MICROWAVE ENGINEERING 2

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Abstract. We give an analysis of the impedance of the surface current and potential on the horizontal and vertical faces found in cavity magnetrons in the cavity/TM mode, using results from [2], and develop a method of tuning the magnetron so that the resonant and responsive modes match in all directions.

Lemma 0.1. *The impedance Z_{xz} of a small receding strip of length z centred at $(x,0)$ on the top horizontal face is approximately;*

$$Z_{xz} = Q_1 x z^2 e^{i\theta}$$

where;

$$Q_1 = (-1)^{m+r}(k^2 + \frac{(\omega^2-k^2c^2)^2}{\omega^4\epsilon_0^2})^{\frac{1}{2}} \frac{\pi r}{d\omega(a^2+d^2)}$$

$$\theta = \pi + tan^{-1}(\frac{1}{k\epsilon_0} - \frac{c^2 k}{\omega^2\epsilon_0})$$

Proof. We recall from [2], that on the far faces, using the TM mode;

$$Re(\frac{\sigma_f}{\epsilon_0}) = 2(-1)^r(1 + \frac{k^2c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}} sin(\frac{\pi m x}{a}) sin(\frac{\pi n y}{b}) cos(\omega t + \phi)$$

where $tan(\phi) = -\frac{c^2 k}{\omega^2 \epsilon_0}$

Using the notation of [2], we have that on the horizontal faces;

$$\frac{\sigma_f}{\epsilon_0} = E'^{*,\perp} - E^{*,\perp}$$

$$= (e'_{2,\omega}e^{i(kz-\omega t)} + e'_{2,-\omega}e^{i(kz-\omega t)}) - (e_{2,\omega}e^{i(kz-\omega t)} - e_{2,-\omega}e^{i(kz+\omega t)})$$

$$= (e'_{2,\omega} - e_{2,\omega})e^{i(kz-\omega t)} + (e'_{2,-\omega} + e_{2,-\omega})e^{i(kz+\omega t)}$$

$$= (\frac{ike'_{3y}}{\frac{\omega^2}{c^2}-k^2} + \frac{c^2}{\omega^2\epsilon_0}p_y)e^{i(kz-\omega t)} + (\frac{ike'_{3y}}{\frac{\omega^2}{c^2}-k^2} - \frac{c^2}{\omega^2\epsilon_0}p_y)e^{i(kz+\omega t)}$$

$= \frac{2ike'_{3y}}{\frac{\omega^2}{c^2}-k^2}e^{ikz}cos(\omega t) - \frac{2ic^2p_y}{\omega^2\epsilon_0}e^{ikz}sin(\omega t)$

$= \frac{2ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{ikz}cos(\omega t) - \frac{2ic^2}{\omega^2\epsilon_0}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{ikz}sin(\omega t)$

and;

$Re(\frac{\sigma_f}{\epsilon_0}) = -\frac{2k}{\frac{\omega^2}{c^2}-k^2}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})sin(kz)cos(\omega t) + \frac{2c^2}{\omega^2\epsilon_0}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})sin(kz)sin(\omega t)$

$= \frac{2\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})sin(kz)[-\frac{k}{\frac{\omega^2}{c^2}-k^2}cos(\omega t) + \frac{c^2}{\omega^2\epsilon_0}sin(\omega t)]$

$= \frac{2\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})sin(\frac{\pi rz}{d})[-\frac{k}{\frac{\omega^2}{c^2}-k^2}cos(\omega t) + \frac{c^2}{\omega^2\epsilon_0}sin(\omega t)]$

$= \frac{2\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})sin(\frac{\pi rz}{d})[(\frac{k^2}{(\frac{\omega^2}{c^2}-k^2)^2} + \frac{c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}}cos(\omega t+\phi)]$

where $tan(\phi) = \frac{-\frac{c^2}{\omega^2\epsilon_0}}{-\frac{k}{\frac{\omega^2}{c^2}-k^2}} = \frac{c^2(\frac{\omega^2}{c^2}-k^2)}{\omega^2 k\epsilon_0} = \frac{1}{k\epsilon_0} - \frac{c^2k}{\omega^2\epsilon_0}$

and on the vertical faces;

