{-c^2 p_y}{\omega^2 \epsilon_0} = -\frac{ic^2 p_y}{\omega}$

$j_3 = \frac{i\omega}{\mu_0 c^2} e_2 = \frac{i\omega}{\mu_0 c^2} \frac{-ikc^2 p}{\omega^2 \epsilon_0} = \frac{c^2 kp}{\omega}$

which is consistent with ($**$). For (vi), set $b_1 = b_2 = b_3 = 0$. The second claim follows easily by rearranging $(v) - (viii)$.

For the third claim, we have, by (iii) and ($viii$), the form of $\overline{E}$, that;

$\bigtriangledown(\rho) = -\frac{1}{c^2} \frac{\partial \overline{J}}{\partial t}$

$= -\frac{1}{c^2} \frac{-1}{c^2 \mu_0} \frac{\partial^2 \overline{E}}{\partial t^2}$

$= \frac{\epsilon_0}{c^2} (-\omega^2) \overline{e}(x, y) e^{i(kz - \omega t)}$

$= \frac{-\epsilon_0 \omega^2}{c^2} \overline{E}$

so that;

$\bigtriangledown(\frac{c^2}{\epsilon_0 \omega^2} \rho) = -\overline{E}$

Letting $V = \frac{c^2 \rho}{\epsilon_0 \omega^2}$, by ($i$), we have that $\Box^2(V) = 0$. By (v) and Stokes's theorem, the electric potential given by;

$V'(\overline{r}) = -\int_O^{\overline{r}} \overline{E} \cdot d\overline{l}$

for a choice of path $\overline{l}$ from a fixed reference point O is well defined, with $\bigtriangledown(V') = -\overline{E}$. As $\bigtriangledown(V - V') = \overline{0}$, we then have that $V = V' + d(t)$. Fix a reference point $O = (x_0, y_0, z_0)$, then, using the path $\overline{l}(t) = (x_0 + t(x - x_0), y_0, z_0)$, with $\overline{l}'(t) = (x - x_0, 0, 0)$, the corresponding potential V' at (x, y_0, z_0) is given by;

$V'(x, y_0, z_0) = -\int_O^{(x, y_0, z_0)} \overline{E} \cdot d\overline{l}$

$= -\int_0^1 \overline{E}(x_0 + t(x - x_0), y_0, z_0) \cdot (x - x_0, 0, 0) dt$

$= -\int_0^1 e_1(x,y)e^{i(kz-\omega t)}|_{x_0+t(x-x_0),y_0,z_0}(x-x_0)dt$

$= \int_0^1 \frac{c^2 p_x(x_0+t(x-x_0),y_0)}{\omega^2\epsilon_0}e^{i(kz_0-\omega t)}(x-x_0)dt$

$= \frac{c^2 p(x,y_0)}{\omega^2\epsilon_0}e^{i(kz_0-\omega t)} - \frac{c^2 p(x_0,y_0)}{\omega^2\epsilon_0}e^{i(kz_0-\omega t)}$

$= V(x,y_0,z_0) + d(t)$

$= \frac{c^2 p(x,y_0)}{\omega^2\epsilon_0}e^{i(kz_0-\omega t)} + d(t)$

so that;

$d(t) = -\frac{c^2 p(x_0,y_0)}{\omega^2\epsilon_0}e^{i(kz_0-\omega t)}$

$= -\frac{c^2\rho(x_0,y_0,z_0,t)}{\omega^2\epsilon_0}$

and;

$V'(\overline{r},t) = V(\overline{r},t) + d(t) = \frac{c^2}{\omega^2\epsilon_0}(\rho(\overline{r},t) - \rho(O,t))$

$= \frac{c^2}{\omega^2\epsilon_0}[p(x,y)e^{ikz} - p(x_0,y_0)e^{ikz_0}]e^{-i\omega t}$

The fourth claim is shown in [3] and [6], solving Maxwell's equations, we require that;

$e_1' = \frac{i}{\frac{\omega^2}{c^2}-k^2}(ke_{3x}' + \omega b_{3y}')$

$e_2' = \frac{i}{\frac{\omega^2}{c^2}-k^2}(ke_{3y}' - \omega b_{3x}')$

$b_1' = \frac{i}{\frac{\omega^2}{c^2}-k^2}(kb_{3x}' - \frac{\omega}{c^2}e_{3y}')$

$b_2' = \frac{i}{\frac{\omega^2}{c^2}-k^2}(kb_{3y}' + \frac{\omega}{c^2}e_{3x}')$

and;

$e_{3,xx}' + e_{3,yy}' + (\frac{\omega^2}{c^2} - k^2)e_3' = 0$

$b_{3,xx}' + b_{3,yy}' + (\frac{\omega^2}{c^2} - k^2)b_3' = 0$

The proof that this can be solved in $\mathcal{R}^2$ with $\overline{b}' \neq 0$ will be shown in the next lemma.

$\square$

Definition 0.2. *We call an electromagnetic pair* $(\overline{E}, \overline{B})$ *resonant, if it corresponds to a nontrivial charge and current* $(\rho, \overline{J})$ *satisfying the first set of equations in Lemma 0.1. We call an electromagnetic pair* $(\overline{E}, \overline{B})$ *responsive, if it corresponds to zero charge and current, satisfying the second set of equations in Lemma 0.1, with* $\overline{B} \neq \overline{0}$.

