FUNCTIONS ANALYTIC AT INFINITY AND NORMALITY

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Abstract. This paper explores the notion of analytic at infinity and normality. We prove that we can define a Fourier transform for normal functions.

Definition 0.1. *A smooth real function $f:\mathcal{R}\rightarrow\mathcal{R}$ is non oscillatory if it is eventually monotone, that is there exist $r\in\mathcal{R}_{>0}$ for which $f|_{(r,\infty)}$ is increasing or decreasing and similarly for $f|_{(-\infty,r)}$. We say that f is of very moderate decrease if there exists a constant $C\in\mathcal{R}_{>0}$ for which $|f(x)|\leq\frac{C}{|x|}$ for $|x|>1$. We say that f is of moderate decrease if there exists a constant $C\in\mathcal{R}_{>0}$ for which $|f(x)|\leq\frac{C}{|x|^2}$ for $|x|>1$.*

Definition 0.2. *Given a smooth real function $f:\mathcal{R}\setminus W\rightarrow\mathcal{R}$, where W is a bounded closed set of $\mathcal{R}$, we say that f is analytic at infinity, if there exist $\{\epsilon_1,\epsilon_2\}\subset\mathcal{R}_{>0}$ such that, for $0<x<\epsilon_1$;*

$f(\frac{1}{x})=a(x)$

where $a(x)=\sum_{n=1}^{\infty}a_nx^n$, $a_n\in\mathcal{R}$, $n\geq 1$, is an absolutely convergent real power series on the interval $(0,\epsilon_1)$, and for $-\epsilon_2<x<0$;

$f(\frac{1}{x})=b(x)$

where $b(x)=\sum_{n=1}^{\infty}b_nx^n$, $b_n\in\mathcal{R}$, $n\geq 1$, is an absolutely convergent real power series on the interval $(-\epsilon_2,0)$.

Lemma 0.3. *The functions $f_d(x)=\frac{1}{|x-d|}$ where $d\in\mathcal{R}$ are analytic at infinity.*

Proof. If $d>0$, we have that for $0<x<\frac{1}{d}$, using the formula for a geometric progression;

$f_d(\frac{1}{x}) = \frac{1}{|\frac{1}{x}-d|}$

$= \frac{|x|}{|1-dx|}$

$= \frac{x}{1-dx}$

$x\sum_{n=0}^{\infty}(dx)^n$

$= \sum_{n=0}^{\infty} d^n x^{n+1}$

$= \sum_{n=1}^{\infty} d^{n-1} x^n$

and, for $-\frac{1}{d} < x < 0$

$f_d(\frac{1}{x}) = \frac{1}{|\frac{1}{x}-d|}$

$= \frac{|x|}{|1-dx|}$

$= -\frac{x}{1-dx}$

$-x\sum_{n=0}^{\infty}(dx)^n$

$= -\sum_{n=0}^{\infty} d^n x^{n+1}$

$= -\sum_{n=1}^{\infty} d^{n-1} x^n$

The facts that f_d are smooth, for $x \neq d$, and the cases $d = 0$ and $d < 0$ are left to the reader.

$\square$

Lemma 0.4. *If $f : \mathcal{R} \setminus W \rightarrow \mathcal{R}$ is analytic at infinity, then so is f', moreover f' is of moderate decrease.*

Proof. As f is analytic at infinity, we have that, for $0 < x < \epsilon_1$;

$f(\frac{1}{x}) = \sum_{n=1} a_n x^n$

so that, by the chain rule, and the fact that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is absolutely convergent for $0 < x < \epsilon_1$;

$-\frac{f'(\frac{1}{x})}{x^2} = \sum_{n=1}^{\infty} na_n x^{n-1}$

so that, rearranging;

$f'(\frac{1}{x}) = -x^2 \sum_{n=1}^{\infty} na_n x^{n-1}$

$= -\sum_{n=1}^{\infty} na_n x^{n+1}$

$= -\sum_{n=2}^{\infty} (n-1)a_{n-1}x^n \ (C)$

A similar calculation holds for $-\epsilon_2 < x < 0$, to show that f is analytic at infinity. From (C), using the proof of Lemma 0.5, we can show that f' is of moderate decrease.

□

Lemma 0.5. *If f is analytic at infinity, then it is of very moderate decrease and non-oscillatory.*

Proof. By the definition, we have that, for $0 < x < \epsilon_1$, that;

$f(\frac{1}{x}) = x^n u(x)$

where, assuming f is non vanishing on $(\frac{1}{\epsilon_1}, \infty))$, $n \geq 1$ and $u(0) \neq 0$, and $|u(x)| \leq M$ on $(0, \epsilon_1)$, so that;

$|f(\frac{1}{x})| \leq Mx^n$

$|f(x) \leq \frac{M}{x^n}$

for $x > \frac{1}{\epsilon_1}$. Similar considerations apply for $x < 0$, so that;

$|f(x)| \leq \frac{N}{|x|^m}$

for $x < -\frac{1}{\epsilon_2}$, $m \geq 1$, $N \in \mathcal{R}_{>0}$. Without loss of generality, assuming that $max(\epsilon_1, \epsilon_2) < 1$, taking $D = max(M, N)$, $r = max(\frac{1}{\epsilon_1}, \frac{1}{\epsilon_2})$, $p = min(m, n)$, we obtain that;

$|f(x)| \leq \frac{D}{|x|^p} \leq \frac{D}{|x|}$

for $|x|>r$. As f is smooth, $|f|_{[-r,r]}|\leq M<\frac{Mr}{|x|}$, so f is of very moderate decrease, taking $C=max(D,Mr)$.