$\frac{\sigma_f}{\epsilon_0} = E'^{*,\perp} - E^{*,\perp}$

$= (e'_{1,\omega}e^{i(kz-\omega t)} + e'_{1,-\omega}e^{i(kz-\omega t)}) - (e_{1,\omega}e^{i(kz-\omega t)} - e_{1,-\omega}e^{i(kz+\omega t)})$

$= (e'_{1,\omega} - e_{1,\omega})e^{i(kz-\omega t)} + (e'_{1,-\omega} + e_{1,-\omega})e^{i(kz+\omega t)}$

$= (\frac{ike'_{3x}}{\frac{\omega^2}{c^2}-k^2} + \frac{c^2}{\omega^2\epsilon_0}p_x)e^{i(kz-\omega t)} + (\frac{ike'_{3x}}{\frac{\omega^2}{c^2}-k^2} - \frac{c^2}{\omega^2\epsilon_0}p_x)e^{i(kz+\omega t)}$

$= \frac{2ike'_{3x}}{\frac{\omega^2}{c^2}-k^2}e^{ikz}cos(\omega t) - \frac{2ic^2p_x}{\omega^2\epsilon_0}e^{ikz}sin(\omega t)$

$= \frac{2ik}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{ikz}cos(\omega t) - \frac{2ic^2}{\omega^2\epsilon_0}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{ikz}sin(\omega t)$

and;

$Re(\frac{\sigma_f}{\epsilon_0}) = -\frac{2k}{\frac{\omega^2}{c^2}-k^2}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})sin(kz)cos(\omega t) + \frac{2c^2}{\omega^2\epsilon_0}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})sin(kz)sin(\omega t)$

$= \frac{2\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})sin(kz)[-\frac{k}{\frac{\omega^2}{c^2}-k^2}cos(\omega t) + \frac{c^2}{\omega^2\epsilon_0}sin(\omega t)]$

$= \frac{2\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})sin(\frac{\pi rz}{d})[-\frac{k}{\frac{\omega^2}{c^2}-k^2}cos(\omega t) + \frac{c^2}{\omega^2\epsilon_0}sin(\omega t)]$

$= \frac{2\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})sin(\frac{\pi rz}{d})[(\frac{k^2}{(\frac{\omega^2}{c^2}-k^2)^2} + \frac{c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}}cos(\omega t+\phi)]$

where $tan(\phi)=\frac{-\frac{c^2}{\omega^2\epsilon_0}}{-\frac{k}{\frac{\omega^2}{c^2}-k^2}}=\frac{c^2(\frac{\omega^2}{c^2}-k^2)}{\omega^2 k\epsilon_0}=\frac{1}{k\epsilon_0}-\frac{c^2k}{\omega^2\epsilon_0}$

Again, we recall from [2], that, on the far faces, using the TM mode;

$Re(\mu_0\overline{K}_f)=2(-1)^r\frac{\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}(-\frac{\pi m}{a}cos(\frac{\pi mx}{a})sin(\frac{\pi ny}{b}),\frac{\pi n}{b}sin(\frac{\pi mx}{a})cos(\frac{\pi ny}{b}))$

$cos(\omega t+\phi)$

where $\phi=-\frac{\pi}{2}$.

and, on the horizontal faces;

$\mu_0(\overline{K}_f\times\hat{\overline{n}})=\overline{B}'^{*,||}-\overline{B}^{*,||}$

$=(b'_{1,\omega},b'_{3,\omega})e^{i(kz-\omega t)}+(b'_{1,-\omega},b'_{3,-\omega})e^{i(kz+\omega t)}-((b_{1,\omega},b_{3,\omega})e^{i(kz-\omega t)}$