Lemma 0.3. *Rectangular Waveguide*

Given a rectangular waveguide with the cross section having coordinates at $\{(-a,-b),(-a,b),(a,-b),(a,b)\}$*, then, if the boundary is a perfect conductor, with* $\overline{E}_1 = \overline{0}$ *and* $\overline{B}_1 = \overline{0}$ *inside the conductor, we can find a resonant solution* $(\overline{E}, \overline{B})$*, inside the waveguide, satisfying the boundary conditions;*

$\overline{E}^{||} = \overline{0},\ B^{\perp} = 0$

Similarly, we can find responsive solutions $(\overline{E}', \overline{B}')$*, outside the waveguide, satisfying the boundary conditions;*

$\overline{E}'^{||} = \overline{0},\ B'^{\perp} = 0$

for both the TM and TE modes. In the TM and TE modes of the responsive solution, for the surface charge and current given by;

$\frac{\sigma_f}{\epsilon_0} = E'^{\perp} - E^{\perp}$

$\mu_0(\overline{K}_f \times \hat{n}) = \overline{B}'^{||} - \overline{B}^{||}$

the continuity equation holds in the form;

$div(\overline{K}_f) + (\overline{J}' - \overline{J}) \cdot \hat{n} = -\frac{\partial \sigma_f}{\partial t}$

Proof. For the first claim, we find a resonant $(\rho, \overline{J}, \overline{E}, \overline{B})$ as in the previous lemma in the interior of the waveguide. Without loss of generality, we can assume the interior is a vacuum. The boundary condition dictates that $e_1 = e_3 = 0$ on the horizontal faces and $e_2 = e_3 = 0$ on the vertical faces. As is done in [3], and using the notation in the previous lemma, this is achieved by the solution;

$p(x,y) = sin(\frac{\pi m x}{a})sin(\frac{\pi n y}{b})$, $|x| \leq a, |y| \leq b$, $\{m, n \subset \mathcal{N}\}$

with $-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + (\frac{\omega^2}{c^2} - k^2) = 0$

Then $e_3 = -\frac{ikc^2 p}{\omega^2 \epsilon_0}$ vanishes on all the faces of the waveguide, while;

$e_1 = -\frac{c^2}{\omega^2\epsilon_0}p_x = -\frac{c^2}{\omega^2\epsilon_0}\frac{\pi m}{a}cos(\frac{\pi m x}{a})sin(\frac{\pi n y}{b})$

$e_2 = -\frac{c^2}{\omega^2\epsilon_0}p_y = -\frac{c^2}{\omega^2\epsilon_0}\frac{\pi n}{b}sin(\frac{\pi m x}{a})cos(\frac{\pi n y}{b})$

vanish on the horizontal and vertical faces respectively, as required. Clearly $B^{\perp} = 0$ on the boundary as $b_1 = b_2 = b_3 = 0$.

For the next claim, the boundary condition dictates that $e_1' = e_3' = 0$ on the horizontal faces, $e_2' = e_3' = 0$ on the vertical faces, $b_2' = 0$ on the horizontal faces, $b_1' = 0$ on the vertical faces. We can achieve this with the TM (transverse magnetic) mode, defined by;

$e_3'(x,y) = sin(\frac{\pi m x}{a})sin(\frac{\pi n y}{b})$, $|x| \geq a$ or $|y| \geq b$, $\{m, n \subset \mathcal{N}\}$

with $-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + (\frac{\omega^2}{c^2} - k^2) = 0$

and $b_3'(x,y) = 0$. From the equations of the previous lemma, we must have that;

$e_1' = \frac{ik}{\frac{\omega^2}{c^2}-k^2}e_{3x}'$

$e_2' = \frac{ik}{\frac{\omega^2}{c^2}-k^2}e_{3y}'$

$b_1' = \frac{-i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e_{3y}'$

$b_2' = \frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e_{3x}'$

Clearly, e_3' vanishes on the horizontal and vertical faces, while e_{3x}' vanishes on the horizontal faces and e_{3y}' vanishes on the vertical faces, so e_1' vanishes on the horizontal faces and e_2' vanishes on the vertical faces, and similarly, b_2' vanishes on the horizontal faces, b_1' vanishes on the vertical faces. Note that $\overline{b} \neq \overline{0}$, so the solution is responsive.

We can also achieve the boundary condition with the TE (transverse electric) mode, defined by;

$e'_3 = 0$

$b'_3 = cos(\frac{\pi m x}{a})cos(\frac{\pi n y}{b})$, $|x| \geq a$ or $|y| \geq b$, $\{m, n \subset \mathcal{N}\}$

with $-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + (\frac{\omega^2}{c^2} - k^2) = 0$

From the equations of the previous lemma, we must have that;

$e'_1 = \frac{i\omega}{\frac{\omega^2}{c^2}-k^2} b'_{3y}$

$e'_2 = \frac{-i\omega}{\frac{\omega^2}{c^2}-k^2} b'_{3x}$

$b'_1 = \frac{ik}{\frac{\omega^2}{c^2}-k^2} b'_{3x}$

$b'_2 = \frac{ik}{\frac{\omega^2}{c^2}-k^2} b'_{3y}$

Clearly, e'_3 vanishes on the horizontal and vertical faces, while b'_{3x} vanishes on the vertical faces and b'_{3y} vanishes on the horizontal faces, so e'_1 vanishes on the horizontal faces and e'_2 vanishes on the vertical faces, and similarly, b'_2 vanishes on the horizontal faces, b'_1 vanishes on the vertical faces. Note that $\overline{b} \neq 0$, so the solution is responsive.

