MICROWAVE ENGINEERING 3

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ABSTRACT. We give an explanation of charge and current driven radiation inside spherical magnetrons, using the equations found in [10], and by verifying compatibility with the TM and TE modes used in microwave engineering.

Lemma 0.1. *There exist* $(\rho, \overline{J}, \overline{E}, \overline{B})$ *satisfying;*

(i). $\square^2(\rho) = 0$.

(ii). $\square^2(\overline{J}) = \overline{0}$.

(iii). $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$.

(iv). $\frac{\partial \rho}{\partial t} + \bigtriangledown \cdot \overline{J} = 0$.

(v). $\square^2(\overline{E}) = \bigtriangledown \times \overline{E} = \overline{0}$

(vi). $\overline{B} = \overline{0}$

(vii). $\bigtriangledown \cdot \overline{E} = \frac{\rho}{\epsilon_0}$

$(viii)$ $\frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} + \mu_0 \overline{J} = \overline{0}$

such that;

$\rho(x, y, z, t) = p(x, y, z)e^{-i\omega t}$

$\overline{J} = \overline{j}(x, y, z)e^{-i\omega t}$, $\overline{j} = (j_1, j_2, j_3)$.

$\overline{E} = \overline{e}(x, y, z)e^{-i\omega t}$, $\overline{e} = (e_1, e_2, e_3)$.

$\overline{B} = \overline{b}(x, y, z)e^{-i\omega t}$, $\overline{b} = (b_1, b_2, b_3)$.

In particularly, Maxwell's equations are satisfied for $(\rho, \overline{J}, \overline{E}, \overline{B})$.

Let $(V', \overline{A}')$ *be the global potentials defined by Jefimenko's equations;*

$V'(\overline{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\overline{r}', t_r)}{\mathfrak{r}} d\tau'$

$\overline{A}'(\overline{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\overline{J}(\overline{r}', t_r)}{\mathfrak{r}} d\tau'$

Then $V' = v'(x, y, z)e^{-i\omega t}$, $\overline{A}' = \overline{a}'(x, y, z)e^{-i\omega t}$, $\overline{a}' = (a_1', a_2', a_3')$.

A similar claim holds for the causal fields $\{\overline{E}', \overline{B}'\}$ *of Jefimenko's equations.*
We have that;

$p(x, y, z) = P(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$

where;

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})R = 0$

$\frac{1}{sin(\theta)}\frac{d}{d\theta}(sin(\theta)\frac{d\Theta}{d\theta}) + (l(l+1) - \frac{m^2}{sin^2(\theta)})\Theta = 0$

$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$ (C)

for constants $\{m, l\} \subset \mathcal{N}$.

The components $\{j_r, j_\theta, j_\phi, e_r, e_\theta, e_\phi, b_r, b_\theta, b_\phi\}$ *of* $\{\overline{j}(r, \theta, \phi), \overline{e}(r, \theta, \phi), \overline{b}(r, \theta, \phi)\}$ *can be written in terms of* $\{R, R'\Theta, \Theta', \Phi, \Phi', r, \theta, \phi\}$.

There exist $(0, \overline{0}, \overline{E}', \overline{B}')$ *satisfying Maxwell's equations in vacuum;*

(i). $\bigtriangledown \cdot \overline{E}' = 0$

(ii). $\bigtriangledown \times \overline{E}' = -\frac{\partial \overline{B}'}{\partial t}$

(iii). $\bigtriangledown \cdot \overline{B}' = \overline{0}$

(iv) $\bigtriangledown \times \overline{B}' = \frac{1}{c^2}\frac{\partial \overline{E}'}{\partial t}$

$\overline{E}' = \overline{e}'(x, y, z)e^{-i\omega t}$, $\overline{e}' = (e_1', e_2', e_3')$.

$\overline{B}' = \overline{b}'(x,y,z)e^{-i\omega t}$, $\overline{b}' = (b'_1, b'_2, b'_3)$.

with $\overline{B}' \neq \overline{0}$

We have that $rb'_r e^{-i\omega t} =< \overline{B'}, \overline{r} >$ *and* $re'_r e^{-i\omega t} =< \overline{E'}, \overline{r} >$ *satisfy the wave equation and;*

$rb'_r(x,y,z) = rb'_r(r,\theta,\phi) = R_1(r)\Theta_1(\theta)\Phi_1(\phi)$

where;

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR_1}{dr}) + (\frac{\omega^2}{c^2} - \frac{l'(l'+1)}{r^2})R_1 = 0$

$\frac{1}{sin(\theta)}\frac{d}{d\theta}(sin(\theta)\frac{d\Theta}{d\theta}) + (l'(l'+1) - \frac{m'^2}{sin^2(\theta)})\Theta_1 = 0$

$\frac{d^2\Phi_1}{d\phi^2} + m'^2\Phi_1 = 0$ $(C1)$

for constants $\{m', l'\} \subset \mathcal{R}$.

A similar result holds for re'_r.

The components $\{e'_r, e'_\theta, e'_\phi, b'_r, b'_\theta, b'_\phi\}$ *of* $\{\overline{e}'(r,\theta,\phi), \overline{b}'(r,\theta,\phi)\}$ *can be written in terms of* $\{R, R', \Theta, \Theta', \Phi, \Phi', r, \theta, \phi\}$.

In particularly, for the TE mode;

$b'_r = \frac{rb'_r}{r}$

$b'_\theta = \frac{1}{l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2 b'_r)$

$b'_\phi = \frac{1}{l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2 b'_r)$

$e'_r = 0$

$e'_\theta = \frac{i\omega}{l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2 b'_r)$

$e'_\phi = -\frac{i\omega}{l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2 b'_r)$ (X)

and for the TM mode;

$e'_r = \frac{re'_r}{r}$

$e'_\theta = \frac{1}{l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2 e'_r)$

$e'_\phi = \frac{1}{l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2 e'_r)$

$b'_r = 0$

$b'_\theta = -\frac{i\omega}{c^2 l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2 e'_r)$

$b'_\phi = \frac{i\omega}{c^2 l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2 e'_r)$ (Y)

The continuity equation holds on the sphere $S(\overline{0}, w)$, for both the TE and TM modes. Moreover, if we restrict to the cases where the current $\overline{J}$ vanishes on the sphere $S(\overline{0}, w)$, the continuity equation holds and we can calculate the surface impedance in particular cases.

Proof. The proof of the first part is similar to [10]. For (i), we have, substituting $p(x, y, z)e^{-i\omega t}$ for ρ, that;

$[p_{xx} + p_{yy} + p_{zz}]e^{-i\omega t} = \frac{1}{c^2}p(-\omega^2)e^{-i\omega t}$

so we require that $p_{xx} + p_{yy} + p_{zz} + \frac{\omega^2}{c^2}p = 0$, $(*)$.

The proof that this can be solved in $\mathcal{R}^3$ is shown in [5], using spherical polar coordinates. For (iii), we have, substituting $p(x, y, z)e^{-i\omega t}$ for ρ, and $\overline{j}(x, y, z)e^{-i\omega t}$ for $\overline{J}$, that;

$(p_x, p_y, p_z)e^{-i\omega t} = -\frac{1}{c^2}(j_1, j_2, j_3)(-i\omega)e^{-i\omega t}$

so that;

$j_1 = \frac{c^2}{i\omega}p_x = -\frac{ic^2}{\omega}p_x$

$j_2 = \frac{c^2}{i\omega}p_y = -\frac{ic^2}{\omega}p_y$

$j_3 = \frac{c^2}{i\omega}p_z = -\frac{ic^2}{\omega}p_z$ $(**)$

If p satisfies $(*)$, differentiating, so do p_x, p_y and p_z, then, from $(**)$, the components $\{j_1, j_2, j_3\}$ satisfy $(*)$ and (ii) is satisfied. For (iv), we have, substituting again, and using $(**)$, that;

$-i\omega p e^{-i\omega t} = -(j_{1x} + j_{2x} + j_{3x})e^{-i\omega t}$

$= -(\frac{c^2}{i\omega}p_{xx} + \frac{c^2}{i\omega}p_{yy} + \frac{c^2}{i\omega}p_{zz})e^{-i\omega t}$

so that;

$-\frac{c^2}{i\omega}p_{xx} - \frac{c^2}{i\omega}p_{yy} - \frac{c^2}{i\omega}p_{zz} + i\omega p = 0$

and multiplying by $-\frac{i\omega}{c^2}$;

$p_{xx} + p_{yy} + p_{zz} + \frac{\omega^2}{c^2}p = 0$

which is $(*)$. As all the steps are reversible, we obtain (iv). For $(viii)$, we require that;

$-\frac{i\omega}{c^2}\overline{e}e^{-i\omega t} = -\mu_0\overline{j}e^{-i\omega t}$

so that;

$e_1 = -\frac{i\mu_0 c^2}{\omega}j_1$

$e_2 = -\frac{i\mu_0 c^2}{\omega}j_2$

$e_3 = -\frac{i\mu_0 c^2}{\omega}j_3$

and, using $(**)$

$e_1 = -\frac{i\mu_0 c^2}{\omega}\frac{-ic^2}{\omega}p_x = -\frac{\mu_0 c^4}{\omega^2}p_x$

$e_2 = -\frac{i\mu_0 c^2}{\omega}\frac{-ic^2}{\omega}p_y = -\frac{\mu_0 c^4}{\omega^2}p_y$

$e_3 = -\frac{i\mu_0 c^2}{\omega}\frac{-ic^2}{\omega}p_z = -\frac{\mu_0 c^4}{\omega^2}p_z$ (A)

For (vi), we just set $b_1 = b_2 = b_3 = 0$. For (v), we have from (A), that $\overline{E} = -\frac{\mu_0 c^4}{\omega^2}\bigtriangledown(\rho)$, so that $\bigtriangledown \times \overline{E} = \overline{0}$ and as $\{p_x, p_y, p_z\}$ satisfy $(*)$, so do $\{e_1, e_2, e_3\}$, so that $\Box^2\overline{E} = \overline{0}$, and (v) is satisfied. For (vii), we have, using (A) and $(*)$, that;

$div(\overline{E})=(e_{1x}+e_{2y}+e_{3z})e^{-i\omega t}$

$=-\frac{\mu_0 c^4}{\omega^2}(p_{xx}+p_{yy}+p_{zz})e^{-i\omega t}$

$=-\frac{\mu_0 c^4}{\omega^2}\frac{-\omega^2}{c^2}pe^{-i\omega t}$

$=\mu_0 c^2 pe^{-i\omega t}$

$=\frac{1}{\epsilon_0 c^2}c^2 pe^{-i\omega t}$

$=\frac{\rho}{\epsilon_0}$

so that (vii) is satisfied. The second claim follows easily by rearranging $(v)-(viii)$.

For the potentials claim, it follows by differentiating under the integral sign, and using the fact that $t_r=t-\frac{|\overline{r}'-\overline{r}|}{c}$, that;

$\frac{\partial V'}{\partial t}=\frac{1}{4\pi\epsilon_0}\int\frac{\dot{\rho}(\overline{r}',t_r)}{\mathfrak{r}}d\tau'$

$=-\frac{i\omega}{4\pi\epsilon_0}\int\frac{\rho(\overline{r}',t_r)}{\mathfrak{r}}d\tau'$

$=-i\omega V'$

Using Peano's theorem on the uniqueness of solutions of first order differential equations, we then must have that;

$V'(x,y,z,t)=v'(x,y,z)e^{-i\omega t}$ (AA)

and, similarly;

$\overline{A}'(x,y,z,t)=\overline{a}'(x,y,z)e^{-i\omega t}$

The claim on $\{\overline{E}',\overline{B}'\}$ is similar, using Jefimenko's equations which only depend on $\{\rho,\overline{J}\}$ and derivatives.

The formulae (C) can be found in [5], once we have $(*)$. When $R=j_l(\frac{\omega r}{c})$, where j_l is a Bessel function of the first kind of order l, $\Theta=P_l^m(cos(\theta))$ where P_l^m is the associated Legendre polynomial, and

$\Phi = sin(m\phi)$ or $cos(m\phi)$, we denote by $p_{m,l,s}$ or $p_{m,l,c}$ the corresponding fundamental solutions, see the discussion in [5].

Let $\{\hat{\overline{r}}, \hat{\overline{\theta}}, \hat{\overline{\phi}}\}$ be the standard orthonormal spherical frame, then we have that, using the above calculation;

$< \overline{J}, \hat{\overline{r}} >=< \overline{j}, \hat{\overline{r}} > e^{-i\omega t}$

$= \frac{-ic^2}{\omega} < \bigtriangledown(p), \hat{\overline{r}} > e^{-i\omega t}$

$= \frac{-ic^2}{\omega}(\frac{\partial p}{\partial r}[\hat{\overline{r}} \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{r}}])e^{-i\omega t}$

so that;

$j_r = \frac{-ic^2}{\omega}(\frac{\partial p}{\partial r}[\hat{\overline{r}} \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{r}}])$

$= \frac{-ic^2}{\omega}(\frac{\partial p}{\partial r} + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{r}}])$

Similarly;

$j_\theta = \frac{-ic^2}{\omega}(\frac{\partial p}{\partial r}[\hat{\overline{r}} \centerdot \hat{\overline{\theta}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\theta}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\theta}}])$

$= \frac{-ic^2}{\omega}(\frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\theta}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\theta}}])$

$j_\phi = \frac{-ic^2}{\omega}(\frac{\partial p}{\partial r}[\hat{\overline{r}} \centerdot \hat{\overline{\phi}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\phi}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\phi}}])$

$= \frac{-ic^2}{\omega}(\frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\phi}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\phi}}])$ (F)

A similar calculation shows that;

$e_r = -\frac{\mu_0 c^4}{\omega^2}(\frac{\partial p}{\partial r}[\hat{\overline{r}} \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial\ x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{r}}])$

$= -\frac{\mu_0 c^4}{\omega^2}(\frac{\partial p}{\partial r} + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{r}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial\ x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{r}}])$

$e_\theta = -\frac{\mu_0 c^4}{\omega^2}(\frac{\partial p}{\partial r}[\hat{r} \centerdot \hat{\overline{\theta}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\theta}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial\ x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\theta}}])$

$= -\frac{\mu_0 c^4}{\omega^2}(\frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\theta}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial\ x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\theta}}])$

$e_\phi = -\frac{\mu_0 c^4}{\omega^2}(\frac{\partial p}{\partial r}[\hat{r} \centerdot \hat{\overline{\phi}}] + \frac{\partial p}{\partial \theta}[(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) \centerdot \hat{\overline{\phi}}] + \frac{\partial p}{\partial \phi}[(\frac{\partial \phi}{\partial\ x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \centerdot \hat{\overline{\phi}}])$

$= -\frac{\mu_0 c^4}{\omega^2}(\frac{\partial p}{\partial\theta}[(\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y},\frac{\partial\theta}{\partial z})\centerdot\hat{\overline{\phi}}]+\frac{\partial p}{\partial\phi}[(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z})\centerdot\hat{\overline{\phi}}])$ (E)

Clearly, we have that $b_r = b_\theta = b_\phi = 0$.

The next claim is then clear, calculating $\{\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y},\frac{\partial\theta}{\partial z},\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\}$ and the orthonormal frame in terms of $\{r,\theta,\phi\}$, as well as the terms $\{\frac{\partial p}{\partial r},\frac{\partial p}{\partial\theta},\frac{\partial p}{\partial r}\}$ in terms of $\{R,R',\Theta,\Theta',\Phi,\Phi'\}$.

For the boundary conditions at the boundary of the cavity magnetron with radius we need $\{e_\theta,e_\phi,b_r\}$ to vanish at the boundary, which we can achieve with $\frac{\partial p}{\partial\theta}=\frac{\partial p}{\partial\phi}=0$, as $b_r=0$. By the explicit form of p in (C), and the calculations in (E), if the magnetron has radius w, this is achieved when $R=j_l(\frac{\omega r}{c})|_{\delta S(\overline{0},w)}=0$, so that $\frac{\omega w}{c}\in Z_l$, $\omega\in\frac{cZ_l}{w}$, where $Z_l = Zero(j_l)$, the zero set of the corresponding Bessel function. In this case, we also have by $(E),(F)$, that $j_\theta=j_\phi=0$ at the boundary, and;

$e_r = -\frac{\mu_0 c^4}{\omega^2}\frac{\partial p}{\partial r}|_{S(\overline{0},w)}$

$j_r = \frac{-ic^2}{\omega}\frac{\partial p}{\partial r}|_{S(\overline{0},w)}$

where p is constant on the boundary, as $\frac{\partial p}{\partial\phi}=\frac{\partial p}{\partial\psi}=0$.

The next claim is a special case of the result proved in [10] and left to the reader.