$-(b_{1,-\omega},b_{3,-\omega})e^{i(kz+\omega t)})$

$=(b'_{1,\omega}-b_{1,\omega},b'_{3,\omega}-b_{3,\omega})e^{i(kz-\omega t)}+(b'_{1,-\omega}+b_{1,-\omega},b'_{3,-\omega}+b_{3,-\omega})e^{i(kz+\omega t)}$

$=(b'_{1,\omega},0)e^{i(kz-\omega t)}+(b'_{1,-\omega},0)e^{i(kz+\omega t)}$

$=(-\frac{i\omega e'_{3y}}{c^2(\frac{\omega^2}{c^2}-k^2)}e^{i(kz-\omega t)}+\frac{i\omega e'_{3y}}{c^2(\frac{\omega^2}{c^2}-k^2)}e^{i(kz+\omega t)},0)$

$=(\frac{i\omega e'_{3y}}{c^2(\frac{\omega^2}{c^2}-k^2)},0)2isin(\omega t)e^{ikz}$

$=(-\frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{ikz}sin(\omega t),0)$

so that;

$\mu_0\overline{K}_f=(0,\frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})e^{ikz}sin(\omega t))$

$Re(\mu_0\overline{K}_f)=(0,\frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})cos(kz)sin(\omega t))$

$=(0,\frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})cos(\frac{\pi rz}{d}))sin(\omega t)$

$=(0,\frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n sin(\frac{\pi mx}{a})cos(\frac{\pi rz}{d}))cos(\omega t+\phi)$

where $\phi=-\frac{\pi}{2}$.

and, on the vertical faces;

$\mu_0(\overline{K}_f \times \hat{\overline{n}}) = \overline{B}'^{*,||} - \overline{B}^{*,||}$

$= (b'_{2,\omega}, b'_{3,\omega})e^{i(kz-\omega t)} + (b'_{2,-\omega}, b'_{3,-\omega})e^{i(kz+\omega t)} - ((b_{2,\omega}, b_{3,\omega})e^{i(kz-\omega t)}$

$-(b_{2,-\omega}, b_{3,-\omega})e^{i(kz+\omega t)})$

$= (b'_{2,\omega} - b_{2,\omega}, b'_{3,\omega} - b_{3,\omega})e^{i(kz-\omega t)} + (b'_{2,-\omega} + b_{2,-\omega}, b'_{3,-\omega} + b_{3,-\omega})e^{i(kz+\omega t)}$

$= (b'_{2,\omega}, 0)e^{i(kz-\omega t)} + (b'_{2,-\omega}, 0)e^{i(kz+\omega t)}$

$= (\frac{i\omega e'_{3x}}{c^2(\frac{\omega^2}{c^2}-k^2)}e^{i(kz-\omega t)} - \frac{i\omega e'_{3x}}{c^2(\frac{\omega^2}{c^2}-k^2)}e^{i(kz+\omega t)}, 0)$

$= (-\frac{i\omega e'_{3x}}{c^2(\frac{\omega^2}{c^2}-k^2)}, 0)2isin(\omega t)e^{ikz}$

$= (\frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{ikz}sin(\omega t), 0)$

so that;

$\mu_0\overline{K}_f = (0, \frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})e^{ikz}sin(\omega t))$

$Re(\mu_0\overline{K}_f) = (0, \frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi ny}{b}(-1)^m sin(\frac{\pi ny}{b})cos(kz)sin(\omega t))$

$= (0, \frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})cos(\frac{\pi rz}{d}))sin(\omega t)$

$= (0, \frac{2\omega}{c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi m}{a}(-1)^m sin(\frac{\pi ny}{b})cos(\frac{\pi rz}{d}))cos(\omega t + \phi)$

where $\phi = -\frac{\pi}{2}$.