For the resonant field, we have that;

$p_x = \frac{\pi m}{a} cos(\frac{\pi m x}{a}) sin(\frac{\pi n y}{b})$

$p_y = \frac{\pi n}{b} sin(\frac{\pi m x}{a}) cos(\frac{\pi n y}{b})$

and for the responsive field in the TM mode;

$e'_{3x} = \frac{\pi m'}{a} cos(\frac{\pi m' x}{a}) sin(\frac{\pi n' y}{b})$

$e'_{3y} = \frac{\pi n'}{b} sin(\frac{\pi m' x}{a}) cos(\frac{\pi n' y}{b})$

so that, for the TM mode;

$\frac{\sigma_f}{\epsilon_0} = E'^{\perp} - E^{\perp}$

$= (-e_2 + e'_2)e^{i(kz-\omega t)}$ (on the horizontal faces)

$= [\frac{c^2}{\omega^2\epsilon_0}p_y + \frac{ik}{\frac{\omega^2}{c^2}-k^2}e'_{3y}]e^{i(kz-\omega t)}$

$= [\frac{c^2}{\omega^2\epsilon_0}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a}) + \frac{ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n'}{b}(-1)^{n'} sin(\frac{\pi m'x}{a})]e^{i(kz-\omega t)}$

$= (-1)^n\frac{\pi n}{b}[\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2}]sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$ $(m = m', n = n')$

and;

$\frac{\sigma_f}{\epsilon_0} = (-e_1 + e'_1)e^{i(kz-\omega t)}$ (on the vertical faces)

$= [\frac{c^2}{\omega^2\epsilon_0}p_x + \frac{ik}{\frac{\omega^2}{c^2}-k^2}e'_{3x}]e^{i(kz-\omega t)}$

$= [\frac{c^2}{\omega^2\epsilon_0}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b}) + \frac{ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m'}{a}(-1)^{m'} sin(\frac{\pi n'y}{b})]e^{i(kz-\omega t)}$

$= (-1)^m\frac{\pi m}{a}[\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2}]sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$ $(m = m', n = n')$

and;

$\mu_0(\overline{K}_f \times \hat{\overline{n}}) = \overline{B}'^{||} - \overline{B}^{||}$

$= [-(b_1, b_3) + (b'_1, b'_3)]e^{i(kz-\omega t)}$ (on the horizontal faces)

$= [(0, 0) + (b'_1, 0)]e^{i(kz-\omega t)}$

$= (b'_1, 0)e^{i(kz-\omega t)}$

$= (-\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3y}e^{i(kz-\omega t)}, 0)$

$= (-\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}, 0)$

so that;

$\mu_0\overline{K}_f = (0, \frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{i(kz-\omega t)})$

and;

$\mu_0(\overline{K}_f \times \hat{\overline{n}}) = [-(b_2, b_3) + (b'_2, b'_3)]e^{i(kz-\omega t)}$ (on the vertical faces)

$[(0,0)+(b'_2,0)]e^{i(kz-\omega t)}$

$=(b'_2 e^{i(kz-\omega t)},0)$

$=(\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3x}e^{i(kz-\omega t)},0)$

$=(\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{i(kz-\omega t)},0)$

so that;

$\mu_0\overline{K}_f=(0,\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{i(kz-\omega t)})$

By the continuity equation on the boundary, see [1], we must have that, on the horizontal faces;

$div(\overline{K}_f)+(\overline{J}'-\overline{J})\centerdot\hat{n}$

$=div(\overline{K}_f)-\overline{J}\centerdot\hat{n}$

$=div(\overline{K}_f)-j_2 e^{i(kz-\omega t)}$

$=div[(0,\frac{i\omega}{\mu_0 c^2(\frac{\omega}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{i(kz-\omega t)})]+\frac{ic^2 p_y}{\omega}e^{i(kz-\omega t)}$

$=ik\frac{i\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}+\frac{ic^2}{\omega}(-1)^n\frac{\pi n}{b}sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$

$=[ik\frac{i\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}+\frac{ic^2}{\omega}](-1)^n\frac{\pi n}{b}sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$

$=-\frac{\partial\sigma_f}{\partial t}$

$=-\frac{\partial}{\partial t}(\epsilon_0(-1)^n\frac{\pi n}{b}[\frac{c^2}{\omega^2\epsilon_0}+\frac{ik}{\frac{\omega^2}{c^2}-k^2}]sin(\frac{\pi mx}{a})e^{i(kz-\omega t)})$

$=i\omega\epsilon_0(-1)^n\frac{\pi n}{b}[\frac{c^2}{\omega^2\epsilon_0}+\frac{ik}{\frac{\omega^2}{c^2}-k^2}]sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$

$=[i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0}+\frac{ik}{\frac{\omega^2}{c^2}-k^2})](-1)^n\frac{\pi n}{b}sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$

so that;

$ik\frac{i\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}+\frac{ic^2}{\omega}$

$=-\frac{\omega k\epsilon_0}{\frac{\omega^2}{c^2}-k^2}+\frac{ic^2}{\omega}$

$= i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2})$

$= -\frac{\omega k\epsilon_0}{\frac{\omega^2}{c^2}-k^2} + \frac{ic^2}{\omega}$

By the continuity equation on the boundary again, we must have that, on the vertical faces;

$div(\overline{K}_f) + (\overline{J}' - \overline{J}) \centerdot \hat{n}$

$= div(\overline{K}_f) - \overline{J} \centerdot \hat{n}$

$= div(\overline{K}_f) - j_1 e^{i(kz-\omega t)}$

$= div[(0, \frac{i\omega}{\mu_0 c^2(\frac{\omega}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{i(kz-\omega t)})] + \frac{ic^2 p_x}{\omega}e^{i(kz-\omega t)}$

$= ik\frac{i\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{i(kz-\omega t)} + \frac{ic^2}{\omega}(-1)^m\frac{\pi m}{a}sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$