For the next claim, $rb'_r e^{-i\omega t} = <\overline{B'},\overline{r}>$ satisfies the wave equation, as;

$\square^2(<\overline{B'},\overline{r}>)$

$=<\square^2\overline{B'},\overline{r}>+<\overline{B'},\square^2\overline{r}>+\bigtriangledown\centerdot\overline{B}'$

$=0$

The equations for the components in the TE and TM modes can be found in [5], and we assume they hold on the exterior of the sphere $S(\overline{0},w)$. For the boundary conditions at the boundary of the cavity magnetron, we need $\{e_\theta,e_\phi,b_r\}$ to vanish at the boundary again. In the TE mode case, from (X), we can achieve this with $\frac{\partial r^2 b_r}{\partial\theta}=\frac{\partial r^2 b_r}{\partial\phi}=0$, and $r^2 b_r = 0$ at the boundary. By the explicit form of

rb_r in $(C1)$, if the magnetron has radius w, this is again achieved when $R = j_{l'}(\frac{\omega r}{c})|_{\delta S(\bar{0},w)} = 0$, so that $\frac{\omega w}{c} \in Z_{l'}$, $\omega \in \frac{cZ_{l'}}{w}$, where $Z_{l'} = Zero(j_{l'})$, the zero set of the corresponding Bessel function. In the TM mode case, from (Y), we can achieve this with $\frac{\partial r^2 e_r}{\partial r} = 0$, as $b_r = 0$ in the TM mode. By the explicit form of re_r in $(C1)$, if the magnetron has radius w, this is achieved when $\frac{\partial rR}{\partial r} = \frac{\partial r j_{l'}(\frac{\omega r}{c})}{\partial r}|_{\delta S(\bar{0},w)} = 0$.

In the TE case, we have that the surface charge σ_f is given by;

$\frac{\sigma_f}{\epsilon_0} = \overline{E}'^{\perp} - \overline{E}^{\perp}$

$= e'_r e^{-i\omega t} - e_r e^{-i\omega t}$

$= -e_r e^{-i\omega t}$

$= \frac{\mu_0 c^4}{\omega^2}\frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

while in the TM case, we have that;

$\frac{\sigma_f}{\epsilon_0} = \overline{E}'^{\perp} - \overline{E}^{\perp}$

$= e'_r e^{-i\omega t} - e_r e^{-i\omega t}$

$= e'_r e^{-i\omega t} + \frac{\mu_0 c^4}{\omega^2}\frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

where re'_r satisfies the relations above.

In the TE case, we have that the surface current $\overline{K}_f$ is given by;

$\mu_0(\overline{K}_f \times \hat{r}) = \overline{B}'^{||} - \overline{B}^{||}$

$= \overline{B}'^{||}$

$= (b'_\theta \hat{\overline{\theta}} + b'_\phi \hat{\overline{\phi}}) e^{-i\omega t}$

$= (\frac{1}{l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2 b'_r)\hat{\overline{\theta}} + \frac{1}{l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2 b'_r)\hat{\overline{\phi}}) e^{-i\omega t}$

where rb'_r satisfies the relations above. It follows that;

$\mu_0 \overline{K}_f = (\frac{1}{l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2 b'_r)\hat{\overline{\phi}} - \frac{1}{l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2 b'_r)\hat{\overline{\theta}}) e^{-i\omega t}$

In the TM case, we have that;

$\mu_0(\overline{K}_f \times \hat{r}) = \overline{B}'^{||} - \overline{B}^{||}$

$= \overline{B}'^{||}$

$= (-\frac{i\omega}{c^2 l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2 e'_r)\hat{\overline{\theta}} + \frac{i\omega}{c^2 l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2 e'_r)\hat{\overline{\phi}})e^{-i\omega t}$

where re'_r satisfies the relations above. It follows that;

$\mu_0\overline{K}_f = (-\frac{i\omega}{c^2 l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2 e'_r)\hat{\overline{\phi}} - \frac{i\omega}{c^2 l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2 e'_r)\hat{\overline{\theta}})e^{-i\omega t}$

In the TE case, we have that;

$\bigtriangledown_{S(\bar{0},w)} \cdot \mu_0\overline{K}_f$

$= (\frac{1}{wsin(\theta)}\frac{\partial}{\partial\phi}, \frac{1}{wsin(\theta)}\frac{\partial}{\partial\theta}sin(\theta))\cdot(\frac{1}{l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2 b'_r), -\frac{1}{l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2 b'_r))e^{-i\omega t}$

$= (\frac{1}{l'(l'+1)}\frac{1}{w^2 sin(\theta)}\frac{\partial^2}{\partial\theta\partial\phi}\frac{\partial}{\partial r}(r^2 b'_r) - \frac{1}{l'(l'+1)}\frac{1}{w^2 sin(\theta)}\frac{\partial^2}{\partial\theta\partial\phi}\frac{\partial}{\partial r}(r^2 b'_r))e^{-i\omega t}$

$= 0$

In the TM case, we have that;

$\bigtriangledown_{S(\bar{0},w)} \cdot \mu_0\overline{K}_f$

$= (\frac{1}{wsin(\theta)}\frac{\partial}{\partial\phi}, \frac{1}{wsin(\theta)}\frac{\partial}{\partial\theta}sin(\theta))\cdot(-\frac{i\omega}{c^2 l'(l'+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2 e'_r), -\frac{i\omega}{c^2 l'(l'+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2 e'_r))e^{-i\omega t}$

$= -\frac{i\omega}{c^2 l'(l'+1)}(\frac{1}{w^2 sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}(r^2 e'_r) + \frac{1}{w^2 sin(\theta)}\frac{\partial}{\partial\theta}(sin(\theta)\frac{\partial}{\partial\theta}(r^2 e'_r)))e^{-i\omega t}$

In the TE and TM cases, we have that;

$(\overline{J}' - \overline{J}) \cdot \hat{n}$

$= -\overline{J} \cdot \hat{n}$

$= -j_r e^{-i\omega t}$

$= \frac{ic^2}{\omega}\frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

In the TE case, we have that;

$\frac{\partial \sigma_f}{\partial t}$

$= -i\omega \frac{\epsilon_0 \mu_0 c^4}{\omega^2} \frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

$= -\frac{ic^2}{\omega} \frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

while in the TM case, we have that;

$\frac{\partial \sigma_f}{\partial t}$

$= (-i\omega\epsilon_0 e'_r - \frac{ic^2}{\omega} \frac{\partial p}{\partial r}|_{S(\bar{0},w)}) e^{-i\omega t}$

It follows that in the TE case;

$\bigtriangledown_{S(\bar{0},w)} \cdot \overline{K}_f + (\overline{J}' - \overline{J}) \cdot \hat{n}$

$= 0 + \frac{ic^2}{\omega} \frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

$= -\frac{\partial \sigma_f}{\partial t}$

so the continuity equation holds on the boundary. In the TM case, we have that;

$\bigtriangledown_{S(\bar{0},w)} \cdot \overline{K}_f + (\overline{J}' - \overline{J}) \cdot \hat{n} + \frac{\partial \sigma_f}{\partial t}$

$= -\frac{i\omega}{\mu_0 c^2 l'(l'+1)} (\frac{1}{w^2 sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}(r^2 e'_r) + \frac{1}{w^2 sin(\theta)} \frac{\partial}{\partial \theta}(sin(\theta) \frac{\partial}{\partial \theta}(r^2 e'_r))) e^{-i\omega t} + \frac{ic^2}{\omega} \frac{\partial p}{\partial r}|_{S(\bar{0},w)} e^{-i\omega t}$

$+(-i\omega\epsilon_0 e'_r - \frac{ic^2}{\omega} \frac{\partial p}{\partial r}|_{S(\bar{0},w)}) e^{-i\omega t}$

$= -\frac{i\omega}{\mu_0 c^2 l'(l'+1)} (\frac{1}{w^2 sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}(r^2 e'_r) + \frac{1}{w^2 sin(\theta)} \frac{\partial}{\partial \theta}(sin(\theta) \frac{\partial}{\partial \theta}(r^2 e'_r))) e^{-i\omega t}$

$-i\omega\epsilon_0 e'_r e^{-i\omega t}$

$= -\frac{i\omega\epsilon_0}{l'(l'+1)} (\frac{1}{w^2 sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}(r^2 e'_r) + \frac{1}{w^2 sin(\theta)} \frac{\partial}{\partial \theta}(sin(\theta) \frac{\partial}{\partial \theta}(r^2 e'_r))) e^{-i\omega t}$

$-\frac{i\omega\epsilon_0}{l'(l'+1)} (\frac{1}{r} \frac{\partial}{\partial r}(r^2 \frac{\partial(re'_r)}{\partial r}) + \frac{\omega^2 r^2}{c^2} e'_r) e^{-i\omega t}$

$= 0$

as;

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial(re'_r)}{\partial r})+\frac{1}{r^2 sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}(re'_r)+\frac{1}{r^2 sin(\theta)}\frac{\partial}{\partial\theta}(sin(\theta)\frac{\partial}{\partial\theta}(re'_r))+\frac{\omega^2}{c^2}re'_r = 0$

and we can multiply by r.

We follow the notation in [11], and denote by;

$\overline{J}_{l_0,k_0} = \sum_{-l_0\leq m\leq l_0}\overline{U}(l_0,m,k_0)\gamma_{l_0,m,k_0}e^{-ik_0ct}$

for $l_0 = 1$, where;

$\overline{U}(l_0,m_0,k_0) = (\frac{2}{\pi})^{\frac{1}{2}}\frac{i^{l_0}k_0^2}{4\pi}\overline{W}(l_0,m)^*$

$= i(\frac{2}{\pi})^{\frac{1}{2}}\frac{k_0^2}{4\pi}\overline{W}(1,m)^*$

and $k_0 \in \frac{S_{l_0}}{w}$, for the zero set of j_{l_0}. Then $\overline{J}$ vanishes on the sphere $S(\overline{0},w)$ and satisfies the radial transform condition, so we can find ρ_{l_0,k_0} such that $(\rho_{l_0,k_0},\overline{J}_{l_0,k_0})$ satisfy $(i)-(iv)$. To calculate ρ_{l_0,k_0}, we have that;

$\rho_{l_0,k_0}(\overline{x},t) = \int_{S(\overline{0},k_0)} f(\overline{k})e^{i(\overline{k}\centerdot\overline{x}-k_0ct)}dS(\overline{0},k_0)$

where $f(\overline{k}) = \frac{(\overline{k},F(\overline{k}))}{c|\overline{k}|} = \frac{|\overline{k}|}{c}$

so that, using the calculation in [12] or [4];

$\rho_{l_0,k_0}(\overline{x},t) = \frac{k_0}{c}e^{-ik_0ct}\int_{S(\overline{0},k_0)} e^{i\overline{k}\centerdot\overline{x}}dS(\overline{0},k_0)(\overline{k})$

$= \frac{k_0^3}{c}e^{-ik_0ct}\int_{S(\overline{0},1)} e^{i(\overline{l}\centerdot k_0\overline{x}}dS(\overline{0},1)(\overline{l})$

$= \frac{k_0^3}{c}e^{-ik_0ct}\frac{(2\pi)^{\frac{3}{2}}}{|k_0\overline{x}|^{\frac{1}{2}}}J_{\frac{1}{2}}(|k_0\overline{x}|)$

$= \frac{4k_0^3\pi}{c}j_0(k_0|\overline{x}|)$

$= \frac{k_0^3}{c}e^{-ik_0ct}4\pi\frac{sin(|k_0\overline{x}|)}{|k_0\overline{x}|}$

$= \frac{4\pi k_0^3}{c}e^{-i\omega_0t}\frac{sin(|k_0\overline{x}|)}{|k_0\overline{x}|}$ (PP)

where $\omega_0 = k_0 c$

We can complete $(\rho_{k_0,l_0}, \overline{J}_{k_0,l_0})$ to a tuple $(\rho_{k_0,l_0}, \overline{J}_{k_0,l_0}, \overline{E}_{k_0,l_0}, \overline{B}_{k_0,l_0})$ satisfying $(i)-(viii)$ as follows. For $(viii)$, we let $\overline{E}_{k_0,l_0} = e_{k_0,l_0} e^{-i\omega_0 t}$ so that;

$-i\omega_0 e_{k_0,l_0} = -\frac{1}{\epsilon_0} j_{k_0,l_0}$

$e_{k_0,l_0} = -\frac{i}{\epsilon_0 \omega_0} j_{k_0,l_0}$

$\overline{E}_{k_0,l_0} = -\frac{i}{\epsilon_0 \omega_0} \overline{J}_{k_0,l_0}$. Then, as;

$\frac{1}{c^2} \frac{\partial \overline{J}_{k_0,l_0}}{\partial t} = -\frac{i\omega_0}{c^2} \overline{J}_{k_0,l_0} = -\bigtriangledown(\rho_{k_0,l_0})$

we have that $\bigtriangledown \times \overline{E}_{k_0,l_0} = \bigtriangledown \times \overline{J}_{k_0,l_0} = \bigtriangledown \times \bigtriangledown(\rho_{k_0,l_0}) = \overline{0}$

and, as $\Box^2 \overline{J}_{k_0,l_0} = \overline{0}$, $\Box^2 \overline{E}_{k_0,l_0} = \overline{0}$, so that (v) holds.

We have that;

$\bigtriangledown \cdot \overline{E}_{k_0,l_0} = \bigtriangledown \cdot -\frac{i}{\epsilon_0 \omega_0} \overline{J}_{k_0,l_0}$

$= \frac{i}{\epsilon_0 \omega_0} \frac{\partial \rho_{k_0,l_0}}{\partial t}$

$= \frac{i}{\epsilon_0 \omega_0} (-i\omega_0) \rho_{k_0,l_0}$

$= \frac{\rho_{k_0,l_0}}{\epsilon_0}$

so that (vii) is satisfied. Setting $\overline{B} = \overline{0}$, we obtain (vi). Observe that by the calculation (PP), ρ_{k_0,l_0} is a scalar multiple of the form considered before the introduction of $\overline{J}$ vanishing at the boundary with the Bessel function defined by $l = 0$ and with $m = 0$. As the set of relations $(i)-(iv)$ hold for both $\overline{J}_{k_0,l_0}$ and $\overline{J}$, where $\overline{J}$ is defined from ρ_{k_0,l_0} using $(**)$ at the beginning of the paper, we must have that;

$\frac{\partial \overline{J}_{k_0,l_0} - \overline{J}}{\partial t} = \overline{0}$

$\Box^2 (\overline{J}_{k_0,l_0} - \overline{J}) = \overline{0}$

so that;

$\bigtriangledown^2(\overline{J}_{k_0,l_0}-\overline{J})=\overline{0}$

and;

$\overline{J}_{k_0,l_0}=\overline{J}+\overline{c}(t)$, by boundedness and the fact that the difference is harmonic at a given time t. Using the relation (iv) again, we must have that $\overline{c}'(t)=\overline{0}$, so that $\overline{c}(t)=\overline{c}$ is time independent. By the fact that the difference $\overline{J}_{k_0,l_0}-\overline{J}$ is of the form $\overline{j}(x,y,z)e^{-ik_0ct}$, we must have that $\overline{c}=\overline{0}$ so that $\overline{J}_{k_0,l_0}=\overline{J}$. We can then use the calculation above to verify the continuity equation at the boundary.

By construction $\overline{E}_{k_0,l_0}|_{S(\overline{0},w)}=\overline{0}$, in particular, the components $\{e_{k_0,l_0,\theta},e_{k_0,l_0,\phi}\}$ vanish at the boundary of the magnetron, so that $\overline{E}^{||}_{k_0,l_0}=\overline{0}$ and clearly $\overline{B}^{\perp}_{k_0,l_0}=0$ as well. As above, in the TE mode case, from (X), we can achieve compatibility of the boundaty condition with $\frac{\partial r^2b_r}{\partial\theta}=\frac{\partial r^2b_r}{\partial\phi}=0$, and $r^2b_r=0$ at the boundary. By the explicit form of rb_r in $(C1)$, if the magnetron has radius w, we achieve this when $R=j_{l_0}(\frac{\omega_0 r}{c})|_{\delta S(\overline{0},w)}=0$, we consider the simplest solution $p_{l_0,m_0,c}$, with $l_0=1$, $m_0=0$. In the TM mode case, from (Y), we can achieve this with $\frac{\partial r^2e_r}{\partial r}=0$, as $b_r=0$ in the TM mode. By the explicit form of re_r in $(C1)$, if the magnetron has radius w, this is achieved when $\frac{\partial rR}{\partial r}=\frac{\partial rj_{l_0'}(\frac{\omega_0' r}{c})}{\partial r}|_{\delta S(\overline{0},w)}=0$. Note that we can achieve this condition with a single Bessel function by Rolle's theorem and the fact that the Bessel functions j_l have infinitely many zeros for $l\geq 0$. We cannot, however achieve this condition with $j_{l_0'}$, for $l_0=l_0'$ unless $\omega=0$, as all the non-zero roots are simple.