Denoting the top horizontal face by H_1. As $\rho = 0$ outside the magnetron, by Jefimenko's equations, we have that the causal potential V on H_1 due to the TM mode is identically zero. Similarly, by the calculation in [2], the potential due to the charge and current configuration inside the magnetron is given by;

$Re((V'_{k,\omega,m,n} + V'_{k,-\omega,m,n}))(x, y, z, t)$

$= Re(\frac{c^2}{\omega^2\epsilon_0}[p(x, y)e^{ikz} - p(x_0, y_0)e^{ikz_0}](e^{-i\omega t} + e^{i\omega t}))$

$= Re(\frac{2c^2}{\omega^2\epsilon_0}[p(x,y)e^{ikz} - p(x_0,y_0)e^{ikz_0}]cos(\omega t))$

$= \frac{2c^2}{\omega^2\epsilon_0}[sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})cos(kz) - sin(\frac{\pi mx_0}{a})sin(\frac{\pi ny_0}{b})cos(kz_0)]cos(\omega t)$

$= \frac{2c^2}{\omega^2\epsilon_0}[sin(\frac{\pi mx}{a})sin(\frac{\pi ny}{b})cos(\frac{\pi rz}{d}) - sin(\frac{\pi mx_0}{a})sin(\frac{\pi ny_0}{b})cos(\frac{k\pi rz_0}{d})]cos(\omega t)$

Without loss of generality, choosing a reference point on the face H_1, we may assume that the potential is identically zero again. It remains to calculate the potential due to the surface charge. We can assume that that for $\{\overline{x}, \overline{x}'\} \subset H_1$, $\frac{|\overline{x}'-\overline{x}|}{c} \simeq 0$. Using Jefimenko's equations, and the fact that the continuity equation holds on H_1, see [2] and [3], we have on H_1, with coordinates (x,z) that the potential due to the top horizontal face is given by:

$V(x,z) = \int_{|x'|\leq a}\int_{|z'|\leq d} \frac{Re(\sigma_f)(\overline{x}',t_r)}{|\overline{x}'-\overline{x}|}dx'dz'$

$\simeq \int_{|x'|\leq a}\int_{|z'|\leq d} \frac{Re(\sigma_f)(\overline{x}',t)}{|\overline{x}'-\overline{x}|}dx'dz'$

$= R_1 cos(\omega t+\phi)\int_{|x'|\leq a}\int_{|z'|\leq d} \frac{sin(\frac{\pi mx'}{a})sin(\frac{\pi rz'}{d})}{|\overline{x}'-\overline{x}|}dx'dz'$

$= R_1 cos(\omega t+\phi)\int_{|x'|\leq a}\int_{|z'|\leq d} \frac{sin(\frac{\pi m(x'-x+x)}{a})sin(\frac{\pi r(z'-z+z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

$\simeq R_1 cos(\omega t+\phi)I_1(x,z)$

where

$I_1 = \int_{|x'|\leq a}\int_{|z'|\leq d} \frac{[sin(\frac{\pi m(x'-x)}{a})cos(\frac{\pi mx}{a})+cos(\frac{\pi m(x'-x)}{a})sin(\frac{\pi mx}{a})][sin(\frac{\pi r(z'-z)}{d})cos(\frac{\pi rz}{d})+cos(\frac{\pi r(z'-z)}{d})sin(\frac{\pi rz}{d})]}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}$

$dx'dz'$

and $R_1 = \epsilon_0\frac{2\pi n}{b}(-1)^n(\frac{k^2}{(\frac{\omega^2}{c^2}-k^2)^2} + \frac{c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}}$.
We have that;

$I_1 = I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}$

where;

$I_{1,1} = cos(\frac{\pi mx}{a})cos(\frac{\pi rz}{d})\int_{|x'|\leq a}\int_{|z'|\leq d} \frac{sin(\frac{\pi m(x'-x)}{a})sin(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

$I_{1,2} = cos(\frac{\pi mx}{a})sin(\frac{\pi rz}{d})\int_{|x'|\leq a}\int_{|z'|\leq d} \frac{sin(\frac{\pi m(x'-x)}{a})cos(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