$= [ik\frac{i\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)} + \frac{ic^2}{\omega}](-1)^m\frac{\pi m}{a}sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$

$= -\frac{\partial\sigma_f}{\partial t}$

$= -\frac{\partial}{\partial t}(\epsilon_0(-1)^m\frac{\pi m}{a}[\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2}]sin(\frac{\pi ny}{b})e^{i(kz-\omega t)})$

$= i\omega\epsilon_0(-1)^m\frac{\pi m}{a}[\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2}]sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$

$= [i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2})](-1)^m\frac{\pi m}{a}sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$

so that;

$ik\frac{i\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)} + \frac{ic^2}{\omega}$

$= -\frac{\omega k\epsilon_0}{\frac{\omega^2}{c^2}-k^2} + \frac{ic^2}{\omega}$

$= i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0} + \frac{ik}{\frac{\omega^2}{c^2}-k^2})$

$= -\frac{\omega k\epsilon_0}{\frac{\omega^2}{c^2}-k^2} + \frac{ic^2}{\omega}$

again.

For the responsive field in the TE mode, we have that, for $m = m', n = n'$;

$b'_{3x} = -\frac{\pi m}{a} sin(\frac{\pi m x}{a}) cos(\frac{\pi n y}{b})$

$b'_{3y} = -\frac{\pi n}{b} cos(\frac{\pi m x}{a}) sin(\frac{\pi n y}{b})$

so that, for the TE mode, with $m = m', n = n'$;

$\frac{\sigma_f}{\epsilon_0} = E'^{\perp} - E^{\perp}$

$= (-e_2 + e'_2) e^{i(kz-\omega t)}$ (on the horizontal faces)

$= [\frac{c^2}{\omega^2 \epsilon_0} p_y - \frac{i\omega}{\frac{\omega^2}{c^2} - k^2} b'_{3x}] e^{i(kz-\omega t)}$

$= [\frac{c^2}{\omega^2 \epsilon_0} \frac{\pi n}{b} (-1)^n sin(\frac{\pi m x}{a}) + \frac{i\omega}{\frac{\omega^2}{c^2} - k^2} \frac{\pi m}{a} (-1)^n sin(\frac{\pi m x}{a})] e^{i(kz-\omega t)}$

$= (-1)^n [\frac{c^2}{\omega^2 \epsilon_0} \frac{\pi n}{b} + \frac{i\omega}{\frac{\omega^2}{c^2} - k^2} \frac{\pi m}{a}] sin(\frac{\pi m x}{a}) e^{i(kz-\omega t)}$

and, for $m = m', n = n'$;

$\frac{\sigma_f}{\epsilon_0} = (-e_1 + e'_1) e^{i(kz-\omega t)}$ (on the vertical faces)

$= [\frac{c^2}{\omega^2 \epsilon_0} p_x + \frac{i\omega}{\frac{\omega^2}{c^2} - k^2} b'_{3y}] e^{i(kz-\omega t)}$

$= [\frac{c^2}{\omega^2 \epsilon_0} \frac{\pi m}{a} (-1)^m sin(\frac{\pi n y}{b}) + \frac{i\omega}{\frac{\omega^2}{c^2} - k^2} \frac{-\pi n}{b} (-1)^m sin(\frac{\pi n y}{b})] e^{i(kz-\omega t)}$

$= (-1)^m [\frac{c^2}{\omega^2 \epsilon_0} \frac{\pi m}{a} - \frac{i\omega}{\frac{\omega^2}{c^2} - k^2} \frac{\pi n}{b}] sin(\frac{\pi n y}{b}) e^{i(kz-\omega t)}$

We have, on the horizontal faces;

$\mu_0(\overline{K}_f \times \hat{\overline{n}}) = [-(b_1, b_3) + (b'_1, b'_3)] e^{i(kz-\omega t)}$

$[(0,0) + (\frac{ik}{\frac{\omega^2}{c^2} - k^2} b'_{3x}, cos(\frac{\pi m x}{a}) cos(\frac{\pi n y}{b}))] e^{i(kz-\omega t)}$

$= (\frac{ik}{\frac{\omega^2}{c^2} - k^2} \frac{-\pi m}{a} sin(\frac{\pi m x}{a}) cos(\frac{\pi n y}{b}), cos(\frac{\pi m x}{a}) cos(\frac{\pi n y}{b})) e^{i(kz-\omega t)}$

$= (\frac{-ik}{\frac{\omega^2}{c^2} - k^2} (-1)^n \frac{\pi m}{a} sin(\frac{\pi m x}{a}), (-1)^n cos(\frac{\pi m x}{a})) e^{i(kz-\omega t)}$

so that;

$\mu_0\overline{K}_f=((-1)^n cos(\frac{\pi mx}{a}),\frac{ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}(-1)^n sin(\frac{\pi mx}{a}))e^{i(kz-\omega t)}$

We have, on the vertical faces;

$\mu_0(\overline{K}_f\times\hat{\overline{n}})=[-(b_2,b_3)+(b_2',b_3')]e^{i(kz-\omega t)}$

$[(0,0)+(\frac{ik}{\frac{\omega^2}{c^2}-k^2}b_{3y}',cos(\frac{\pi mx}{a})cos(\frac{\pi ny}{b}))]e^{i(kz-\omega t)}$

$=(\frac{ik}{\frac{\omega^2}{c^2}-k^2}\frac{-\pi n}{b}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b}),cos(\frac{\pi mx}{a})cos(\frac{\pi ny}{b}))e^{i(kz-\omega t)}$