In the TE case, we have that the surface charge $\frac{\sigma_{k_0,l_0}}{\epsilon_0}$ is given by;

$\overline{E}'^{\perp}_{k_0,l_0}-\overline{E}^{\perp}_{k_0,l_0}$

$=\overline{E}'^{\perp}_{k_0,l_0}$

$=e'_{k_0,l_0,r}e^{-i\omega_0 t}$

$=0$

by definition of the TE mode and the fact that $\overline{E}=\overline{0}$ at the boundary $S(\overline{0},w)$.

In the TM case, we have that the surface charge $\frac{\sigma_{k_0,l_0,f}}{\epsilon_0}$ is given by;

$\overline{E}'^{\perp}_{k_0,l_0}-\overline{E}^{\perp}_{k_0,l_0}$

$=\overline{E}'^{\perp}_{k_0,l_0}$

$=e'_{k_0,l_0,r}e^{-i\omega'_0 t}$

where $re'_{k_0,l_0,r}$ satisfies the usual relations with $R=j_{l'_0}(\frac{\omega'_0 r}{c})$.

In the TE case, we have that the surface current $\overline{K}_{k_0,l_0,f}$ is given by;

$\mu_0(\overline{K}_{k_0,l_0,f}\times\hat{r})=\overline{B}'^{||}_{k_0,l_0}-\overline{B}^{||}_{k_0,l_0}$

$=\overline{B}'^{||}_{k_0,l_0}$

$=(b'_{k_0,l_0,\theta}\hat{\overline{\theta}}+b'_{k_0,l_0,\phi}\hat{\overline{\phi}})e^{-i\omega_0 t}$

$=(\frac{1}{l_0(l_0+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2b'_{k_0,l_0,r})\hat{\overline{\theta}}+\frac{1}{l_0(l_0+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2b'_{k_0,l_0,r})\hat{\overline{\phi}})e^{-i\omega_0 t}$

where $rb'_{k_0,l_0,r}$ satisfies the relations above. It follows that;

$\mu_0\overline{K}_{k_0,l_0,f}=(\frac{1}{l_0(l_0+1)}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r^2b'_{k_0,l_0,r})\hat{\overline{\phi}}-\frac{1}{l_0(l_0+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r^2b'_{k_0,l_0,r})\hat{\overline{\theta}})e^{-i\omega_0 t}$

In the TM case, we have that;

$\mu_0(\overline{K}_{k_0,l_0,f}\times\hat{r})=\overline{B}'^{||}_{k_0,l_0}-\overline{B}^{||}_{k_0,l_0}$

$=\overline{B}'^{||}_{k_0,l_0}$

$=(-\frac{i\omega'_0}{c^2l'_0(l'_0+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2e'_{k_0,l_0,r})\hat{\overline{\theta}}+\frac{i\omega'_0}{c^2l'_0(l'_0+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2e'_r)\hat{\overline{\phi}})e^{-i\omega'_0 t}$

where $re'_{k_0,l_0,r}$ satisfies the relations above. It follows that;

$\mu_0\overline{K}_{k_0,l_0,f}=(-\frac{i\omega'_0}{c^2l'_0(l'_0+1)}\frac{1}{rsin(\theta)}\frac{\partial}{\partial\phi}(r^2e'_{k_0,l_0,r})\hat{\overline{\phi}}-\frac{i\omega'_0}{c^2l'_0(l'_0+1)}\frac{1}{r}\frac{\partial}{\partial\theta}(r^2e'_{k_0,l_0,r})\hat{\overline{\theta}})e^{-i\omega'_0 t}$

It follows that in the TE case, if we fix a circle S_{θ_0} on the sphere given by $\theta=\theta_0$, we have that the current along S_{θ_0} in the direction of

$\bar{\hat{\phi}}$ is given by, when $m_0 = 0$, $l_0 = 1$;

$$\mu_0 \overline{K}_{k_0,l_0,f}|_{S_{\theta_0}} = \frac{1}{2w}\frac{\partial^2}{\partial\theta\partial r}(rR_{k_0,l_0}(r)\Theta_{k_0,l_0}(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\bar{\hat{\phi}}e^{-i\omega_0 t}$$

$$= \frac{1}{2w}(R_{k_0,l_0}(r)\Theta'_{k_0,l_0}(\theta)\Phi_{k_0,l_0}(\phi)+R'_{k_0,l_0}(r)\Theta'_{k_0,l_0}(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\bar{\hat{\phi}}e^{-i\omega_0 t}$$

$$= \frac{1}{2w}(R'_{k_0,l_0}(r)\Theta'_{k_0,l_0}(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\bar{\hat{\phi}}e^{-i\omega_0 t},\ (R_{k_0,l_0}(w) = 0)$$

$$= \frac{1}{2w}j_1'(\frac{\omega_0 r}{c})(P_1^0(cos(\theta)))'|_{w,\theta_0,\phi}e^{-i\omega_0 t}\bar{\hat{\phi}}$$

$$= \frac{1}{2w}j_1'(\frac{\omega_0 r}{c})(cos(\theta))'|_{w,\theta_0,\phi}e^{-i\omega_0 t}\bar{\hat{\phi}}$$

$$= -\frac{sin(\theta_0)\omega_0}{2wc}j_1'(\frac{\omega_0 w}{c})e^{-i\omega_0 t}\bar{\hat{\phi}}$$

which is alternating current of amplitude $\frac{sin(\theta_0)\omega_0}{2wc}j_1'(\frac{\omega_0 w}{c})$ and frequency $\frac{\omega_0}{2\pi}$.

By the above, we have that the surface charge in the TE mode is zero, so the potential due to the surface charge on the sphere $S(\bar{0}, k_0)$ is also zero, by Jefimenko's equations. As $\rho = 0$ outside the magnetron, again by Jefimenko's equations, the causal potential on the sphere $S(\bar{0}, w)$, due to the TE mode, is again zero. The potential due to the charge inside the magnetron is found using the method of [10]. We have that, using the calculation above;

$$V_{k_0,l_0}(\bar{x}, t) = \frac{c^2\epsilon_0\rho_{k_0,l_0}(\bar{x},t)}{\omega^2}$$

$$= \frac{4\pi k_0^3 c^2\epsilon_0}{c\omega_0^2}e^{-i\omega_0 t}\frac{sin(|k_0\bar{x}|)}{|k_0\bar{x}|}$$

$$= \frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}e^{-i\omega_0 t}\frac{sin(|k_0\bar{x}|)}{|k_0\bar{x}|}$$

so the surface $S(\bar{0}, w)$ is an equipotential $\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}e^{-i\omega_0 t}\frac{sin(k_0 w)}{k_0 w}$

In particularly, if we ground $\phi = 0$ and take real parts, the impedance Z_{θ_0} along S_{θ_0} is given by;

$$Z_{\theta_0} = \frac{\frac{4\pi k_0^3 c\epsilon_0\mu_0}{\omega_0^2}e^{-i\omega_0 t}\frac{sin(k_0 w)}{k_0 w}}{\frac{sin(\theta_0)\omega_0}{2wc}j_1'(\frac{\omega_0 w}{c})e^{-i\omega_0 t}}$$

$$= \frac{\frac{4\pi k_0^3}{c\omega_0^2}\frac{sin(k_0w)}{k_0w}}{-\frac{sin(\theta_0)\omega_0}{2wc}j_1'(\frac{\omega_0 w}{c})}$$

$$= \frac{8\pi}{c^2\omega_0}\frac{sin(k_0w)}{sin(\theta_0)j_1'(\frac{\omega_0 w}{c})}$$

The cases when $l_0 \neq 1$ mean changing the frequency ω_0 to a new ω_0', but the cases can be computed using the formula for the derivative of an associated Legendre polynomial, when $-l_0 \leq m_0 \leq l_0$, $l_0 \geq 1$, see [14], with the convention that $P_l^m = 0$ for $|m| > l$. The quoted formula assumes the Condon-Shortley phase factor $(-1)^{m_0}$ which is not used here, but the formula is not effected;

$$(x^2-1)\frac{d}{dx}(P_{l_0}^{m_0}(x)) = l_0 x P_{l_0}^{m_0}(x) - (l_0+m_0)P_{l_0-1}^{m_0}(x)$$

which gives that;

$$P_{l_0}^{m_0}(cos(\theta))' = \frac{sin(\theta)}{sin^2(\theta)}(l_0 cos(\theta)P_{l_0}^{m_0}(cos(\theta)) - (l_0+m_0)P_{l_0-1}^{m_0}(cos(\theta)))$$

$$= l_0 cot(\theta)P_{l_0}^{m_0}(cos(\theta)) - (l_0+m_0)cosec(\theta)P_{l_0-1}^{m_0}(cos(\theta))$$

It follows that in the TE case, if we fix a circle S_{θ_0} on the sphere given by $\theta = \theta_0$, we have that the current along S_{θ_0} in the direction of $\hat{\overline{\phi}}$ is given in general for the basic solutions $p_{l_0',m_0',c}$, for $l_0' \geq 2$, $-l_0' \leq m_0' \leq l_0'$ by;

$$\mu_0\overline{K}_{k_0,l_0,f}|_{S_{\theta_0}} = \frac{1}{l_0'(l_0'+1)w}\frac{\partial^2}{\partial\theta\partial r}(rR_{k_0,l_0}(r)\Theta_{k_0,l_0}(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\hat{\overline{\phi}}e^{-i\omega_0' t}$$

$$= \frac{1}{l_0'(l_0'+1)w}(R_{k_0,l_0}(r)\Theta_{k_0,l_0}'(\theta)\Phi_{k_0,l_0}(\phi)+R_{k_0,l_0}'(r)\Theta_{k_0,l_0}'(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\hat{\overline{\phi}}e^{-i\omega_0' t}$$

$$= \frac{1}{l_0'(l_0'+1)w}(R_{k_0,l_0}'(r)\Theta_{k_0,l_0}'(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\hat{\overline{\phi}}e^{-i\omega_0' t}$$

$$= \frac{1}{l_0'(l_0'+1)w}j_{l_0'}'(\frac{\omega_0' r}{c})(P_{l_0}^{m_0}(cos(\theta_0)))'cos(m_0\phi)|_{w,\theta_0,\phi}e^{-i\omega_0' t}\hat{\overline{\phi}}$$

$$= \frac{1}{l_0'(l_0'+1)w}j_{l_0'}'(\frac{\omega_0' r}{c})cos(m_0'\phi)(l_0'cot(\theta_0)P_{l_0'}^{m_0'}(cos(\theta_0))-(l_0'+m_0')cosec(\theta_0)P_{l_0'-1}^{m_0'}(cos(\theta_0)))e^{-i\omega_0' t}\hat{\overline{\phi}}$$

$$= j_{l_0'}'(\frac{\omega_0' r}{c})e^{-i\omega_0' t}cos(m_0'\phi)\hat{\overline{\phi}}(\frac{1}{(l_0'+1)w}cot(\theta_0)P_{l_0'}^{m_0'}(cos(\theta_0))-\frac{(l_0'+m_0')}{l_0'(l_0'+1)w}cosec(\theta_0)P_{l_0'-1}^{m_0'}(cos(\theta_0)))$$

We leave it as an exercise to compute the impedance following the method below.

..... Similarly, in the TM case, if we fix the circle S_{θ_0} on the sphere given by $\theta=\theta_0$, we have that the current $\mu_0 I_{\theta_0}$ along S_{θ_0} in the direction of $\hat{\overline{\phi}}$ is given by;

$$-\frac{i\omega'_0}{c^2 l'_0(l'_0+1)}\frac{w}{wsin(\theta)}\frac{\partial}{\partial\phi}(R_{k_0,l_0}(r)\Theta_{k_0,l_0}(\theta)\Phi_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\hat{\overline{\phi}})e^{-i\omega'_0 t}$$

$$=-\frac{i\omega'_0}{c^2 l'_0(l'_0+1)}\frac{1}{sin(\theta)}(R_{k_0,l_0}(r)\Theta_{k_0,l_0}(\theta)\Phi'_{k_0,l_0}(\phi))|_{w,\theta_0,\phi}\hat{\overline{\phi}})e^{-i\omega'_0 t}$$

We consider the case $l'_0\neq 1$, $-l'_0\leq m'_0\leq l'_0$ remembering that we require $\frac{\partial}{\partial r}(rj_{l'_0}(\frac{\omega'_0 r}{c}))|_{\delta S(\overline{0},w)}=0$, which we cannot achieve with $l'_0=1$. We consider the basic solutions $p_{l'_0,m'_0,c}$.

$$\mu_0 I_{\theta_0}=-\frac{i\omega'_0}{l'_0(l'_0+1)c^2}\frac{1}{sin(\theta_0)}j_{l'_0}(\frac{\omega'_0 w}{c})((P^{m'_0}_{l'_0})(cos(\theta_0)))cos(m'_0\phi)'\hat{\overline{\phi}}e^{-i\omega'_0 t}$$

$$=\frac{im'_0\omega'_0}{l'_0(l'_0+1)c^2}\frac{1}{sin(\theta_0)}j_{l'_0}(\frac{\omega'_0 w}{c})((P^{m'_0}_{l'_0})(cos(\theta_0)))sin(m'_0\phi)\hat{\overline{\phi}}e^{-i\omega'_0 t}$$

As $\rho=0$ outside the magnetron, again by Jefimenko's equations, the causal potential on the sphere $S(\overline{0},w)$, due to the TM mode, is again zero. We can ignore the potential due to the surface charge in the TM mode, by Jefimenko's equations. As before, $S(\overline{0},w)$ is an equipotential;

$$V_{k_0,l_0}=\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}e^{-i\omega_0 t}\frac{sin(k_0 w)}{k_0 w}$$

due to the configuration inside the magnetron. We consider the $2m'_0$ points $\phi\in\{\frac{k\pi}{m'_0}:-m'_0\leq k\leq m'_0-1\}$ on the circle defined by $\theta=\theta_0$. Then the average current between the points $\phi=\frac{j\pi}{m'_0},\phi=\frac{(j+1)\pi}{m'_0}$, $-m'_0\leq j\leq m'_0-1$ mod m'_0 is;

$$\frac{m'_0}{\mu_0\pi}\int_{\frac{j\pi}{m'_0}}^{\frac{(j+1)\pi}{m'_0}}\frac{im'_0\omega'_0}{l'_0(l'_0+1)c^2}j_{l'_0}(\frac{\omega'_0 w}{c})((P^{m'_0}_{l'_0})(cos(\theta_0)))sin(m'_0\phi)\hat{\overline{\phi}}e^{-i\omega'_0 t}d\phi$$

$$=\frac{m'_0}{\mu_0\pi}\frac{im'_0\omega'_0}{l'_0(l'_0+1)c^2}j_{l'_0}(\frac{\omega'_0 w}{c})e^{-i\omega'_0 t}((P^{m'_0}_{l'_0})(cos(\theta_0)))\hat{\overline{\phi}}\int_{\frac{j\pi}{m'_0}}^{\frac{(j+1)\pi}{m'_0}}sin(m'_0\phi)d\phi$$

$$=\frac{2(-1)^j m'^2_0}{\mu_0\pi}\frac{i\omega'_0}{l'_0(l'_0+1)c^2}j_{l'_0}(\frac{\omega'_0 w}{c})e^{-i\omega'_0 t}((P^{m'_0}_{l'_0})(cos(\theta_0)))\hat{\overline{\phi}}$$

whereas if we ground the m_0 points corresponding to $\phi\in\{\frac{(2s-m_0)\pi}{m_0}:0\leq s\leq m_0-1\}$, the potential difference across the $2m_0$ regions is $\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}e^{-i\omega_0 t}\frac{sin(k_0 w)}{k_0 w}$.