$I_{1,3}=sin(\frac{\pi mx}{a})cos(\frac{\pi rz}{d})\int_{|x'|\leq a}\int_{|z'|\leq d}\frac{cos(\frac{\pi m(x'-x)}{a})sin(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

$I_{1,4}=sin(\frac{\pi mx}{a})sin(\frac{\pi rz}{d})\int_{|x'|\leq a}\int_{|z'|\leq d}\frac{cos(\frac{\pi m(x'-x)}{a})cos(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

We assume that (x,z) is located near the centre of the face, so that $x\omega\simeq 0$, $z\omega\simeq 0$ and, as $\frac{\pi m}{a}<\omega$, $\frac{\pi r}{d}<\omega$;

$sin(\frac{\pi mx}{a})\simeq sin(\frac{\pi rz}{d})\simeq 0$

$cos(\frac{\pi mx}{a})\simeq cos(\frac{\pi rz}{d})\simeq 1$

so that $I_{1,2}\simeq I_{1,3}\simeq I_{1,4}\simeq 0$;

$I_1\simeq cos(\frac{\pi mx}{a})cos(\frac{\pi rz}{d})\int_{|x'|\leq a}\int_{|z'|\leq d}\frac{sin(\frac{\pi m(x'-x)}{a})sin(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

$\simeq\int_{|x'|\leq a}\int_{|z'|\leq d}\frac{sin(\frac{\pi m(x'-x)}{a})sin(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

Without loss of generality, assume that $x>0$, $z>0$, then, using the asymmetry of sine, the symmetry of $[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}$ and the fact that;

$sin(\frac{\pi r(-d-z)}{d})=sin(-\pi r-\frac{\pi rz}{d})$

$=cos(-\pi r)sin(\frac{-\pi rz}{d})$

$\simeq(-1)^r\frac{-\pi rz}{d}$

$=(-1)^{r+1}\frac{\pi rz}{d}$

$-d+z-z=d$

$sin(\frac{\pi m(-a-x)}{a})=sin(-\pi m-\frac{\pi mx}{a})$

$=cos(-\pi m)sin(\frac{-\pi mx}{a})$

$\simeq(-1)^m\frac{-\pi mx}{a}$

$=(-1)^{m+1}\frac{\pi mx}{a}$

$-d+z-z=d$

we have that;

$I_1=\int_{|x'|\leq a}\int_{-d}^{-d+z}\frac{sin(\frac{\pi m(x'-x)}{a})sin(\frac{\pi r(z'-z)}{d})}{[(x'-x)^2+(z'-z)^2]^{\frac{1}{2}}}dx'dz'$

$\simeq(-1)^{r+1}\frac{\pi rz}{d}z\int_{|x'|\leq a}\frac{sin(\frac{\pi m(x'-x)}{a})}{[(x'-x)^2+d^2]^{\frac{1}{2}}}dx'$

$=(-1)^{r+1}\frac{\pi rz^2}{d}\int_{|x'|\leq a}\frac{sin(\frac{\pi m(x'-x)}{a})}{[(x'-x)^2+d^2]^{\frac{1}{2}}}dx'$

$=(-1)^{r+1}\frac{\pi rz^2}{d}\int_{-a}^{-a+x}\frac{sin(\frac{\pi m(x'-x)}{a})}{[(x'-x)^2+d^2]^{\frac{1}{2}}}dx'$

$\simeq(-1)^{r+1}\frac{\pi rz^2}{d}(-1)^{m+1}\frac{\pi mx}{a}x\frac{1}{a^2+d^2}$

$=(-1)^{r+m}\frac{\pi^2 mrx^2z^2}{ad(a^2+d^2)}$

and, towards the centre of the horizontal face H_1 of the magnetron;

$V(x,z)\simeq S_1cos(\omega t+\phi)x^2z^2$

where $S_1=(-1)^{n+m+r}\epsilon_0(\frac{k^2}{(\frac{\omega^2}{c^2}-k^2)^2}+\frac{c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}}\frac{2\pi^3 mnr}{abd(a^2+d^2)}$