$=(\frac{-ik}{\frac{\omega^2}{c^2}-k^2}(-1)^m\frac{\pi n}{b}sin(\frac{\pi ny}{b}),(-1)^m cos(\frac{\pi ny}{b}))e^{i(kz-\omega t)}$

so that;

$\mu_0\overline{K}_f=((-1)^{m+1}cos(\frac{\pi ny}{b}),\frac{-ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}(-1)^m sin(\frac{\pi ny}{b}))e^{i(kz-\omega t)}$

By the continuity equation on the boundary again, we must have that, on the horizontal faces;

$div(\overline{K}_f)+(\overline{J}'-\overline{J})\cdot\hat{n}$

$=div(\overline{K}_f)-\overline{J}\cdot\hat{n}$

$=div(\overline{K}_f)-j_2e^{i(kz-\omega t)}$

$=div[\frac{(-1)^n}{\mu_0}cos(\frac{\pi mx}{a}),\frac{ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}\frac{(-1)^n}{\mu_0}sin(\frac{\pi mx}{a})]e^{i(kz-\omega t)}+\frac{ic^2p_y}{\omega}e^{i(kz-\omega t)}$

$=[\frac{(-1)^{n+1}}{\mu_0}sin(\frac{\pi mx}{a})\frac{\pi m}{a}+\frac{ik(ik)}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}\frac{(-1)^n}{\mu_0}sin(\frac{\pi mx}{a})+\frac{ic^2}{\omega}(-1)^n\frac{\pi n}{b}sin(\frac{\pi mx}{a})]e^{i(kz-\omega t)}$

$=[-\frac{\pi m}{\mu_0 a}-\frac{k^2}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a\mu_0}+\frac{ic^2}{\omega}\frac{\pi n}{b}](-1)^n sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$

$=-\frac{\partial\sigma_f}{\partial t}$

$=-\frac{\partial}{\partial t}(\epsilon_0(-1)^n[\frac{c^2}{\omega^2\epsilon_0}\frac{\pi n}{b}+\frac{i\omega}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}]sin(\frac{\pi mx}{a})e^{i(kz-\omega t)})$

$=[i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0}\frac{\pi n}{b}+\frac{i\omega}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a})](-1)^n sin(\frac{\pi mx}{a})e^{i(kz-\omega t)}$

so that;

$-\frac{\pi m}{\mu_0 a}-\frac{k^2}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{\mu_0 a}+\frac{ic^2}{\omega}\frac{\pi n}{b}$

$= -\frac{\frac{\omega^2}{c^2}}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{\mu_0 a} + \frac{ic^2}{\omega}\frac{\pi n}{b}$

$= -\frac{\omega^2\epsilon_0}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a} + \frac{ic^2}{\omega}\frac{\pi n}{b}$

$= i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0}\frac{\pi n}{b} + \frac{i\omega}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a})$

$= -\frac{\omega^2\epsilon_0}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a} + \frac{ic^2}{\omega}\frac{\pi n}{b}$

By the continuity equation on the boundary again, we must have that, on the vertical faces;

$div(\overline{K}_f) + (\overline{J}' - \overline{J}) \cdot \hat{n}$

$= div(\overline{K}_f) - \overline{J} \cdot \hat{n}$

$= div(\overline{K}_f) - j_1 e^{i(kz-\omega t)}$

$= div[\frac{(-1)^{m+1}}{\mu_0}cos(\frac{\pi ny}{b}), -\frac{ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}\frac{(-1)^m}{\mu_0}sin(\frac{\pi ny}{b})]e^{i(kz-\omega t)} + \frac{ic^2 p_x}{\omega}e^{i(kz-\omega t)}$

$= [\frac{(-1)^m}{\mu_0}sin(\frac{\pi ny}{b})\frac{\pi n}{b} + \frac{ik(-ik)}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}\frac{(-1)^m}{\mu_0}sin(\frac{\pi ny}{b}) + \frac{ic^2}{\omega}(-1)^m\frac{\pi m}{a}sin(\frac{\pi ny}{b})]e^{i(kz-\omega t)}$

$= [\frac{\pi n}{\mu_0 b} + \frac{k^2}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b\mu_0} + \frac{ic^2}{\omega}\frac{\pi m}{a}](-1)^m sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$

$= -\frac{\partial\sigma_f}{\partial t}$

$= -\frac{\partial}{\partial t}(\epsilon_0(-1)^m[\frac{c^2}{\omega^2\epsilon_0}\frac{\pi m}{a} - \frac{i\omega}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}]sin(\frac{\pi ny}{b})e^{i(kz-\omega t)})$

$= [i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0}\frac{\pi m}{a} - \frac{i\omega}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b})](-1)^m sin(\frac{\pi ny}{b})e^{i(kz-\omega t)}$

so that;

$\frac{\pi n}{\mu_0 b} + \frac{k^2}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{\mu_0 b} + \frac{ic^2}{\omega}\frac{\pi m}{a}$

$= \frac{\frac{\omega^2}{c^2}}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{\mu_0 b} + \frac{ic^2}{\omega}\frac{\pi m}{a}$

$= \frac{\omega^2\epsilon_0}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b} + \frac{ic^2}{\omega}\frac{\pi m}{a}$

$= i\omega\epsilon_0(\frac{c^2}{\omega^2\epsilon_0}\frac{\pi m}{a} - \frac{i\omega}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b})$

$= \frac{\omega^2\epsilon_0}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b} + \frac{ic^2}{\omega}\frac{\pi m}{a}$

□

Definition 0.4. *For (k,ω,m,n) with;*

$-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}+(\frac{\omega^2}{c^2}-k^2)=0$

we denote by $(\overline{E}_{k,\omega,m,n},\overline{0})$ the resonant solution found above in the interior of the waveguide, and by $(\overline{E}'_{k,\omega,m,n},\overline{B}'_{k,\omega,m,n})$ the responsive solutions found above in the TM and TE modes.