Taking real parts, we have that the average current is given by;

$\frac{2(-1)^j m_0'^2}{\mu_0\pi}\frac{\omega_0'}{l_0'(l_0'+1)c^2}j_{l_0'}(\frac{\omega_0' w}{c})sin(\omega_0' t)((P_{l_0'}^{m_0'})(cos(\theta_0)))$

whereas the potential is;

$\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}\frac{sin(k_0 w)}{k_0 w}cos(\omega_0 t)$

We have that;

$$cos(\omega_0' t)-cos(\omega_0 t)=-2sin(\frac{(\omega_0+\omega_0')}{2}t)sin(\frac{(\omega_0-\omega_0')}{2}t)$$

so that if we apply a voltage;

$$V'(t)=-\frac{8\pi k_0^3 c\epsilon_0}{\omega_0^2}\frac{sin(k_0 w)}{k_0 w}sin(\frac{(\omega_0+\omega_0')}{2}t)sin(\frac{(\omega_0-\omega_0')}{2}t)$$

to the sphere boundary, the total sphere potential is;

$\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}\frac{sin(k_0 w)}{k_0 w}cos(\omega_0' t)$

and the impedance in the $2m_0$ regions is;

$$Z_{j,\theta_0}=i\frac{\frac{4\pi k_0^3 c\epsilon_0\mu_0}{\omega_0^2}\frac{sin(k_0 w)}{k_0 w}}{\frac{2(-1)^j m_0'^2}{\pi}\frac{\omega_0'}{l_0'(l_0'+1)c^2}j_{l_0'}(\frac{\omega_0' w}{c})((P_{l_0'}^{m_0'})(cos(\theta_0)))}$$
$$=i\frac{2\pi^2(-1)^j l_0'(l_0'+1)sin(\frac{\omega_0 w}{c})}{m_0'^2 cw\omega_0' j_{l_0'}(\frac{\omega_0' w}{c})((P_{l_0'}^{m_0'})(cos(\theta_0)))}$$

V' can be generated from an AC potential of frequency $\frac{(\omega_0+\omega_0')}{4\pi}$, with a variable transformer, in which the sliding contact determining the turns ratio varies as $sin(\frac{(\omega_0-\omega_0')}{2}t)$. Alternatively, the potentials;

$\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}\frac{sin(k_0 w)}{k_0 w}cos(\omega_0' t)$

$\frac{4\pi k_0^3 c\epsilon_0}{\omega_0^2}\frac{sin(k_0 w)}{k_0 w}cos(\omega_0 t)$

can be generated directly using an RL or RC circuit, tuned to the correct resonant frequency, and then combined using a mixer. Notice that the approximation to the current becomes better with large m_0'.

□

Lemma 0.2. *Let* $(\rho,\overline{J},\overline{E},\overline{B})$ *be the configuration found in Lemma 0.1, and let* $(\overline{E}',\overline{B}')$ *be the causal fields generated by Jefimenko's equations for the current and charge* $(\rho,\overline{J})$ *restricted to* $B(\overline{0},w)$. *Then* *on $B^\circ(\overline{0},w)$, $\overline{E}' = \overline{E}+\overline{E}_0$, $\overline{B}' = \overline{B}_0$ where $(\overline{E}_0,\overline{B}_0)$ is a solution to Maxwell's equation in vacuum, and on $B(\overline{0},w)^c$, $(\overline{E}',\overline{B}')$ is a solution to Maxwell's equation in vacuum.*

Proof. By the proof in [11], we have that $(\rho,\overline{J},\overline{E}',\overline{B}')$ satisfy Maxwell's equations on $B^\circ(\overline{0},w)$ and $(0,\overline{0},\overline{E}',\overline{B}')$ satisfy Maxwell's equations on $B(\overline{0},w)^c$. By the proof in [11], we can find $(\overline{E}_0,\overline{B}_0)$ satisfying Maxwell's equations in vacuum on $B^\circ(\overline{0},w)$, such that;

$\bigtriangledown\times(\overline{E}'-\overline{E}_0)=\overline{0}$

We then have that $\frac{\partial(\overline{B}'-\overline{B}_0)}{\partial t} = \bigtriangledown\times(\overline{E}'-\overline{E}_0)=\overline{0}$

so that $(\overline{B}'-\overline{B}_0)$ is magnetostatic. By the proof of Lemma 0.1 and a careful examination of the proof in [11], we have that;

$\overline{B}'-\overline{B}_0$

is of the form $\overline{b}''(x,y,z)e^{-i\omega t}$, so that $-i\omega\overline{b}''=\overline{0}$, $\overline{b}''=\overline{0}$ and $\overline{B}'=\overline{B}_0$. We have that;

$(\overline{E}'-\overline{E}_0-\overline{E},\overline{B}'-\overline{B}_0-\overline{B}) = (\overline{E}'-\overline{E}_0-\overline{E},\overline{0}-\overline{0})$

$=(\overline{E}'-\overline{E}_0-\overline{E},\overline{0})$

is a solution to Maxwell's equation in vacuum, on the ball $B(\overline{0},w)$, so that, by Maxwell's fourth equation;

$\frac{\partial(\overline{E}'-\overline{E}_0-\overline{E})}{\partial t} = \bigtriangledown\times\overline{0}-\overline{0}=\overline{0}$

Again, using the explicit form $\overline{e}''(x,y,z)e^{-i\omega t}$ for $\overline{E}'-\overline{E}_0-\overline{E}$, it follows that $\overline{E}'-\overline{E}_0=\overline{E}$.

□

Lemma 0.3. *Let $\overline{E}$ be a field, of the form $\overline{e}(x,y,z)e^{-i\omega t}$ with the property that $\square^2(\overline{E})=\overline{0}$ and $\bigtriangledown\cdot\overline{E}=0$, or equivalently $\bigtriangledown^2(\overline{e})=-\frac{\omega^2}{c^2}\overline{e}$ and $\bigtriangledown\cdot\overline{e}=0$, then there exists a unique field $\overline{B}$ of the form $\overline{b}e^{-i\omega t}$ such that the pair $(\overline{E},\overline{B})$ satisfies Maxwell's equations in free space.*

Proof. Clearly (i) of Maxwell's equations is satisfied. Let $\overline{B}=\overline{b}e^{-i\omega t}$, where $\overline{b}=-\frac{i}{\omega}\bigtriangledown\times\overline{e}$. For (ii), we have that;

$\bigtriangledown \times \overline{E} = (\bigtriangledown \times \overline{e})e^{-i\omega t}$

$= i\omega(\frac{-i}{\omega}) \bigtriangledown \times \overline{e}e^{-i\omega t}$

$= i\omega\overline{b}e^{-i\omega t}$

$= -\frac{\partial \overline{B}}{\partial t}$

For (iii), we have that;

$\bigtriangledown \cdot \overline{B} = \bigtriangledown \cdot (\overline{b}e^{-i\omega t})$

$= (\bigtriangledown \cdot (-\frac{i}{\omega} \bigtriangledown \times \overline{e}))e^{-i\omega t})$

$= 0$

For (iv), we have, by the properties of $\overline{e}$ that;

$\bigtriangledown \times \overline{B} = \bigtriangledown \times (\overline{b}e^{-i\omega t})$

$= (\bigtriangledown \times (-\frac{i}{\omega} \bigtriangledown \times \overline{e}))e^{-i\omega t})$

$= -\frac{i}{\omega}(\bigtriangledown \times \bigtriangledown \times \overline{e})e^{-i\omega t})$

$= -\frac{i}{\omega}(\bigtriangledown(\bigtriangledown \cdot \overline{e}) - \bigtriangledown^2(\overline{e}))e^{-i\omega t}$

$-\frac{i}{\omega}(- \bigtriangledown^2 (\overline{e}))e^{-i\omega t}$

$= \frac{i}{\omega}\frac{-\omega^2}{c^2}\overline{e}e^{-i\omega t}$

$-\frac{i\omega}{c^2}\overline{e}e^{-i\omega t}$

$= \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t}$

For uniqueness, let $(\overline{E}, \overline{B}_1)$ and $(\overline{E}, \overline{B}_2)$ be two pairs of the above form, so that, subtracting, $(\overline{0}, \overline{B}_1 - \overline{B}_2)$ is a solution to Maxwell's equation in vacuum. By (ii);

$\frac{\partial(\overline{B}_1 - \overline{B}_2)}{\partial t} = -i\omega(\overline{B}_1 - \overline{B}_2)$

$= -(\bigtriangledown \times \overline{0})$

$= \overline{0}$

so that $\overline{B}_1 = \overline{B}_2$.

$\square$

Lemma 0.4. *If $\overline{V}$ is a vector potential of the form $\overline{v}(x,y,z)e^{-i\omega t}$, with the property that $\square^2(\overline{V}) = 0$, or equivalently $\bigtriangledown^2(\overline{v}) = -\frac{\omega^2}{c^2}\overline{v}$, then if $\overline{E} = \bigtriangledown \times \overline{V}$, we have that $\overline{E}$ satisfies the properties in Lemma 0.3. Given boundary conditions $\{\overline{f}, \overline{g}\}$ on $\delta S(\overline{0}, w)$, if;*

$\bigtriangledown \times \overline{v}|_{\delta S(\overline{0},w)} = \overline{f}$

$-\frac{i}{\omega}(\bigtriangledown \times \bigtriangledown \times \overline{v})|_{\delta S(\overline{0},w)} = \overline{g}$

then the corresponding fields $\{\overline{E}, \overline{B}\}$ are continuous with fields $\{\overline{f}e^{-i\omega t}, \overline{g}e^{-i\omega t}\}$ on $B(\overline{0}, w)$. These boundary conditions can be satisfied for $\overline{v}$ with the above property, if $\overline{g} = \overline{0}$ and $\overline{f}^r = \overline{f}^2 = 0$. In particular, these boundary conditions are satisfied for the configuration from Lemma 0.1, when $\overline{J}^r|_{\delta B(\overline{0},w)} = 0$ or when $\overline{J}|_{\delta B(\overline{0},w)}$, in which case we obtain a 2-dimensional family of solutions.

Proof. The first claim follows easily, noting that;

$\bigtriangledown \cdot \overline{E} = \bigtriangledown \cdot (\bigtriangledown \times \overline{V})$

$= 0$

$\square^2(\overline{E}) = \square^2(\bigtriangledown \times \overline{V}) = \bigtriangledown \times \square^2(\overline{V})$

$= \bigtriangledown \times \overline{0}$

$= \overline{0}$

We can write $\overline{v}$ in the form;

$\overline{v}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(v_{lm}^r(r)\overline{Y}_{lm}(r,\theta,\phi) + v_{lm}^1(r)\overline{\Psi}_{lm}(r,\theta,\phi) + v_{lm}^2(r)\overline{\Phi}_{lm}(r,\theta,\phi))$

where $\{\overline{Y}_{lm}, \overline{\Psi}_{lm}, \overline{\Phi}_{lm}\}$ are vector spherical harmonics, see [2].

Then;

$\bigtriangledown^2(\overline{v}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^r_{lm}}{dr})\overline{Y}_{lm}+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^1_{lm}}{dr})\overline{\Psi}_{lm}+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^2_{lm}}{dr})\overline{\Phi}_{lm}$

$+v^r_{lm}(-\frac{1}{r^2}(2+l(l+1))\overline{Y}_{lm}+\frac{2}{r^2}\overline{\Psi}_{lm})+v^1_{lm}(\frac{2}{r^2}l(l+1)\overline{Y}_{lm}-\frac{1}{r^2}l(l+1)\overline{\Psi}_{lm})$

$+v^2_{lm}(-\frac{1}{r^2}l(l+1)\overline{\Phi}_{lm})$

$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^r_{lm}}{dr}+v^r_{lm}(-\frac{1}{r^2}(2+l(l+1)))+v^1_{lm}(\frac{2}{r^2}l(l+1)))\overline{Y}_{lm}$

$+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^1_{lm}}{dr} + v^r_{lm}\frac{2}{r^2} - v^1_{lm}\frac{1}{r^2}l(l+1))\overline{\Psi}_{lm}$

$+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^2_{lm}}{dr} - v^2_{lm}\frac{1}{r^2}l(l+1))\overline{\Phi}_{lm}$

so that equating coefficients, the condition $\bigtriangledown^2(\overline{v}) = -\frac{\omega^2}{c^2}\overline{v}$, becomes;

$(i).\ \frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^r_{lm}}{dr} + v^r_{lm}(-\frac{1}{r^2}(2+l(l+1))) + v^1_{lm}(\frac{2}{r^2}l(l+1)) = -\frac{\omega^2}{c^2}v^r_{lm}$

$(ii).\ \frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^1_{lm}}{dr} + v^r_{lm}\frac{2}{r^2} - v^1_{lm}\frac{1}{r^2}l(l+1) = -\frac{\omega^2}{c^2}v^1_{lm}$

$(iii).\ \frac{1}{r^2}\frac{d}{dr}r^2\frac{dv^2_{lm}}{dr} - v^2_{lm}\frac{1}{r^2}l(l+1) = -\frac{\omega^2}{c^2}v^2_{lm}$

or equivalently;

$(i).\ (v^r_{lm})'' + \frac{2}{r}(v^r_{lm})' + (\frac{\omega^2}{c^2} - \frac{2+l(l+1)}{r^2})v^r_{lm} + \frac{2l(l+1)}{r^2}v^1_{lm} = 0$

$(ii).\ (v^1_{lm})'' + \frac{2}{r}(v^1_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})v^1_{lm} + \frac{2}{r^2}v^r_{lm} = 0$

$(iii).\ (v^2_{lm})'' + \frac{2}{r}(v^2_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})v^2_{lm} = 0\ (P)$

Letting $\overline{w} = (v^r_{lm}, (v^r_{lm})', v^1_{lm}, (v^1_{lm})', v^2_{lm}, (v^2_{lm})')$, we can write these conditions in the form;

$\overline{w}' = M\overline{w}$

where M is a matrix, with;

$M_{12} = 1,\ M_{1j} = 0,\ j = 1$ or $3 \leq j \leq 6$

$M_{34} = 1,\ M_{3j} = 0,\ 1 \leq j \leq 2,\ 4 \leq j \leq 6$

$M_{56}=1$, $M_{5j}=0$, $1\leq j\leq 5$

$M_{21}=-(\frac{\omega^2}{c^2}-\frac{2+l(l+1)}{r^2})$, $M_{22}=-\frac{2}{r}$, $M_{23}=-\frac{2l(l+1)}{r^2}$

$M_{2j}=0$, $4\leq j\leq 6$

$M_{43}=-(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})$, $M_{44}=-\frac{2}{r}$, $M_{41}=-\frac{2}{r^2}$

$M_{4j}=0$, $j=2$, $5\leq j\leq 6$

$M_{65}=-(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})$, $M_{66}=-\frac{2}{r}$, $M_{6j}=0$, $1\leq j\leq 4$

By the vector valued version of Peano's existence and uniqueness theorem, this has a unique solution given the initial values of $\overline{w}$ at w. We have that;

$\bigtriangledown\times\overline{v}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\bigtriangledown\times(v_{lm}^r\overline{Y}_{lm})+\bigtriangledown\times(v_{lm}^1\overline{\Psi}_{lm})+\bigtriangledown\times(v_{lm}^2\overline{\Phi}_{lm}))$

$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\frac{1}{r}v_{lm}^r\overline{\Phi}_{lm}+(\frac{dv_{lm}^1}{dr}+\frac{1}{r}v_{lm}^1)\overline{\Phi}_{lm}+((-\frac{l(l+1)}{r})v_{lm}^2\overline{Y}_{lm}$

$-(\frac{dv_{lm}^2}{dr}+\frac{1}{r}v_{lm}^2)\overline{\Psi}_{lm})$

$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\frac{l(l+1)}{r})v_{lm}^2\overline{Y}_{lm}-(\frac{dv_{lm}^2}{dr}+\frac{1}{r}v_{lm}^2)\overline{\Psi}_{lm}$

$+(\frac{dv_{lm}^1}{dr}+\frac{1}{r}v_{lm}^1-\frac{1}{r}v_{lm}^r)\overline{\Phi}_{lm}$

so the first boundary condition becomes;

$(a).\ -\frac{l(l+1)}{w})v_{lm}^2(w)=\overline{f}_{lm}^r(w)$

$(b)\ -(\frac{dv_{lm}^2}{dr}(w)+\frac{1}{w}v_{lm}^2(w))=\overline{f}_{lm}^1(w)$

$(c)\ (\frac{dv_{lm}^1}{dr}(w)+\frac{1}{w}v_{lm}^1(w)-\frac{1}{w}v_{lm}^r(w))=\overline{f}_{lm}^2(w)$

We have that, using (P);

$\bigtriangledown\times\bigtriangledown\times\overline{v}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\bigtriangledown\times((-\frac{l(l+1)}{r})v_{lm}^2\overline{Y}_{lm})$

$-\bigtriangledown\times((\frac{dv_{lm}^2}{dr}+\frac{1}{r}v_{lm}^2)\overline{\Psi}_{lm})+\bigtriangledown\times((\frac{dv_{lm}^1}{dr}+\frac{1}{r}v_{lm}^1-\frac{1}{r}v_{lm}^r)\overline{\Phi}_{lm})$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -\frac{1}{r}(-\frac{l(l+1)}{r} v_{lm}^2)\overline{\Phi}_{lm} - (\frac{d}{dr}(\frac{dv_{lm}^2}{dr} + \frac{1}{r}v_{lm}^2) + \frac{1}{r}(\frac{dv_{lm}^2}{dr} + \frac{1}{r}v_{lm}^2))\overline{\Phi}_{lm}$