An almost identical calculation shows that for the potential $V'(x,z)$ due to the bottom horizontal face H_2 is given by;

$V'(x,z)\simeq S_2cos(\omega t+\phi)x^2z^2$

where $S_2=(-1)^{n+m+r}\epsilon_0(\frac{k^2}{(\frac{\omega^2}{c^2}-k^2)^2}+\frac{c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}}\frac{2\pi^3 mnr}{abd(a^2+d^2+4b^2)}$

(Similar calculations for the vertical faces $\{V_1,V_2\}$ and the far faces $\{F_1,F_2\}$.)

By a similar approximation, and using the above calculation, we have that, towards the centre of H_1, the current is given by;

$I(x,z)\simeq(0,\frac{2\omega}{\mu_0c^2(\frac{\omega^2}{c^2}-k^2)}\frac{\pi n}{b}(-1)^n\frac{\pi mx}{a})cos(\omega t+\phi)$

$=T_1cos(\omega t+\psi)(0,x)$

where $T_1 = (-1)^n \frac{2\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)} \frac{\pi^2 mn}{ab}$

Let W_x be a small receding strip centred closed to the centre of the vertical face H_1 of length z, the the potential across the strip, is approximately;

$V_{xz} = V(x,z) - V(x,0)$

$= V(x,z)$

$= S_1 cos(\omega t + \phi)x^2 z^2$

while the current through the strip is approximately;

$I_{xz} = T_1 cos(\omega t + \psi)x$

so that the impedance Z_{xz} is given by;

$Z_{xz} = \frac{V'_{xz}}{I'_{xz}}$

$= \frac{S_1 x^2 z^2 e^{i(\omega t+\phi)}}{T_1 e^{i(\omega t+\psi)} x}$

$= \frac{S_1}{T_1} x z^2 e^{i(\phi-\psi)}$

$= Q_1 x z^2 e^{i(\phi-\psi)}$

where;

$$Q_1 = \frac{(-1)^{n+m+r}\epsilon_0(\frac{k^2}{(\frac{\omega^2}{c^2}-k^2)^2}+\frac{c^4}{\omega^4\epsilon_0^2})^{\frac{1}{2}} \frac{2\pi^3 mnr}{abd(a^2+d^2)}}{(-1)^n \frac{2\omega}{\mu_0 c^2(\frac{\omega^2}{c^2}-k^2)} \frac{\pi^2 mn}{ab}}$$

$= (-1)^{m+r}(k^2 + \frac{(\omega^2-k^2c^2)^2}{\omega^4\epsilon_0^2})^{\frac{1}{2}} \frac{\pi r}{d\omega(a^2+d^2)}$

We have that;

$dV_z(x,z) \simeq S_1 cos(\omega t + \phi)x^2 d(z^2)$

$\simeq S_1 cos(\omega t + \phi)x^2 2z dz$

while $I(x,z) = T_1 cos(\omega t + \psi)x$, so that the local impedance $dZ(x,z)$ is given by;

$\frac{dV_z(x,z)}{I(x,z)} = \frac{S_1 e^{i(\omega t+\phi)} x^2 2z dz}{T_1 e^{i(\omega t+\psi)} x}$

$= \frac{S_1}{T_1} 2xzdz e^{i(\phi-\psi)}$

so that as impedance is summable in a series circuit, we have that, for small z;

$Z(x,z) = \sum_{0\leq z'\leq z} dZ(x,z')$

$\simeq \sum_{0\leq z'\leq z} \frac{S_1}{T_1} 2xz'dz' e^{i(\phi-\psi)}$

$\simeq \int_0^z \frac{S_1}{T_1} 2xz'dz' e^{i(\phi-\psi)}$

$= \frac{S_1}{T_1} xz^2 e^{i(\phi-\psi)}$

which agrees with our previous result.

□

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