Lemma 0.5. *Cavity Magnetron*

Given a cavity magnetron with the corners having coordinates at;

$\{(-a,-b,d),(-a,b,d),(a,-b,d),(a,b,d),(-a,-b,-d),(-a,b,-d),(a,-b,-d),(a,b,-d)\}$

then, if the boundary is a perfect conductor, with $\overline{E}_1=\overline{0}$ and $\overline{B}_1=\overline{0}$ inside the conductor, we can find a resonant solution $(\overline{E},\overline{B})$, inside the magnetron, satisfying the boundary conditions;

$\overline{E}^{||}=\overline{0},\ B^{\perp}=0$

Similarly, we can find responsive solutions $(\overline{E}',\overline{B}')$, outside the magnetron, satisfying the boundary conditions;

$\overline{E}'^{||}=\overline{0},\ B'^{\perp}=0$

for the TM mode. In the TM mode of the responsive solution, for the surface charge and current given by;

$\frac{\sigma_f}{\epsilon_0}=E'^{\perp}-E^{\perp}$

$\mu_0(\overline{K}_f\times\hat{n})=\overline{B}'^{||}-\overline{B}^{||}$

the continuity equation holds in the form;

$div(\overline{K}_f)+(\overline{J}'-\overline{J})\cdot\hat{n}=-\frac{\partial\sigma_f}{\partial t}$

Proof. For fixed $m,n\in\mathcal{N}$, in the interior of the magnetron, we let;

$(\overline{E},\overline{0})=(Re(\overline{E}_{k,\omega,m,n}-\overline{E}_{k,-\omega,m,n}),\overline{0})$

and in the exterior of the magnetron, using the TM modes, we let;

$(\overline{E}', \overline{B}') = (Re(\overline{E}'_{k,\omega,m,n} + \overline{E}'_{k,-\omega,m,n}), Re(\overline{B}'_{k,\omega,m,n} + \overline{B}'_{k,-\omega,m,n}))$

We let;

$(\overline{E}^*, \overline{0}) = (\overline{E}_{k,\omega,m,n} - \overline{E}_{k,-\omega,m,n}, \overline{0})$

$(\overline{E}'^*, \overline{B}'^*) = (\overline{E}'_{k,\omega,m,n} + \overline{E}'_{k,-\omega,m,n}, \overline{B}'_{k,\omega,m,n} + \overline{B}'_{k,-\omega,m,n})$

Using the calculation in Lemma 0.3, we have that;

$\rho(x, y, z, t) = Re(\rho_{k,\omega,m,n} - \rho_{k,-\omega,m,n})$

$Re(sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})e^{i(kz-\omega t)} - sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})e^{i(kz+\omega t)})$

$= Re(2sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})(sin(kz)sin(\omega t) - icos(kz)sin(\omega t)))$

$= 2sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})sin(kz)sin(\omega t)$

$e_1(x, y, z, t) = Re(e_{1,k,\omega,m,n} - e_{1,k,\omega,m,n})$

$= Re(-\frac{c^2}{\omega^2\epsilon_0}\rho_{k,\omega,m,n,x} + \frac{c^2}{\omega^2\epsilon_0}\rho_{k,-\omega,m,n,x})$

$= -\frac{c^2}{\omega^2\epsilon_0}Re(\rho_{k,\omega,m,n,x} - \rho_{k,-\omega,m,n,x})$

$= -\frac{c^2}{\omega^2\epsilon_0}\rho_x(x, y, z, t)$

$= -\frac{2c^2}{\omega^2\epsilon_0}\frac{\pi m}{a}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})sin(kz)sin(\omega t)$

$e_2(x, y, z, t) = Re(e_{2,k,\omega,m,n} - e_{2,k,\omega,m,n})$

$= Re(-\frac{c^2}{\omega^2\epsilon_0}\rho_{k,\omega,m,n,y} + \frac{c^2}{\omega^2\epsilon_0}\rho_{k,-\omega,m,n,y})$

$= -\frac{c^2}{\omega^2\epsilon_0}Re(\rho_{k,\omega,m,n,y} - \rho_{k,-\omega,m,n,y})$

$= -\frac{c^2}{\omega^2\epsilon_0}\rho_y(x, y, z, t)$

$= -\frac{2c^2}{\omega^2\epsilon_0}\frac{\pi n}{b}sin(\frac{\pi mx}{a})cos(\frac{\pi ny}{b})sin(kz)sin(\omega t)$

so that, with the choice $k = \frac{r\pi}{d}$, $r \in \mathcal{N}$, $r = r'$, we have that $e_1 = e_2 = 0$ on the far faces, defined by $z = d$ and $z = -d$, so that

$\overline{E}^{||} = \overline{0}$ on the far faces. As $\overline{B} = \overline{0}$, we have that $B^{\perp} = 0$ on the far faces as well. By linearity, and taking the real parts, we can use the calculation of Lemma 0.3, to see that $\overline{E}^{||} = \overline{0}$ and $B^{\perp} = 0$, on the vertical and horizontal faces as well.