$+(-\frac{l(l+1)}{r}(\frac{dv_{lm}^1}{dr} + \frac{1}{r}v_{lm}^1 - \frac{1}{r}v_{lm}^r))\overline{Y}_{lm} - (\frac{d}{dr}(\frac{dv_{lm}^1}{dr} + \frac{1}{r}v_{lm}^1 - \frac{1}{r}v_{lm}^r)$

$+\frac{1}{r}(\frac{dv_{lm}^1}{dr} + \frac{1}{r}v_{lm}^1 - \frac{1}{r}v_{lm}^r))\overline{\Psi}_{lm}$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l}([-l(l+1)(\frac{1}{r}(v_{lm}^1)' + \frac{1}{r^2}v_{lm}^1 - \frac{1}{r^2}v_{lm}^r)]\overline{Y}_{lm}$

$+[-(v_{lm}^1)'' + \frac{1}{r^2}v_{lm}^1 - \frac{1}{r}(v_{lm}^1)' - \frac{1}{r^2}v_{lm}^r + \frac{1}{r}(v_{lm}^r)' + \frac{1}{r}(v_{lm}^1)' + \frac{1}{r^2}v_{lm}^1$

$-\frac{1}{r^2}v_{lm}^r]\overline{\Psi}_{lm} + [\frac{l(l+1)}{r^2}v_{lm}^2 - (v_{lm}^2)'' + \frac{1}{r^2}v_{lm}^2 - \frac{1}{r}(v_{lm}^2)' - \frac{1}{r}(v_{lm}^2)' - \frac{1}{r^2}v_{lm}^2]\overline{\Phi}_{lm})$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l}([-l(l+1)(\frac{1}{r}(v_{lm}^1)' + \frac{1}{r^2}v_{lm}^1 - \frac{1}{r^2}v_{lm}^r)]\overline{Y}_{lm}$

$+[-(v_{lm}^1)'' + \frac{1}{r}(v_{lm}^r)' + \frac{2}{r^2}v_{lm}^1 - \frac{2}{r^2}v_{lm}^r]\overline{\Psi}_{lm} + [-(v_{lm}^2)'' - \frac{2}{r}(v_{lm}^2)' + \frac{l(l+1)}{r^2}v_{lm}^2]\overline{\Phi}_{lm})$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l}([-l(l+1)(\frac{1}{r}(v_{lm}^1)' + \frac{1}{r^2}v_{lm}^1 - \frac{1}{r^2}v_{lm}^r)]\overline{Y}_{lm}$

$+[\frac{2}{r}(v_{lm}^1)' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})v_{lm}^1 + \frac{2}{r^2}v_{lm}^r + \frac{1}{r}(v_{lm}^r)' + \frac{2}{r^2}v_{lm}^1 - \frac{2}{r^2}v_{lm}^r]\overline{\Psi}_{lm}$

$+[\frac{2}{r}(v_{lm}^2)' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})v_{lm}^2 - \frac{2}{r}(v_{lm}^2)' + \frac{l(l+1)}{r^2}v_{lm}^2]\overline{\Phi}_{lm})$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l}([-l(l+1)(\frac{1}{r}(v_{lm}^1)' + \frac{1}{r^2}v_{lm}^1 - \frac{1}{r^2}v_{lm}^r)]\overline{Y}_{lm}$

$+[\frac{2}{r}(v_{lm}^1)' + \frac{1}{r}(v_{lm}^r)' + (\frac{\omega^2}{c^2} + \frac{2-l(l+1)}{r^2})v_{lm}^1]\overline{\Psi}_{lm}$

$+[\frac{\omega^2}{c^2}v_{lm}^2]\overline{\Phi}_{lm})$

so the second boundary condition becomes;

$(d).\ \frac{il(l+1)}{\omega}(\frac{1}{w}(v_{lm}^1)'(w) + \frac{1}{w^2}v_{lm}^1(w) - \frac{1}{w^2}v_{lm}^r(w)) = \overline{g}_{lm}^r(w)$

$(e).\ -\frac{2i}{w\omega}(v_{lm}^1)'(w) - \frac{i}{w\omega}(v_{lm}^r)'(w) - \frac{i}{\omega}(\frac{\omega^2}{c^2} + \frac{2-l(l+1)}{w^2})v_{lm}^1(w) = \overline{g}_{lm}^1(w)$

$(f).\ -\frac{i}{\omega}\frac{\omega^2}{c^2}v_{lm}^2(w) = \overline{g}_{lm}^2(w)$

We can write the two boundary conditions in the form;

$N\overline{w}|_w = \overline{s}$

where $\overline{w}$ is as above, and;

$\overline{s}=(\overline{f}^r_{lm}(w),\overline{f}^1_{lm}(w),\overline{f}^2_{lm}(w),\overline{g}^r_{lm}(w),\overline{g}^1_{lm}(w),\overline{g}^2_{lm}(w))$

and N is a matrix, with;

$N_{15}=-\frac{l(l+1)}{w}$, $N_{1j}=0$, $j=6$ or $1\leq j\leq 4$

$N_{25}=-\frac{1}{w}$, $N_{26}=-1$, $N_{2j}=0$, $1\leq j\leq 4$

$N_{65}=-\frac{i}{\omega}\frac{\omega^2}{c^2}$, $N_{6j}=0$, $j=6$ or $1\leq j\leq 4$

$N_{31}=-\frac{1}{w}$, $N_{33}=\frac{1}{w}$, $N_{34}=1$, $N_{3j}=0$, $j=2$ or $5\leq j\leq 6$

$N_{41}=-\frac{i(l+1)}{\omega}\frac{1}{w^2}$, $N_{43}=\frac{i(l+1)}{\omega}\frac{1}{w^2}$, $N_{44}=\frac{i(l+1)}{\omega}\frac{1}{w}$ $N_{4j}=0$, $j=2$ or $5\leq j\leq 6$

$N_{52}=-\frac{i}{\omega}\frac{1}{w}$, $N_{53}=-\frac{i}{\omega}(\frac{\omega^2}{c^2}+\frac{2-l(l+1)}{w^2})$, $N_{54}=-\frac{i}{\omega}\frac{2}{w}$, $N_{5j}=0$, $j=1$ or $5\leq j\leq 6$

If $\overline{g}=\overline{0}$ and $\overline{f}^r=\overline{f}^2=0$, then;

$\overline{s}=(0,\overline{f}^1_{lm}(w),0,0,0,0)$

and we obtain a solution by setting;

$v^2_{lm}=0$

$(v^2_{lm})'=-\overline{f}^1_{lm}(w)$

$-v^r_{lm}+v^1_{lm}+w(v^1_{lm})'=0$

$(v^r_{lm})'+w(\frac{\omega^2}{c^2}+\frac{2-l(l+1)}{w^2})v^1_{lm}+2(v^1_{lm})'=0$

which is a 2 dimensional family, as we are free to choose v^1_{lm} and $(v^1_{lm})'$. Using the fact that, for the configuration $(\rho,\overline{J},\overline{E},\overline{B})$ inside the magnetron;

$\bigtriangledown \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = \overline{0}$

$\bigtriangledown \times \overline{B} = \mu_0 \overline{J} + \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} = \overline{0}$

we obtain, at the boundary;

$(\bigtriangledown \times \overline{E})^r_{lm} = -\frac{l(l+1)}{w}(\overline{E})^2_{lm} = 0$

so that $(\overline{E})^2_{lm}(w) = 0$, and;

$\mu_0(\overline{J})^r_{lm} - \frac{i\omega}{c^2}(\overline{E})^r_{lm}$

so that, with the hypothesis that $\overline{J}^r|_{\delta B(\overline{0},w)} = 0$ or $\overline{J}|_{\delta B(\overline{0},w)} = \overline{0}$, we obtain that $(\overline{E})^r_{lm}(w) = 0$, as required.

$\square$

Lemma 0.5. *If V and $\overline{A}$ are potentials of the form $v(x,y,z)e^{-i\omega t}$ and $\overline{a}(x,y,z)e^{-i\omega t}$, with the property that and $\bigtriangledown \cdot \overline{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$ and $\square^2(\overline{A}) = \overline{0}$, or equivalently $\bigtriangledown \cdot \overline{a} = \frac{i\omega}{c^2}v$ and $\bigtriangledown^2(\overline{a}) = -\frac{\omega^2}{c^2}\overline{a}$, then if;*

$\overline{E} = -\bigtriangledown(V) - \frac{\partial \overline{A}}{\partial t} = -\bigtriangledown(V) + i\omega\overline{A}$

$\overline{B} = \bigtriangledown \times \overline{A}$

we have that $\{\overline{E}, \overline{B}\}$ satisfy Maxwell's equations in free space on $B(\overline{0},w)^c$. Given boundary conditions $\{\overline{f}, \overline{g}\}$ on $\delta S(\overline{0},w)$, if;

$-\bigtriangledown(v) + i\omega\overline{a}|_{\delta S(\overline{0},w)} = \overline{f}$

$\bigtriangledown \times \overline{a}|_{\delta S(\overline{0},w)} = \overline{g}$

then the corresponding fields $\{\overline{E}, \overline{B}\}$ are continuous with fields $\{\overline{f}e^{-i\omega t}, \overline{g}e^{-i\omega t}\}$ on $B(\overline{0},w)$. These boundary conditions can be satisfied for $\{v, \overline{a}\}$ with the above property, if $\overline{g} = \overline{0}$ and $\overline{f}^r = \overline{f}^2 = 0$. In particular, these boundary conditions are satisfied for the configuration from Lemma 0.1, with $\overline{J}|_{\delta B(\overline{0},w)} = \overline{0}$, in which case we obtain a 2-dimensional family of solutions in the TM mode.

Proof. First observe that if V is of the form $v(x,y,z)e^{-i\omega t}$, then as $\square^2\overline{A}=0$ and $\frac{\partial V}{\partial t}=-i\omega V$, we obtain, using the Lorentz gauge condition, that $\bigtriangledown\centerdot\overline{A}=-\frac{1}{c^2}\frac{\partial V}{\partial t}$;

$$\square^2(V)=\frac{i}{\omega}\square^2(\frac{\partial V}{\partial t})$$

$$=\frac{i}{\omega}\square^2(-c^2\bigtriangledown\centerdot\overline{A})$$

$$=-\frac{c^2 i}{\omega}\bigtriangledown\centerdot(\square^2\overline{A})$$

$$=-\frac{c^2 i}{\omega}\bigtriangledown\centerdot(\overline{0})$$

$$=0$$

The first claim then follows from the result in [11], as the Lorentz gauge condition and wave equations for $(V,\overline{A})$ are satisfied. We can write v in the form;

$$v(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(v_{lm}(r)Y_{lm}(r,\theta,\phi)$$

where the $\{Y_{lm}:l\geq 0,-l\leq m\leq l\}$ are the spherical harmonics. Then;

$$\bigtriangledown^2(v)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dv_{lm}}{dr})-\frac{l(l+1)}{r^2}v_{lm})Y_{lm}$$

so that equating coefficients, the condition $\bigtriangledown^2(\overline{v})=-\frac{\omega^2}{c^2}\overline{v}$, becomes;

$$(i).\ \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dv_{lm}}{dr})-\frac{l(l+1)}{r^2}v_{lm}=\frac{-\omega^2}{c^2}v_{lm}$$

or equivalently;

$$(i).\ (v_{lm})''+\frac{2}{r}(v_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})v_{lm}=0\ (P)$$

We can write $\overline{a}$ in the form;

$$\overline{a}(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{lm}^r(r)\overline{Y}_{lm}(r,\theta,\phi)+a_{lm}^1(r)\overline{\Psi}_{lm}(r,\theta,\phi)+a_{lm}^2(r)\overline{\Phi}_{lm}(r,\theta,\phi))$$

where $\{\overline{Y}_{lm},\overline{\Psi}_{lm},\overline{\Phi}_{lm}\}$ are vector spherical harmonics, see [2].

Then;

$\bigtriangledown \cdot \overline{a} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{da^r_{lm}}{dr} + \frac{2}{r}a^r_{lm} - \frac{l(l+1)}{r}a^1_{lm})Y_{lm}$

so that equating coefficients, the Lorentz gauge condition;

$v = -\frac{ic^2}{\omega} \bigtriangledown \cdot \overline{a}$

becomes;

$(ii).\ v_{lm} = -\frac{ic^2}{\omega}(\frac{da^r_{lm}}{dr} + \frac{2}{r}a^r_{lm} - \frac{l(l+1)}{r}a^1_{lm})$

or equivalently;

$(ii).\ v_{lm} = -\frac{ic^2}{\omega}((a^r_{lm})' + \frac{2}{r}a^r_{lm} - \frac{l(l+1)}{r}a^1_{lm})\ (P2)$

Moreover;

$\bigtriangledown^2(\overline{a}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{r^2}\frac{d}{dr}r^2\frac{da^r_{lm}}{dr})\overline{Y}_{lm}+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{da^1_{lm}}{dr})\overline{\Psi}_{lm}+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{da^2_{lm}}{dr})\overline{\Phi}_{lm}$

$+a^r_{lm}(-\frac{1}{r^2}(2+l(l+1))\overline{Y}_{lm}+\frac{2}{r^2}\overline{\Psi}_{lm})+a^1_{lm}(\frac{2}{r^2}l(l+1)\overline{Y}_{lm}-\frac{1}{r^2}l(l+1)\overline{\Psi}_{lm})$

$+a^2_{lm}(-\frac{1}{r^2}l(l+1)\overline{\Phi}_{lm})$

$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{r^2}\frac{d}{dr}r^2\frac{da^r_{lm}}{dr}+a^r_{lm}(-\frac{1}{r^2}(2+l(l+1)))+a^1_{lm}(\frac{2}{r^2}l(l+1)))\overline{Y}_{lm}$

$+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{da^1_{lm}}{dr} + a^r_{lm}\frac{2}{r^2} - a^1_{lm}\frac{1}{r^2}l(l+1))\overline{\Psi}_{lm}$

$+(\frac{1}{r^2}\frac{d}{dr}r^2\frac{da^2_{lm}}{dr} - a^2_{lm}\frac{1}{r^2}l(l+1))\overline{\Phi}_{lm}$

so that equating coefficients again, the condition $\bigtriangledown^2(\overline{a}) = -\frac{\omega^2}{c^2}\overline{a}$, becomes;

$(iii).\ \frac{1}{r^2}\frac{d}{dr}r^2\frac{da^r_{lm}}{dr} + a^r_{lm}(-\frac{1}{r^2}(2 + l(l+1))) + a^1_{lm}(\frac{2}{r^2}l(l+1)) = -\frac{\omega^2}{c^2}a^r_{lm}$

$(iv).\ \frac{1}{r^2}\frac{d}{dr}r^2\frac{da^1_{lm}}{dr} + a^r_{lm}\frac{2}{r^2} - a^1_{lm}\frac{1}{r^2}l(l+1) = -\frac{\omega^2}{c^2}a^1_{lm}$

$(v).\ \frac{1}{r^2}\frac{d}{dr}r^2\frac{da^2_{lm}}{dr} - a^2_{lm}\frac{1}{r^2}l(l+1) = -\frac{\omega^2}{c^2}a^2_{lm}$

or equivalently;

$(iii).\ (a^r_{lm})'' + \frac{2}{r}(a^r_{lm})' + (\frac{\omega^2}{c^2} - \frac{2+l(l+1)}{r^2})a^r_{lm} + \frac{2l(l+1)}{r^2}a^1_{lm} = 0$

(iv). $(a^1_{lm})''+\frac{2}{r}(a^1_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})a^1_{lm}+\frac{2}{r^2}a^r_{lm}=0$

(v). $(a^2_{lm})''+\frac{2}{r}(a^2_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})a^2_{lm}=0$ (Q)

Letting $\overline{w}=(a^r_{lm},(a^r_{lm})',a^1_{lm},(a^1_{lm})',a^2_{lm},(a^2_{lm})')$, we can write conditions $(iii),(iv),(v)$ in the form;

$\overline{w}'=M\overline{w}$

where M is a 6×6 matrix, with;

$M_{12}=1$, $M_{1j}=0$, $j=1$ or $3\leq j\leq 6$

$M_{34}=1$, $M_{3j}=0$, $1\leq j\leq 3$, $5\leq j\leq 6$

$M_{56}=1$, $M_{5j}=0$, $1\leq j\leq 5$

$M_{21}=-(\frac{\omega^2}{c^2}-\frac{2+l(l+1)}{r^2})$, $M_{22}=-\frac{2}{r}$, $M_{23}=-\frac{2l(l+1)}{r^2}$, $M_{2j}=0$

$4\leq j\leq 6$

$M_{43}=-(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})$, $M_{44}=-\frac{2}{r}$, $M_{41}=-\frac{2}{r^2}$, $M_{4j}=0$, $j=2$

or $5\leq j\leq 6$

$M_{66}=-\frac{2}{r}$, $M_{65}-(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})$, $M_{6j}=0$, $1\leq j\leq 4$

By the vector valued version of Peano's existence and uniqueness theorem, this has a unique solution given the initial values of $\overline{w}$ at w. We have that;

$-\bigtriangledown(v)=-\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{dv_{lm}}{dr}\overline{Y}_{lm}+\frac{v_{lm}}{r}\overline{\Psi}_{lm}$

$i\omega\overline{a}=i\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a^r_{lm}\overline{Y}_{lm}+a^1_{lm}\overline{\Psi}_{lm}+a^2_{lm}\overline{\Phi}_{lm})$

$\bigtriangledown\times\overline{a}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\bigtriangledown\times(a^r_{lm}\overline{Y}_{lm})+\bigtriangledown\times(a^1_{lm}\overline{\Psi}_{lm})+\bigtriangledown\times(a^2_{lm}\overline{\Phi}_{lm}))$

$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\frac{1}{r}a^r_{lm}\overline{\Phi}_{lm}+(\frac{da^1_{lm}}{dr}+\frac{1}{r}a^1_{lm})\overline{\Phi}_{lm}+((-\frac{l(l+1)}{r})a^2_{lm}\overline{Y}_{lm}$

$-(\frac{da^2_{lm}}{dr}+\frac{1}{r}a^2_{lm})\overline{\Psi}_{lm})$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-\frac{l(l+1)}{r}) a_{lm}^2 \overline{Y}_{lm} - (\frac{da_{lm}^2}{dr} + \frac{1}{r} a_{lm}^2) \overline{\Psi}_{lm}$

$+(\frac{da_{lm}^1}{dr} + \frac{1}{r} a_{lm}^1 - \frac{1}{r} a_{lm}^r) \overline{\Phi}_{lm}$

so the boundary conditions become;

$(a).\ -\frac{dv_{lm}}{dr}(w) + i\omega a_{lm}^r(w) = \overline{f}_{lm}^r(w)$

$(b).\ -\frac{v_{lm}(w)}{w} + i\omega a_{lm}^1(w) = \overline{f}_{lm}^1(w)$

$(c).\ i\omega a_{lm}^2(w) = \overline{f}_{lm}^2(w)$

$(d).\ (-\frac{l(l+1)}{w}) a_{lm}^2(w) = \overline{g}_{lm}^r(w)$

$(e).\ -(\frac{da_{lm}^2}{dr}(w) + \frac{1}{w} a_{lm}^2(w)) = \overline{g}_{lm}^1(w)$

$(f).\ (\frac{da_{lm}^1}{dr}(w) + \frac{1}{w} a_{lm}^1(w) - \frac{1}{w} a_{lm}^r(w)) = \overline{g}_{lm}^2(w)$

and using the two relation $(ii), (P2)$ and (iii);

$v_{lm} = -\frac{ic^2}{\omega}((a_{lm}^r)' + \frac{2}{r} a_{lm}^r - \frac{l(l+1)}{r} a_{lm}^1)$

$(a_{lm}^r)'' + \frac{2}{r}(a_{lm}^r)' + (\frac{\omega^2}{c^2} - \frac{2+l(l+1)}{r^2}) a_{lm}^r + \frac{2l(l+1)}{r^2} a_{lm}^1 = 0$

we have that;

$\frac{dv_{lm}^r}{dr} = -\frac{ic^2}{\omega}((a_{lm}^r)'' - \frac{2}{r^2} a_{lm}^r + \frac{2}{r}(a_{lm}^r)' + \frac{l(l+1)}{r^2} a_{lm}^1 - \frac{l(l+1)}{r}(a_{lm}^1)')$

$= -\frac{ic^2}{\omega}(-\frac{2}{r}(a_{lm}^r)' - (\frac{\omega^2}{c^2} - \frac{2+l(l+1)}{r^2}) a_{lm}^r - \frac{2l(l+1)}{r^2} a_{lm}^1 - \frac{2}{r^2} a_{lm}^r + \frac{2}{r}(a_{lm}^r)'$

$+\frac{l(l+1)}{r^2} a_{lm}^1 - \frac{l(l+1)}{r}(a_{lm}^1)')$

$= -\frac{ic^2}{\omega}(-(\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}) a_{lm}^r - \frac{l(l+1)}{r^2} a_{lm}^1 - \frac{l(l+1)}{r}(a_{lm}^1)')$

so we can rewrite $(a), (b)$ as;

$(a)'.\ \frac{ic^2}{\omega}(-(\frac{\omega^2}{c^2} - \frac{l(l+1)}{w^2}) a_{lm}^r(w) - \frac{l(l+1)}{w^2} a_{lm}^1(w) - \frac{l(l+1)}{w}(a_{lm}^1)'(w)) + i\omega a_{lm}^r(w)$

$= \overline{f}_{lm}^r(w)$

$(b)'$. $\frac{ic^2}{w\omega}((a^r_{lm})'(w)+\frac{2}{w}a^r_{lm}(w)-\frac{l(l+1)}{w}a^1_{lm}(w))+i\omega a^1_{lm}(w)=\overline{f}^1_{lm}(w)$

or equivalently;

$(a)'$ $\frac{il(l+1)c^2}{w^2}a^r_{lm}(w)-\frac{ic^2l(l+1)}{w^2\omega}a^1_{lm}(w)-\frac{ic^2l(l+1)}{w\omega}(a^1_{lm})'(w)$

$=\overline{f}^r_{lm}(w)$

$(b)'$ $\frac{2ic^2}{w^2\omega}a^r_{lm}(w)+\frac{ic^2}{w\omega}(a^r_{lm})'(w)+(i\omega-\frac{ic^2l(l+1)}{w^2\omega})a^1_{lm}(w)=\overline{f}^1_{lm}(w)$

We can write the boundary conditions (a'), $(b)'$, (c), (d), (e), (f) in the form;

$N\overline{w}|_w=\overline{s}$

where $\overline{w}$ is as above, and;

$\overline{s}=(\overline{f}^r_{lm}(w),\overline{f}^1_{lm}(w),\overline{f}^2_{lm}(w),\overline{g}^r_{lm}(w),\overline{g}^1_{lm}(w),\overline{g}^2_{lm}(w))$

and N is a matrix, with;

$N_{11}=\frac{il(l+1)c^2}{w^2\omega}$, $N_{13}=-\frac{ic^2l(l+1)}{w^2\omega}$, $N_{14}=-\frac{ic^2l(l+1)}{w\omega}$, $N_{1j}=0$

$j=2$ or $5\leq j\leq 6$

$N_{21}=\frac{2ic^2}{w^2\omega}$, $N_{22}=\frac{ic^2}{w\omega}$, $N_{23}=i\omega-\frac{ic^2l(l+1)}{w^2\omega}$, $N_{2j}=0$, $4\leq j\leq 6$

$N_{35}=i\omega$, $N_{3j}=0$, $1\leq j\leq 4$, $j=6$

$N_{45}=-\frac{l(l+1)}{w}$, $N_{4j}=0$, $1\leq j\leq 4$, $j=6$

$N_{55}=-\frac{1}{w}$, $N_{56}=-1$, $N_{5j}=0$, $1\leq j\leq 4$

$N_{61}=-\frac{1}{w}$, $N_{63}=\frac{1}{w}$, $N_{64}=1$, $N_{6j}=0$, $j=2$, $5\leq j\leq 6$

If $\overline{g}=\overline{0}$ and $\overline{f}^r=\overline{f}^2=0$, then;

$\overline{s}=(0,\overline{f}^1_{lm}(w),0,0,0,0)$

and we obtain a solution by setting;

$a^2_{lm}(w) = 0$

$(a^2_{lm})'(w) = 0$

$(a^r_{lm})' = -\frac{iw\omega}{c^2}(-\frac{2ic^2}{w^2\omega}a^r_{lm}(w) + (\frac{ic^2 l(l+1)}{w^2\omega} - i\omega)a^1_{lm}(w) + \overline{f}^1_{lm}(w))$

$= -\frac{2}{w}a^r_{lm}(w) + (\frac{l(l+1)}{w} - \frac{w\omega^2}{c^2})a^1_{lm}(w) - \frac{iw\omega}{c^2}\overline{f}^1_{lm}(w)$

$\frac{il(l+1)c^2}{w^2\omega}a^r_{lm}(w) - \frac{ic^2 l(l+1)}{w^2\omega}a^1_{lm}(w) - \frac{ic^2 l(l+1)}{w\omega}(a^1_{lm})'(w) = 0$

$-\frac{a^r_{lm}(w)}{w} + \frac{a^1_{lm}(w)}{w} + (a^1_{lm})'(w) = 0$

which is a 2-dimensional family, as we are free to choose $a^1_{lm}(w), (a^1_{lm})'(w)$. Using the fact that, for the configuration $(\rho, \overline{J}, \overline{E}, \overline{B})$ inside the magnetron;

$\bigtriangledown \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = \overline{0}$

we obtain, at the boundary;

$(\bigtriangledown \times \overline{E})^r_{lm} = -\frac{l(l+1)}{w}(\overline{E})^2_{lm} = 0$

so that $(\overline{E})^2_{lm}(w) = 0$

and $\overline{B} = \overline{0}$ by properties of the configuration. By equation (v) of (Q) and the fact that $a^2_{lm}(w) = 0, (a^2_{lm})'(w) = 0$, we obtain that $a^2_{lm}(r) = 0$, for $r \geq w$, so that;

$(\overline{B})^r_{lm} = (\bigtriangledown \times \overline{A})^r_{lm} = -\frac{l(l+1)}{r}(\overline{A})^2_{lm} = 0$

and we obtain solutions in the TM mode, with no surface charge or current. Using the fact that;

$\bigtriangledown \times \overline{B} = \mu_0\overline{J} + \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} = \overline{0}$

we obtain, at the boundary;

$\mu_0(\overline{J})^r_{lm} - \frac{i\omega}{c^2}(\overline{E})^r_{lm} = 0$

so that, with the hypothesis that $\overline{J}^r|_{\delta B(\overline{0},w)} = 0$ or $\overline{J}|_{\delta B(\overline{0},w)} = \overline{0}$, we obtain that $(\overline{E})^r_{lm}(w) = 0$, as required.

$\square$

Lemma 0.6. *If $(\overline{E}, \overline{B})$ are fields of the form $e(x,y,z)e^{-i\omega t}$ and $b(x,y,z)e^{-i\omega t}$ satisfying Maxwell's equations in free space, in the region $B(\overline{0},w)^c$, then there exists potentials V and $\overline{A}$ of the form $v(x,y,z)e^{-i\omega t}$ and $\overline{a}(x,y,z)e^{-i\omega t}$, with the property that $\square^2(V) = 0$, $\square^2(\overline{A}) = \overline{0}$ $\bigtriangledown \centerdot \overline{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$, or equivalently $\bigtriangledown^2(v) = -\frac{\omega^2}{c^2}v$,$\bigtriangledown^2(\overline{a}) = -\frac{\omega^2}{c^2}\overline{a}$, $\bigtriangledown \centerdot \overline{a} = \frac{i\omega}{c^2}v$, such that;*

$\overline{E} = -\bigtriangledown(V) - \frac{\partial \overline{A}}{\partial t} = -\bigtriangledown(V) + i\omega\overline{A}$

$\overline{B} = \bigtriangledown \times \overline{A}$

In particularly, the causal field generated by Jefimenko's equations for the charge and current configuration found in Lemma 0.2 is not in the 2-dimensional family found in Lemma 0.5, unless $\overline{J}^r|_{\delta B(\overline{0},w)} = 0$.

Proof. As $\bigtriangledown \centerdot \overline{B} = 0$, or equivalently $\bigtriangledown \centerdot \overline{b} = \overline{0}$, we can find $\overline{A}'$ of the form $\overline{a}'e^{-i\omega t}$ such that $\bigtriangledown \times \overline{A}' = \overline{B}$ (A), by requiring that $\bigtriangledown \times \overline{a}' = \overline{b}$. Then, as $(\overline{E}, \overline{B})$ satisfy Maxwell's equations, we have that;

$\bigtriangledown \times \overline{E} = (\bigtriangledown \times \overline{e})e^{-i\omega t}$

$= -\frac{\partial \overline{B}}{\partial t}$

$= -\frac{\partial(\bigtriangledown \times \overline{A}')}{\partial t}$

$= i\omega(\bigtriangledown \times \overline{a}')e^{-i\omega t}$

so that;

$\bigtriangledown(\overline{e} - i\omega\overline{a}') = 0$

and we can find a scalar v' such that;

$-\bigtriangledown(v') = \overline{e} - i\omega\overline{a}'$

in which case, setting $V'=v'e^{-i\omega t}$, we have that;

$\overline{E}=-\bigtriangledown(V')-\frac{\partial\overline{A}'}{\partial t}$ (B)

Using the proof in [6], p417, as $(\overline{E},\overline{B})$ satisfy Maxwell's equations in free space, we have that;

$\bigtriangledown^2(V')+\frac{\partial(\bigtriangledown\cdot\overline{A}')}{\partial t}=0$

$(\bigtriangledown^2(\overline{A}')-\frac{1}{c^2}\frac{\partial^2\overline{A}}{\partial t^2})-\bigtriangledown(\bigtriangledown\cdot\overline{A}'+\frac{1}{c^2}\frac{\partial V'}{\partial t})=\overline{0}$ (C)

We claim that we can find potentials $(V,\overline{A})$ satisfying $(A),(B)$, of the form $v(x,y,z)e^{-i\omega t}$ and $\overline{a}(x,y,z)e^{-i\omega t}$ such that the additional Lorentz gauge condition;

$\bigtriangledown\cdot\overline{A}=-\frac{1}{c^2}\frac{\partial V}{\partial t}$ (D)

holds, in which case, substituting into (C), we obtain the relations;

$\Box^2(V)=0$

$\Box^2(\overline{A})=\overline{0}$

as required. As in the proof of [6], for a scalar Λ, if $\overline{A}=\overline{A}'+\bigtriangledown(\Lambda)$ and $V=V'-\frac{\partial\Lambda}{\partial t}$, then $(V,\overline{A})$ satisfy $(A),(B)$, so to obtain (D), we require that;

$\bigtriangledown\cdot\overline{A}=\bigtriangledown\cdot(\overline{A}'+\bigtriangledown(\Lambda))$

$=-\frac{1}{c^2}\frac{\partial V}{\partial t}$

$=-\frac{1}{c^2}\frac{\partial(V'-\frac{\partial\Lambda}{\partial t})}{\partial t}$

$=-\frac{1}{c^2}\frac{\partial V'}{\partial t}+\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}$

so that;

$\bigtriangledown^2(\Lambda)-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=-\bigtriangledown\cdot(\overline{A}')-\frac{1}{c^2}\frac{\partial V'}{\partial t}$

Writing Λ in the form $\lambda e^{-i\omega t}$, we require a solution to;

$\bigtriangledown^2(\lambda)+\frac{\omega^2}{c^2}\lambda=-\bigtriangledown\centerdot(\overline{a}')+\frac{i\omega}{c^2}v'$

on $B(\overline{0},w)^c$. Denoting the forcing term $-\bigtriangledown\centerdot(\overline{a}')+\frac{i\omega}{c^2}v'$ by τ, and letting;

$\tau=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tau_{lm}(r)Y_{lm}(\theta,\phi)$

be its expansion in spherical harmonics, expanding;

$\lambda=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\lambda_{lm}(r)Y_{lm}(\theta,\phi)$

in spherical harmonics and equating coefficients, we require that, see (P) in the proof of Lemma 0.5, that;

$(\lambda_{lm})''+\frac{2}{r}(\lambda_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})\lambda_{lm}=\tau_{lm}\ (E)$

in the region $r>w$. This is a second order differential equation, the homogenous version;

$(\lambda_{lm})''+\frac{2}{r}(\lambda_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})\lambda_{lm}=0$

having two independendent solutions $j_l(\frac{\omega r}{c})$ and $n_l(\frac{\omega r}{c})$, where j_l and n_l are the spherical Bessel and Neumann functions of order l. By Abel's theorem, the Wronskian $W(j_l(\frac{\omega r}{c}),n_l(\frac{\omega r}{c}))$ is given by;

$c_0exp(-\int\frac{2}{r}dr)=\frac{c_0}{r^2}$

where c_0 is a constant, and the general solution of (E), given by variation of parameters, see [3], is;

$\lambda_{lm}(r)=c_1j_l(\frac{\omega r}{c})+c_2n_l(\frac{\omega r}{c})+Z_{lm}(r)$

where c_1 and c_2 are constants and;

$Z_{lm}(r)=-j_l(\frac{\omega r}{c})\int\frac{n_l(\frac{\omega r}{c})\tau_{lm}(r)}{W(j_l(\frac{\omega r}{c}),n_l(\frac{\omega r}{c}))}dr+n_l(\frac{\omega r}{c})\int\frac{j_l(\frac{\omega r}{c})\tau_{lm}(r)}{W(j_l(\frac{\omega r}{c}),n_l(\frac{\omega r}{c}))}dr$

$=-\frac{j_l(\frac{\omega r}{c})}{c_0}\int r^2n_l(\frac{\omega r}{c})\tau_{lm}(r)dr+\frac{n_l(\frac{\omega r}{c})}{c_0}\int r^2j_l(\frac{\omega r}{c})\tau_{lm}(r)dr$

The last claim is clear by Lemmas 0.5, 0.2 and 0.1.