Using the calculation in Lemma 0.3, we have that;

$e_3'^*(x, y, z, t) = e'_{3,k,\omega,m,n} + e'_{3,k,-\omega,m,n}$

$sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})e^{i(kz-\omega t)} + sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})e^{i(kz+\omega t)}$

$= 2sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})(cos(kz)cos(\omega t) + isin(kz)cos(\omega t))$

$e_1'(x, y, z, t) = Re(e'_{1,k,\omega,m,n} + e'_{1,k,\omega,m,n})$

$= Re(\frac{ik}{\frac{\omega^2}{c^2}-k^2}e'_{3,k,\omega,m,n,x} + \frac{ik}{\frac{\omega^2}{c^2}-k^2}e'_{3,k,-\omega,m,n,x})$

$= \frac{k}{\frac{\omega^2}{c^2}-k^2}Re(i(e'_{3,k,\omega,m,n,x} + e'_{3,k,-\omega,m,n,x}))$

$= \frac{k}{\frac{\omega^2}{c^2}-k^2}Re(ie'^*_{3,x}(x, y, z, t))$

$= \frac{k}{\frac{\omega^2}{c^2}-k^2}Re(2sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})(icos(kz)cos(\omega t) - sin(kz)cos(\omega t)))_x$

$= \frac{-2k}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})sin(kz)cos(\omega t)$

$e_2'(x, y, z, t) = Re(e'_{2,k,\omega,m,n} + e'_{2,k,\omega,m,n})$

$= Re(\frac{ik}{\frac{\omega^2}{c^2}-k^2}e'_{3,k,\omega,m,n,y} + \frac{ik}{\frac{\omega^2}{c^2}-k^2}e'_{3,k,-\omega,m,n,y})$

$= \frac{k}{\frac{\omega^2}{c^2}-k^2}Re(i(e'_{3,k,\omega,m,n,y} + e'_{3,k,-\omega,m,n,y}))$

$= \frac{k}{\frac{\omega^2}{c^2}-k^2}Re(ie'^*_{3,y}(x, y, z, t))$

$= \frac{k}{\frac{\omega^2}{c^2}-k^2}Re(2sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})(icos(kz)cos(\omega t) - sin(kz)cos(\omega t)))_y$

$= \frac{-2k}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}sin(\frac{\pi mx}{a})cos(\frac{\pi ny}{b})sin(kz)cos(\omega t)$

so that, with the choice $k = \frac{r\pi}{d}$, $r \in \mathcal{N}$, $r = r'$, we have that $e_1' = e_2' = 0$ on the far faces, defined by $z = d$ and $z = -d$, so that $\overline{E}'^{||} = \overline{0}$ on the far faces. By definition of the TM mode, linearity and taking the real part, we have that $b_3' = 0$, in particular, we have that

$B^{\perp} = 0$ on the far faces as well. By linearity, and taking the real parts, we can use the calculation of Lemma 0.3, to see that $\overline{E}'^{||} = \overline{0}$ and $\overline{B}'^{\perp} = 0$, on the vertical and horizontal faces as well.

As verified in Lemma 0.3, we have that the pairs;

$(\overline{E}_{k,\omega,m,n}, \overline{0}, \overline{E}'_{k,\omega,m,n}, \overline{B}'_{k,\omega,m,n})$

$(\overline{E}_{k,-\omega,m,n}, \overline{0}, \overline{E}'_{k,-\omega,m,n}, \overline{B}'_{k,-\omega,m,n})$

satisfy the continuity equation at the boundary of the vertical and horizontal faces, for the associated free charges and currents $(\rho_{f,k,\omega,m,n}, \overline{J}_{f,k,\omega,m,n})$ and $(\rho_{f,k,-\omega,m,n}, \overline{J}_{f,k,-\omega,m,n})$, so that, by linearity, so does the sum;

$(\overline{E}_{k,\omega,m,n} + \overline{E}_{k,-\omega,m,n}, \overline{0}, \overline{E}'_{k,\omega,m,n} + \overline{E}'_{k,-\omega,m,n}, \overline{B}'_{k,\omega,m,n} + \overline{B}'_{k,-\omega,m,n})$

$= (\overline{E}^{*} + 2\overline{E}_{k,-\omega,m,n}, \overline{0}, \overline{E}'^{*}, \overline{B}'^{*})$

for the induced free charge and current;

$(\rho_{f,k,\omega,m,n} + \rho_{f,k,\omega,m,n}, \overline{J}_{f,k,\omega,m,n} + \overline{J}_{f,k,-\omega,m,n})$

We claim that $(2\overline{E}_{k,-\omega,m,n}, \overline{0}, \overline{0}, \overline{0})$ satisfies the continuity equation, $(*)$, for the induced free charge and current;

$\frac{\rho_f}{\epsilon_0} = 0^{\perp} - 2E^{\perp}_{k,-\omega,m,n}$

$\rho_f = -2\epsilon_0 E^{\perp}_{k,-\omega,m,n}$

$\mu_0(\overline{J}_f \times \hat{n}) = \overline{0}^{||} - \overline{0}^{||} = \overline{0}$

$\overline{J}_f = \overline{0}$

The continuity equation is given by;

$div(\overline{J}_f) + (\overline{J}' - \overline{J}) \cdot \hat{n} = -\frac{\partial \rho_f}{\partial t}$

where $\overline{J'} = \overline{0}$, so, by the above, we have to check that;

$-2\overline{J}_{k,-\omega,m,n} \cdot \hat{n} = -\frac{\partial(-2\epsilon_0 E^{\perp}_{k,-\omega,m,n})}{\partial t}$

which follows from;

$\overline{J}_{k,-\omega,m,n}+\epsilon_0\frac{\partial\overline{E}_{k,-\omega,m,n}}{\partial t}=\overline{0}$

This follows from Maxwell's equations and the fact that $\overline{B}=\overline{0}$ inside the magnetron. Hence $(*)$ is shown, and, by linearity, $=(\overline{E}^{*},\overline{0},\overline{E}'^{*},\overline{B}'^{*})$ satisfies the continuity equation at the vertical and horizontal faces.