$\square$

Lemma 0.7. *When $l = 0$ or $l = 1$, the equations from Lemma 0.5;*

(*i*). $(v_{lm})'' + \frac{2}{r}(v_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})v_{lm} = 0 \ (P)$

(*ii*). $(a^r_{lm})'' + \frac{2}{r}(a^r_{lm})' + (\frac{\omega^2}{c^2} - \frac{2+l(l+1)}{r^2})a^r_{lm} + \frac{2l(l+1)}{r^2}a^1_{lm} = 0$

(*iii*). $(a^1_{lm})'' + \frac{2}{r}(a^1_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})a^1_{lm} + \frac{2}{r^2}a^r_{lm} = 0$

(*iv*). $(a^2_{lm})'' + \frac{2}{r}(a^2_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})a^2_{lm} = 0 \ (Q)$

have an explicit general solution in terms of Bessel and Neumann functions.

Proof. When $l = 0$, the equations;

(*i*). $(v_{lm})'' + \frac{2}{r}(v_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})v_{lm} = 0$

(*ii*). $(a^r_{lm})'' + \frac{2}{r}(a^r_{lm})' + (\frac{\omega^2}{c^2} - \frac{2+l(l+1)}{r^2})a^r_{lm} + \frac{2l(l+1)}{r^2}a^1_{lm} = 0$

(*iii*). $(a^1_{lm})'' + \frac{2}{r}(a^1_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})a^1_{lm} + \frac{2}{r^2}a^r_{lm} = 0$

(*iv*). $(a^2_{lm})'' + \frac{2}{r}(a^2_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})a^2_{lm} = 0$

simplify to;

(*i*). $(v_{1m})'' + \frac{2}{r}(v_{1m})' + \frac{\omega^2}{c^2}v_{1m} = 0$

(*ii*). $(a^r_{1m})'' + \frac{2}{r}(a^r_{1m})' + (\frac{\omega^2}{c^2} - \frac{2}{r^2})a^r_{1m} = 0$

(*iii*). $(a^1_{1m})'' + \frac{2}{r}(a^1_{1m})' + \frac{\omega^2}{c^2}a^1_{1m} + \frac{2}{r^2}a^r_{1m} = 0$

(*iv*). $(a^2_{1m})'' + \frac{2}{r}(a^2_{1m})' + \frac{\omega^2}{c^2}a^2_{1m} = 0$

By calculating $(ii) + (iii)$, we obtain that;

$(a^r_{1m} + 2a^1_{1m})'' + \frac{2}{r}(a^r_{1m} + 2a^1_{1m})' + \frac{\omega^2}{c^2}(a^r_{1m} + a^1_{1m}) = 0$

which has the general solution;

$(a^r_{1m} + a^1_{1m})(r) = c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c})$

where j_0 and n_0 are the spherical Bessel and Neumann functions of order 0. It follows that;

$a^r_{lm} = c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}) - a^1_{1m}(r)$ (H)

and substituting into (iii), we obtain that;

$(a^1_{1m})'' + \frac{2}{r}(a^1_{1m})' + \frac{\omega^2}{c^2} a^1_{1m} + \frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}) - a^1_{1m}) = 0$

$(a^1_{1m})'' + \frac{2}{r}(a^1_{1m})' + (\frac{\omega^2}{c^2} - \frac{2}{r^2}) a^1_{1m} = -\frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}))$ (K)

The homogenous version;

$(a^1_{1m})'' + \frac{2}{r}(a^1_{1m})' + (\frac{\omega^2}{c^2} - \frac{2}{r^2}) a^1_{1m} = 0$ (I)

has a general solution;

$a^1_{lm} = c_3 j_1(\frac{\omega r}{c}) + c_4 n_1(\frac{\omega r}{c})$

where j_1 and n_1 are the spherical Bessel and Neumann functions of order 1. By Abel's theorem, the Wronskian $W(j_1(\frac{\omega r}{c}), n_1(\frac{\omega r}{c}))$ is given by;

$c_5 exp(-\int \frac{2}{r} dr) = \frac{c_5}{r^2}$

where c_5 is a constant, and the general solution of (K), given by variation of parameters again, is;

$a^1_{lm}(r) = c_3 j_1(\frac{\omega r}{c}) + c_4 n_1(\frac{\omega r}{c}) + V_{lm}(r)$

where;

$V_{lm}(r) = -j_1(\frac{\omega r}{c}) \int \frac{n_1(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}))]}{W(j_l(\frac{\omega r}{c}), n_l(\frac{\omega r}{c}))} dr + n_1(\frac{\omega r}{c}) \int \frac{j_1(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}))]}{W(j_1(\frac{\omega r}{c}), n_1(\frac{\omega r}{c}))} dr$

$= -\frac{j_1(\frac{\omega r}{c})}{c_5} \int r^2 n_1(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}))] dr$

$+ \frac{n_1(\frac{\omega r}{c})}{c_5} \int r^2 j_1(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c}) + c_2 n_0(\frac{\omega r}{c}))] dr$

$= \frac{2c_1 j_1(\frac{\omega r}{c})}{c_5} \int n_1 j_0(\frac{\omega r}{c}) dr + \frac{2c_2 j_1(\frac{\omega r}{c})}{c_5} \int n_1 n_0(\frac{\omega r}{c}) dr$

$-\frac{2c_1n_1(\frac{\omega r}{c})}{c_5}\int j_1j_0(\frac{\omega r}{c})dr-\frac{2c_2n_1(\frac{\omega r}{c})}{c_5}\int j_1n_0(\frac{\omega r}{c})dr$

so that, substituting into (H), we obtain;

$a^r_{lm}(r)=c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c})-2(c_3j_1(\frac{\omega r}{c})+c_4n_1(\frac{\omega r}{c})+V_{lm}(r))$

as a general solution. The general solutions of (i) and (iv) are given by;

$v_{1m}(r)=c_6j_0(\frac{\omega r}{c})+c_7n_0(\frac{\omega r}{c})$

$a^2_{lm}(r)=c_8j_0(\frac{\omega r}{c})+c_9n_0(\frac{\omega r}{c})$

where c_6, c_7, c_8, c_9 are constants and j_0, n_0 are Bessel and Neumann functions of order 0.

When $l=1$, the equations;

$(i).\ (v_{lm})''+\frac{2}{r}(v_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})v_{lm}=0$

$(ii).\ (a^r_{lm})''+\frac{2}{r}(a^r_{lm})'+(\frac{\omega^2}{c^2}-\frac{2+l(l+1)}{r^2})a^r_{lm}+\frac{2l(l+1)}{r^2}a^1_{lm}=0$

$(iii).\ (a^1_{lm})''+\frac{2}{r}(a^1_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})a^1_{lm}+\frac{2}{r^2}a^r_{lm}=0$

$(iv).\ (a^2_{lm})''+\frac{2}{r}(a^2_{lm})'+(\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2})a^2_{lm}=0$

simplify to;

$(i).\ (v_{1m})''+\frac{2}{r}(v_{1m})'+(\frac{\omega^2}{c^2}-\frac{2}{r^2})v_{1m}=0$

$(ii).\ (a^r_{1m})''+\frac{2}{r}(a^r_{1m})'+(\frac{\omega^2}{c^2}-\frac{4}{r^2})a^r_{1m}+\frac{4}{r^2}a^1_{1m}=0$

$(iii).\ (a^1_{1m})''+\frac{2}{r}(a^1_{1m})'+(\frac{\omega^2}{c^2}-\frac{2}{r^2})a^1_{1m}+\frac{2}{r^2}a^r_{1m}=0$

$(iv).\ (a^2_{1m})''+\frac{2}{r}(a^2_{1m})'+(\frac{\omega^2}{c^2}-\frac{2}{r^2})a^2_{1m}=0$

By calculating $(ii)+2(iii)$, we obtain that;

$(a^r_{1m}+2a^1_{1m})''+\frac{2}{r}(a^r_{1m}+2a^1_{1m})'+\frac{\omega^2}{c^2}(a^r_{1m}+2a^1_{1m})=0$

which has the general solution;

$(a^r_{1m}+2a^1_{1m})(r)=c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c})$

where j_0 and n_0 are the spherical Bessel and Neumann functions of order 0. It follows that;

$a^r_{lm}=c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c})-2a^1_{1m}(r)$ (G)

and substituting into (iii), we obtain that;

$(a^1_{1m})''+\frac{2}{r}(a^1_{1m})'+(\frac{\omega^2}{c^2}-\frac{2}{r^2})a^1_{1m}+\frac{2}{r^2}(c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c})-2a^1_{1m})=0$

$(a^1_{1m})''+\frac{2}{r}(a^1_{1m})'+(\frac{\omega^2}{c^2}-\frac{6}{r^2})a^1_{1m}=-\frac{2}{r^2}(c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c}))$

The homogenous version;

$(a^1_{1m})''+\frac{2}{r}(a^1_{1m})'+(\frac{\omega^2}{c^2}-\frac{6}{r^2})a^1_{1m}=0$ (F)

has a general solution;

$a^1_{lm}=c_3j_2(\frac{\omega r}{c})+c_4n_2(\frac{\omega r}{c})$

where j_2 and n_2 are the spherical Bessel and Neumann functions of order 2. By Abel's theorem, the Wronskian $W(j_2(\frac{\omega r}{c}),n_2(\frac{\omega r}{c}))$ is given by;

$c_5exp(-\int\frac{2}{r}dr)=\frac{c_5}{r^2}$

where c_5 is a constant, and the general solution of (F), given by variation of parameters again, is;

$a^1_{lm}(r)=c_3j_2(\frac{\omega r}{c})+c_4n_2(\frac{\omega r}{c})+T_{lm}(r)$

where;

$T_{lm}(r)=-j_2(\frac{\omega r}{c})\int\frac{n_2(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c}))]}{W(j_l(\frac{\omega r}{c}),n_l(\frac{\omega r}{c}))}dr+n_2(\frac{\omega r}{c})\int\frac{j_2(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c}))]}{W(j_2(\frac{\omega r}{c}),n_2(\frac{\omega r}{c}))}dr$

$=-\frac{j_2(\frac{\omega r}{c})}{c_5}\int r^2n_2(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1j_0(\frac{\omega r}{c})+c_2n_0(\frac{\omega r}{c}))]dr$

$+\frac{n_2(\frac{\omega r}{c})}{c_5}\int r^2 j_2(\frac{\omega r}{c})[-\frac{2}{r^2}(c_1 j_0(\frac{\omega r}{c})+c_2 n_0(\frac{\omega r}{c}))]dr$

$=\frac{2c_1 j_2(\frac{\omega r}{c})}{c_5}\int n_2 j_0(\frac{\omega r}{c})dr+\frac{2c_2 j_2(\frac{\omega r}{c})}{c_5}\int n_2 n_0(\frac{\omega r}{c})dr$

$-\frac{2c_1 n_2(\frac{\omega r}{c})}{c_5}\int j_2 j_0(\frac{\omega r}{c})dr-\frac{2c_2 n_2(\frac{\omega r}{c})}{c_5}\int j_2 n_0(\frac{\omega r}{c})dr$

so that, substituting into (G), we obtain;

$a^r_{lm}(r)=c_1 j_0(\frac{\omega r}{c})+c_2 n_0(\frac{\omega r}{c})-2(c_3 j_2(\frac{\omega r}{c})+c_4 n_2(\frac{\omega r}{c})+T_{lm}(r))$

as a general solution. The general solutions of (i) and (iv) are given by;

$v_{1m}(r)=c_6 j_1(\frac{\omega r}{c})+c_7 n_1(\frac{\omega r}{c})$

$a^2_{lm}(r)=c_8 j_1(\frac{\omega r}{c})+c_9 n_1(\frac{\omega r}{c})$

where c_6, c_7, c_8, c_9 are constants and j_1, n_1 are Bessel and Neumann functions of order 1.

□

Lemma 0.8. *If $(\rho, \overline{J}, \overline{E}, \overline{B})$ is the configuration from Lemma 0.1, obtained as a limit of $(\rho_\delta, \overline{J}_\delta, \overline{E}_\delta, \overline{B}_\delta)$, where $(\rho_\delta, \overline{J}_\delta)$ admit the standard wave equation representation in terms of Fourier transforms, then $\overline{E}$ and $\overline{J}$ are radial. Moreover, $\overline{E}$ and $\overline{J}$ can be expanded in terms of Bessel functions and spherical harmonica of order* 1.

Proof. By (PP) in the proof of Lemma 0.1, we have that;

$\rho(\overline{x},t)=\alpha\frac{4\pi k^3}{c}e^{-i\omega t}\frac{sin(|k\overline{x}|)}{|k\overline{x}|}$

where α is a complex constant and $\omega=kc$. Taking the gradient, and using the fact that;

$\frac{\partial\overline{J}}{\partial t}=-i\omega\overline{J}$

$=-c^2\bigtriangledown(\rho)$

it is clear as ρ is constant on spheres $S(\overline{0},r)$, for $r>0$, that $\overline{J}$ is radial. As $\overline{E}=\frac{1}{i\omega\epsilon_0}\overline{J}$, by Maxwell's fourth equation and $\overline{B}=\overline{0}$, $\overline{E}$ is radial. We have that, by the proof of (PP), that;

$\overline{J}=\alpha\sum_{-1\leq m\leq 1}\overline{U}(1,m,k)\gamma_{1,m,k}e^{-ikct}$

where;

$\overline{U}(1,m,k)=i(\frac{2}{\pi})^{\frac{1}{2}}\frac{k^2}{4\pi}\overline{W}(1,m)^*$

so that, by the calculations in [11], in particularly the spherical expansion of $\hat{\overline{r}}$ and using the fact that the coefficient vectors $\overline{W}(1,m)$, $-1\leq m\leq 1$, are real;

$\overline{J}=\alpha\sum_{-1\leq m\leq 1}i(\frac{2}{\pi})^{\frac{1}{2}}\frac{k^2}{4\pi}\overline{W}(1,m)^*k(\frac{2}{\pi})^{\frac{1}{2}}j_1(kr)Y_{1,m}(\theta,\phi)e^{-ikct}$

$=\alpha dj_1(kr)e^{-ikct}\sum_{-1\leq m\leq 1}\overline{W}(1,m)^*Y_{1,m}(\theta,\phi)$

$=\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\sum_{-1\leq m\leq 1}\overline{W}(1,m)^*Y_{1,m}(\theta,\phi)$

$=\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\sum_{-1\leq m\leq 1}\overline{W}(1,m)Y_{1,m}(\theta,\phi)$

$=\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\hat{\overline{r}}$

where $d=\frac{ik^3}{2\pi^2}=\frac{i\omega^3}{2c^3\pi^2}$ and $\omega=kc$.

It follows that;

$\overline{E}=\frac{1}{i\omega\epsilon_0}\overline{J}=\frac{1}{i\omega\epsilon_0}\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\hat{\overline{r}}$

We have that;

$\overline{E}^r_{lm}(r)=\int_{S(\overline{0},1)}\overline{E}_{lm}\centerdot\overline{Y}_{lm}dS(\overline{0},1)$

$=\frac{1}{i\omega\epsilon_0}\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\int_{S(\overline{0},1)}\hat{\overline{r}}\centerdot\hat{\overline{r}}Y_{lm}dS(\overline{0},1)$

$=\frac{1}{i\omega\epsilon_0}\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\int_{S(\overline{0},1)}Y_{lm}dS(\overline{0},1)$

$=\frac{2\sqrt{\pi}}{i\omega\epsilon_0}\alpha dj_1(\frac{\omega r}{c})e^{-i\omega t}\delta_{0,l}\delta_{0,m}$

and, using the divergence theorem;

$\overline{E}^1_{lm}(r) = \int_{S(\overline{0},1)} \overline{E}_{lm} \cdot \overline{\Psi}_{lm} dS(\overline{0},1)$

$\int_{S(\overline{0},1)} \overline{E}_{lm} \cdot r \bigtriangledown (Y_{lm}) dS(\overline{0},1)$

$= \frac{r}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \int_{S(\overline{0},1)} \hat{\overline{r}} \cdot \bigtriangledown(Y_{lm}) dS(\overline{0},1)$

$= \frac{r}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \int_{S(\overline{0},1)} \bigtriangledown(Y_{lm}) d\overline{S}(\overline{0},1)$

$= \frac{r}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \int_{B(\overline{0},1)} \bigtriangledown^2(Y_{lm}) dB(\overline{0},1)$

$= \frac{r}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \int_{B(\overline{0},1)} -\frac{l(l+1)}{r^2} Y_{lm} dB(\overline{0},1)$

$= -\frac{l(l+1)r}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \int_{S(\overline{0},1)} Y_{lm} dS(\overline{0},1)$

$= -\frac{l(l+1)r}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \delta_{0,l}\delta_{0,m}$

$= 0$

$\overline{E}^2_{lm}(r) = \int_{S(\overline{0},1)} \overline{E}_{lm} \cdot \overline{\Psi}_{lm} dS(\overline{0},1)$

$= \int_{S(\overline{0},1)} \overline{E}_{lm} \cdot (\overline{r} \times \bigtriangledown(Y_{lm})) dS(\overline{0},1)$

$= \frac{1}{i\omega\epsilon_0} \alpha d j_1(\frac{\omega r}{c}) e^{-i\omega t} \int_{S(\overline{0},1)} \hat{\overline{r}} \cdot (\overline{r} \times \bigtriangledown(Y_{lm})) dS(\overline{0},1)$

$= 0$

Using the boundary conditions from Lemma 0.5, if ω is chosen so that $j_1(\frac{\omega w}{c}) = 0$, we obtain a solution by setting;

$a^2_{lm}(w) = 0$

$(a^2_{lm})'(w) = 0$

$(a^r_{lm})' = -\frac{2}{w} a^r_{lm}(w) + (\frac{l(l+1)}{w} - \frac{w\omega^2}{c^2}) a^1_{lm}(w)$

$-\frac{a^r_{lm}(w)}{w} + \frac{a^1_{lm}(w)}{w} + (a^1_{lm})'(w) = 0 \ (X)$

for $(l,m) \neq (0,0)$, and;

$a^2_{00}(w) = 0$

$(a_{00}^2)'(w) = 0$

$(a_{00}^r)' = -\frac{2}{w}a_{00}^r(w) + (\frac{l(l+1)}{w} - \frac{w\omega}{c^2})a_{00}^1(w)$

$-\frac{a_{00}^r(w)}{w} + \frac{a_{00}^1(w)}{w} + (a_{00}^1)'(w) = 0$ (Y)

In the 2-dimensional family of solutions, we can set;

$a_{lm}^1(w) = (a_{lm}^1)'(w) = 0$

for all (l,m). Then, for (l,m), by $(X),(Y)$;

$a_{lm}^r(w) = (a_{lm}^r)'(w) = a_{lm}^1(w) = (a_{lm}^1)'(w)$

$= a_{lm}^2(w) = (a_{lm}^2)'(w) = 0$

and, by Peano's existence and uniqueness theorem, using the conditions $(iii),(iv),(v)$ in Lemma 0.5;

$a_{lm}^r(r) = (a_{lm}^r)'(r) = a_{lm}^1(r) = (a_{lm}^1)'(r)$

$= a_{lm}^2(r) = (a_{lm}^2)'(r) = 0$

for $r \geq w$. By the relation (ii), $(P2)$ in Lemma 0.5, we obtain that $v_{lm}(r) = 0$, for $r \geq w$ as well, so that we obtain the trivial solution.

□

Lemma 0.9. *If* $(\overline{E},\overline{B})$ *are fields of the form* $e(x,y,z)e^{-i\omega t}$ *and* $b(x,y,z)e^{-i\omega t}$ *satisfying Maxwell's equations in free space, in the region* $B(\overline{0},w)^c$*, then there exists potentials* V *and* $\overline{A}$ *of the form* $v(x,y,z)e^{-i\omega t}$ *and* $\overline{a}(x,y,z)e^{-i\omega t}$*, with the properties that;*

$\bigtriangledown^2(V) + \frac{\partial(\bigtriangledown\centerdot\overline{A})}{\partial t} = 0$

$(\bigtriangledown^2(\overline{A}) - \frac{1}{c^2}\frac{\partial^2\overline{A}}{\partial t^2}) - \bigtriangledown(\bigtriangledown\centerdot\overline{A} + \frac{1}{c^2}\frac{\partial V'}{\partial t}) = \overline{0}$ (C)

or equivalently;

$\bigtriangledown^2(v) - i\omega\bigtriangledown\centerdot\overline{a} = 0$

$\bigtriangledown^2(\overline{a})+\frac{\omega^2}{c^2}\overline{a}-\bigtriangledown(\bigtriangledown\centerdot\overline{a}-\frac{i\omega}{c^2}v)=\overline{0}$

such that;

$\overline{E}=-\bigtriangledown(V)-\frac{\partial\overline{A}}{\partial t}=-\bigtriangledown(V)+i\omega\overline{A}$

$\overline{B}=\bigtriangledown\times\overline{A}\ (D)$

Conversely, if we have potentials $(V,\overline{A})$ *satisfying* (C) *and we define the fields* $(\overline{E},\overline{B})$ *by* (D)*, then* $(\overline{E},\overline{B})$ *satisfy Maxwell's equations in free space on* $B(\overline{0},w)^c$*.*

Given boundary conditions $\{\overline{f},\overline{g}\}$ *on* $\delta S(\overline{0},w)$*, if;*

$-\bigtriangledown(v)+i\omega\overline{a}|_{\delta S(\overline{0},w)}=\overline{f}$

$\bigtriangledown\times\overline{a}|_{\delta S(\overline{0},w)}=\overline{g}$

then the corresponding fields $\{\overline{E},\overline{B}\}$ *are continuous with fields* $\{\overline{f}e^{-i\omega t},\overline{g}e^{-i\omega t}\}$ *on* $B(\overline{0},w)$*. These boundary conditions cannot be satisfied for* $\{v,\overline{a}\}$ *with the above property, for the configuration from Lemma 0.8, unless* $\overline{J}|_{\delta S(\overline{0},w)}=\overline{0}$*.*

Proof. The first claim is just the first part of Lemma 0.6, the converse claim just amounts to checking the steps are reversible in the proof of [6].

Again, we can write v in the form;

$v(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(v_{lm}(r)Y_{lm}(r,\theta,\phi)$

where the $\{Y_{lm}:l\geq 0,-l\leq m\leq l\}$ are the spherical harmonics.
Then;

$\bigtriangledown^2(v)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dv_{lm}}{dr})-\frac{l(l+1)}{r^2}v_{lm})Y_{lm}$

Similarly, we write $\overline{a}$ again in the form;

$\overline{a}(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{lm}^r(r)\overline{Y}_{lm}(r,\theta,\phi)+a_{lm}^1(r)\overline{\Psi}_{lm}(r,\theta,\phi)+a_{lm}^2(r)\overline{\Phi}_{lm}(r,\theta,\phi))$

where $\{\overline{Y}_{lm},\overline{\Psi}_{lm},\overline{\Phi}_{lm}\}$ are vector spherical harmonics, see [2].

Then;

$\bigtriangledown \centerdot \overline{a} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (\frac{da^r_{lm}}{dr} + \frac{2}{r} a^r_{lm} - \frac{l(l+1)}{r} a^1_{lm}) Y_{lm}$

so that equating coefficients, the condition;

$\bigtriangledown^2(v) - i\omega \bigtriangledown \centerdot \overline{a} = 0$

becomes;

$(i).\ \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{dv_{lm}}{dr}) - \frac{l(l+1)}{r^2} v_{lm} - i\omega(\frac{da^r_{lm}}{dr} + \frac{2}{r} a^r_{lm} - \frac{l(l+1)}{r} a^1_{lm}) = 0$

or equivalently;

$(i).\ (v_{lm})'' + \frac{2}{r}(v_{lm})' - \frac{l(l+1)}{r^2} v_{lm} - i\omega (a^r_{lm})' - \frac{2i\omega}{r} a^r_{lm} + \frac{i\omega l(l+1)}{r} a^1_{lm} = 0$

We have that;

$\bigtriangledown(v) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{dv_{lm}}{dr} \overline{Y}_{lm} + \frac{v_{lm}}{r} \overline{\Psi}_{lm}$

and by the proof of Lemma 0.4;

$\bigtriangledown \times \bigtriangledown \times \overline{a} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} ([-l(l+1)(\frac{1}{r}(a^1_{lm})' + \frac{1}{r^2} a^1_{lm} - \frac{1}{r^2} a^r_{lm})] \overline{Y}_{lm}$

$+[-(a^1_{lm})'' + \frac{1}{r}(a^r_{lm})' + \frac{2}{r^2} a^1_{lm} - \frac{2}{r^2} a^r_{lm}] \overline{\Psi}_{lm}$

$+[-(a^2_{lm})'' - \frac{2}{r}(a^2_{lm})' + \frac{l(l+1)}{r^2} a^2_{lm}] \overline{\Phi}_{lm})$

so that, equating coefficients again, the condition;

$\bigtriangledown^2(\overline{a}) + \frac{\omega^2}{c^2} \overline{a} - \bigtriangledown(\bigtriangledown \centerdot \overline{a} - \frac{i\omega}{c^2} v) = \overline{0}$

or equivalently;

$-\bigtriangledown \times \bigtriangledown \times \overline{a} + \frac{\omega^2}{c^2} \overline{a} + \frac{i\omega}{c^2} \bigtriangledown (v) = \overline{0}$

becomes;

$(ii).\ -[-l(l+1)(\frac{1}{r}(a^1_{lm})' + \frac{1}{r^2} a^1_{lm} - \frac{1}{r^2} a^r_{lm})] + \frac{\omega^2}{c^2} a^r_{lm} + \frac{i\omega}{c^2}(v_{lm})' = 0$

$(iii).\ -[-(a^1_{lm})'' + \frac{1}{r}(a^r_{lm})' + \frac{2}{r^2} a^1_{lm} - \frac{2}{r^2} a^r_{lm}] + \frac{\omega^2}{c^2} a^1_{lm} + \frac{i\omega}{c^2} \frac{v_{lm}}{r} = 0$

(iv). $-[-(a^2_{lm})'' - \frac{2}{r}(a^2_{lm})' + \frac{l(l+1)}{r^2}a^2_{lm}] + \frac{\omega^2}{c^2}a^2_{lm} = 0$

or equivalently;

(ii). $\frac{l(l+1)}{r}(a^1_{lm})' + \frac{l(l+1)}{r^2}a^1_{lm} + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})a^r_{lm} + \frac{i\omega}{c^2}(v_{lm})' = 0$

(iii). $(a^1_{lm})'' - \frac{1}{r}(a^r_{lm})' + (\frac{\omega^2}{c^2} - \frac{2}{r^2})a^1_{lm} + \frac{2}{r^2}a^r_{lm} + \frac{i\omega}{c^2}\frac{v_{lm}}{r} = 0$

(iv). $(a^2_{lm})'' + \frac{2}{r}(a^2_{lm})' + (\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2})a^2_{lm} = 0$

For $l = 0$, we obtain that;

$(i)(0)$ $(v_{00})'' + \frac{2}{r}(v_{00})' - i\omega(a^r_{00})' - \frac{2i\omega}{r}a^r_{00} = 0$

$(ii)(0)$ $\frac{\omega^2}{c^2}a^r_{00} + \frac{i\omega}{c^2}(v_{00})' = 0$

$(iii)(0)$ $(a^1_{00})'' - \frac{1}{r}(a^r_{00})' + (\frac{\omega^2}{c^2} - \frac{2}{r^2})a^1_{00} + \frac{2}{r^2}a^r_{00} + \frac{i\omega}{c^2}\frac{v_{00}}{r} = 0$

$(iv)(0)$ $(a^2_{00})'' + \frac{2}{r}(a^2_{00})' + \frac{\omega^2}{c^2}a^2_{00} = 0$

and from $(ii)(0)$, we obtain that;

$a^r_{00} = -\frac{i}{\omega}(v_{00})'$

and, differentiating;

$(a^r_{00})' = -\frac{i}{\omega}(v_{00})''$ (A)

Substituting (A) into $(i)(0)$, we see this equation is automatically satisfied, and substituting (A) into (iii), we obtain;

$(a^1_{00})'' + \frac{i}{r\omega}(v_{00})'' + (\frac{\omega^2}{c^2} - \frac{2}{r^2})a^1_{00} - \frac{2i}{r^2\omega}(v_{00})' + \frac{i\omega}{c^2 r}v_{00} = 0$

which rearranging, gives;

$(a^1_{00})'' + (\frac{\omega^2}{c^2} - \frac{2}{r^2})a^1_{00} = -\frac{i}{r\omega}(v_{00})'' + \frac{2i}{r^2\omega}(v_{00})' - \frac{i\omega}{c^2 r}v_{00}$ (B)

Given a smooth choice of v_{00}, (A) has a unique solution for a^r_{00}, and, by Peano's theorem, (B) has a unique solution for a^1_{00}, given a choice of $a^1_{00}(w), (a^1_{00})'(w)$. Similarly, (iv) has a unique solution for a^2_{00}, given a choice of $a_{00}^2(w),(a_{00}^2)'(w)$.

We have that;

$-\bigtriangledown(v)=-\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{dv_{lm}}{dr}\overline{Y}_{lm}+\frac{v_{lm}}{r}\overline{\Psi}_{lm}$

$i\omega\overline{a}=i\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{lm}^r\overline{Y}_{lm}+a_{lm}^1\overline{\Psi}_{lm}+a_{lm}^2\overline{\Phi}_{lm})$

$\bigtriangledown\times\overline{a}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\bigtriangledown\times(a_{lm}^r\overline{Y}_{lm})+\bigtriangledown\times(a_{lm}^1\overline{\Psi}_{lm})+\bigtriangledown\times(a_{lm}^2\overline{\Phi}_{lm}))$

$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\frac{1}{r}a_{lm}^r\overline{\Phi}_{lm}+(\frac{da_{lm}^1}{dr}+\frac{1}{r}a_{lm}^1)\overline{\Phi}_{lm}+((-\frac{l(l+1)}{r})a_{lm}^2\overline{Y}_{lm}$

$-(\frac{da_{lm}^2}{dr}+\frac{1}{r}a_{lm}^2)\overline{\Psi}_{lm})$

$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-\frac{l(l+1)}{r})a_{lm}^2\overline{Y}_{lm}-(\frac{da_{lm}^2}{dr}+\frac{1}{r}a_{lm}^2)\overline{\Psi}_{lm}$

$+(\frac{da_{lm}^1}{dr}+\frac{1}{r}a_{lm}^1-\frac{1}{r}a_{lm}^r)\overline{\Phi}_{lm}$

so the boundary conditions become;

$(a).\ -\frac{dv_{lm}}{dr}(w)+i\omega a_{lm}^r(w)=\overline{f}_{lm}^r(w)$

$(b).\ -\frac{v_{lm}(w)}{w}+i\omega a_{lm}^1(w)=\overline{f}_{lm}^1(w)$

$(c).\ i\omega a_{lm}^2(w)=\overline{f}_{lm}^2(w)$

$(d).\ (-\frac{l(l+1)}{w})a_{lm}^2(w)=\overline{g}_{lm}^r(w)$

$(e).\ -(\frac{da_{lm}^2}{dr}(w)+\frac{1}{w}a_{lm}^2(w))=\overline{g}_{lm}^1(w)$

$(f).\ (\frac{da_{lm}^1}{dr}(w)+\frac{1}{w}a_{lm}^1(w)-\frac{1}{w}a_{lm}^r(w))=\overline{g}_{lm}^2(w)$

and for $l=0,m=0$, using the result of Lemma 0.8, we obtain;

$(a).\ -\frac{dv_{00}}{dr}(w)+i\omega a_{00}^r(w)=\overline{f}_{lm}^r(w)$

$(b).\ -\frac{v_{00}(w)}{w}+i\omega a_{00}^1(w)=0$

$(c).\ i\omega a_{00}^2(w)=0$

(d). $0 = 0$

(e). $-(\frac{da^2_{00}}{dr}(w) + \frac{1}{w}a^2_{00}(w)) = 0$

(f). $(\frac{da^1_{00}}{dr}(w) + \frac{1}{w}a^1_{00}(w) - \frac{1}{w}a^r_{00}(w)) = 0$

where $\overline{f}^r_{lm}(w) = \frac{2\sqrt{\pi}}{i\omega\epsilon_0}\alpha d j_1(\frac{\omega w}{c})e^{-i\omega t}$

From (A), we see that the boundary condition (a) cannot be satisfied unless $j_1(\frac{\omega w}{c}) = 0$, in which case $\overline{J}|_{\delta S(\overline{0},w)} = \overline{0}$.

□

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