For the far faces, we have that the free charge σ_f induced by $\{\overline{E}'^{*},\overline{E}^{*}\}$ is given by;

$\frac{\sigma_f}{\epsilon_0}=E'^{*,\perp}-E^{*,\perp}$

$=(e'_{3,\omega}e^{i(kz-\omega t)}+e_{3,-\omega}e^{i(kz+\omega t)})-(e_{3,\omega}e^{i(kz-\omega t)}-e_{3,-\omega}e^{i(kz+\omega t)})$

$=(e'_{3,\omega}-e_{3,\omega})e^{i(kz-\omega t)}+(e'_{3,-\omega}+e_{3,-\omega})e^{i(kz+\omega t)}$

$=(-1)^r e^{-i\omega t}(e'_{3,\omega}-e_{3,\omega})+(-1)^r e^{i\omega t}(e'_{3,-\omega}+e_{3,-\omega})\ (k=\frac{\pi r}{d})$

$=(-1)^r e^{-i\omega t}(sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})+\frac{ikc^2p}{\omega^2\epsilon_0})+(-1)^r e^{i\omega t}(sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})$

$-\frac{ikc^2p}{\omega^2\epsilon_0})$

$=2(-1)^r cos(\omega t)sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})-2(-1)^r isin(\omega t)\frac{ikc^2}{\omega^2\epsilon_0}sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})$

$=2(-1)^r sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})(cos(\omega t)+\frac{kc^2}{\omega^2\epsilon_0}sin(\omega t))$

$=2(-1)^r(1+\frac{k^2c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}}sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})sin(\omega t+\phi)$

where $tan(\phi)=\frac{\omega^2\epsilon_0}{kc^2}$

while the free current, $\overline{K}_f$ is given by;

$\mu_0(\overline{K}_f\times\hat{n})=\overline{B}'^{*,||}-\overline{B}^{*,||}=\overline{B}'^{*,||}$

$=(b'_{1,\omega},b'_{2,\omega})e^{i(kz-\omega t)}+(b'_{1,-\omega},b'_{2,-\omega})e^{i(kz+\omega t)}$

$=(-\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3y},\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3x})e^{i(kz-\omega t)}$

$+(\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3y},-\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3x})e^{i(kz+\omega t)}$

$= (-1)^r(-\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3y}, \frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3x})e^{-i\omega t}$

$+(-1)^r(\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3y}, -\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}e'_{3x})e^{i\omega t}$

$= 2i(-1)^r sin(\omega t)\frac{i\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}(e'_{3y}, -e'_{3x})$

$= 2(-1)^{r+1} sin(\omega t)\frac{\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}(\frac{\pi n}{b}sin(\frac{\pi mx}{a})cos(\frac{\pi ny}{b}), -\frac{\pi m}{a}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b}))$

so that;

$\mu_0\overline{K}_f = 2(-1)^{r+1} sin(\omega t)\frac{\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}(\frac{\pi m}{a}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b}), -\frac{\pi n}{b}sin(\frac{\pi mx}{a})cos(\frac{\pi ny}{b}))$

$= 2(-1)^r sin(\omega t)\frac{\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}(-\frac{\pi m}{a}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b}), \frac{\pi n}{b}sin(\frac{\pi mx}{a})cos(\frac{\pi ny}{b}))$

We have that, on the far faces;

$(\overline{J}'^* - \overline{J}^*) \cdot \hat{n}$

$= -\overline{J}^* \cdot \hat{n}$

$= -(j_{3,\omega}e^{i(kz-\omega t)} - j_{3,-\omega}e^{i(kz+\omega t)})$

$= -(\frac{c^2kp}{\omega}e^{i(kz-\omega t)} + \frac{c^2kp}{\omega}e^{i(kz+\omega t)})$

$= (-1)^{r+1}\frac{c^2kp}{\omega}(e^{-i\omega t} + e^{i\omega t})$

$= 2(-1)^{r+1}cos(\omega t)\frac{c^2k}{\omega}sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})$

so that, for the continuity equation, we have that;

$div(\overline{K}_f) + (\overline{J}'^* - \overline{J}^*) \cdot \hat{n}$

$= 2(-1)^r sin(\omega t)\frac{\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}(\frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2})sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})$

$+2(-1)^{r+1}cos(\omega t)\frac{c^2k}{\omega}sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})$

$= -\frac{\partial\sigma_f}{\partial t}$

$= -2(-1)^r\epsilon_0 sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})\frac{\partial}{\partial t}(cos(\omega t) + \frac{kc^2}{\omega^2\epsilon_0}sin(\omega t))$

$= 2(-1)^{r+1}\epsilon_0 \omega sin(\frac{\pi m x}{a}) sin(\frac{\pi n y}{b})(-sin(\omega t) + \frac{kc^2}{\omega^2 \epsilon_0} cos(\omega t))$

so that;

$sin(\omega t)\frac{\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}(\frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}) - cos(\omega t)\frac{c^2 k}{\omega}$

$= sin(\omega t)\frac{\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)}(\frac{\omega^2}{c^2} - k^2) - cos(\omega t)\frac{c^2 k}{\omega}$

$= sin(\omega t)\frac{\omega}{\mu_0 c^2} - cos(\omega t)\frac{c^2 k}{\omega}$

$= sin(\omega t)\epsilon_0 \omega - cos(\omega t)\frac{c^2 k}{\omega}$

$= -\epsilon_0 \omega(-sin(\omega t) + \frac{kc^2}{\omega^2 \epsilon_0} cos(\omega t))$

as required.

□

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