By Lemma 0.4, we have that f' is analytic at infinity. If $f'=0$, the result follows. Otherwise, we clam that the zero set of f', $Zero(f')$ is contained in $[-s,s]$, for some $s\in\mathcal{R}_{>0}$, in which case the result again follows. To see this is the case, suppose there exists a sequence of zeros of f', $\{s_n:n\in\mathcal{N}\}$, for which $|s_n|\rightarrow\infty$, Without loss of generality we may assume that $s_n>0$, so that the absolutely convergent power series $a(x)$ has infinitely many zeros in the interval $(0,\epsilon)$, for any $0<\epsilon<\epsilon_1$. Writing $a(x)=x^nu(x)$, with $u(0)\neq 0$, by continuity we can assume that $u(x)\neq 0$, in the interval $(0,\epsilon_3)$, where $0<\epsilon_3<\epsilon_1$. Then $x^nu(x)=0$ iff $x=0$ or $u(x)=0$, iff $x=0$ or $u(x)=0$, iff $x=0$ or $x\notin(0,\epsilon_3)$, which is a contradiction.

$\square$

Remarks 0.6. *The class of non-oscillatory functions, with very moderate decrease, was considered in the paper* [1], *where we proved a Fourier inversion theorem.*

Definition 0.7. *Let $f:\mathcal{R}^2\setminus W\rightarrow\mathcal{R}$ be smooth, with W closed and bounded, we say that f is of very moderate decrease if there exists a constant $C\in\mathcal{R}_{>0}$ such that $|f(x,y)|\leq\frac{C}{|(x,y)|}$, for $|(x,y)|>1$. We say that f is of moderate decrease if there exists a constant $C\in\mathcal{R}_{>0}$ such that $|f(x,y)|\leq\frac{C}{|(x,y)|^2}$, for $|(x,y)|>1$. We say that f is of moderate decrease n, if there exists a constant $C\in\mathcal{R}_{>0}$ such that $|f(x,y)|\leq\frac{C}{|(x,y)|^n}$, for $|(x,y)|>1$, with $n\geq 2$. We say that f is normal, if;*

(i). *For $x\in\mathcal{R}$, $f_x(y)$ is analytic at infinity.*

(ii). *For $y\in\mathcal{R}$, $f_y(x)$ is analytic at infinity.*

(iii). *f is of very moderate decrease.*

(iv). *The higher derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are of moderate decrease.*

(v). *There is a uniform bound val in the number of zeros of $f_x,(f_x)'(f_x)''$ and $f_y,(f_y)',(f_y)''$.*

We say that f is quasi normal if $(i)-(iv)$ hold and $(v)'$, for sufficiently large x, the zeros of f_x are contained in the union of bounded intervals $(-M_x,-N_x)\cup(N_x,M_x)$, with M_x-N_x uniformly bounded in x, and similarly for $\{(f_x)'(f_x)''\}$, with corresponding M'_x, M''_x, N'_x, N''_x, and $\{f_y,(f_y)'(f_y)''\}$, with corresponding $\{M_y, M'_y, M''_y, N_y, N'_y, N''_y\}$. We say that f is quasi split normal if $(i)-(iv)$ hold and $(v)''$, for sufficiently large (x,y), $f(x,y)=f_1(x,y)+f_2(x,y)$, with f_1 and f_2 quasi normal, and $\{f,f_1,f_2\}$ are smooth.

Lemma 0.8. *If $\{a,b\}\subset\mathcal{R}$, and W is a closed ball $B((a,b),s)$, $s>0$, containing (a,b), the function $f(x,y)|_{\mathcal{R}^2\setminus W}$*

where $f(x,y)=\frac{1}{|(x,y)-(a,b)|}$, $(x,y)\neq(a,b)$

is normal.

If $\rho\geq 0$ is continuous with compact support, $\rho\neq 0$, the function;

$f(x,y)=\int_{\mathcal{R}^2}\frac{\rho(x',y')}{|(x,y)-(x',y')|}dx'dy'$

is quasi normal, and if ρ is smooth with compact support, $\rho\neq 0$;

$f(x,y)=\int_{\mathcal{R}^2}\frac{\rho(x',y')}{|(x,y)-(x',y')|}dx'dy'$

is quasi split normal.

Proof. Fix $x_0\in\mathcal{R}$, then;

$f_{x_0}(y)=\frac{1}{((x_0-a)^2+(y-b)^2)^{\frac{1}{2}}}$

Without loss of generality, assuming that $x_0\neq a$, we have that, for $y>0$;

$f_{x_0}(\frac{1}{y})=\frac{1}{((x_0-a)^2+(\frac{1}{y}-b)^2)^{\frac{1}{2}}}$

$=\frac{y}{(y^2(x_0-a)^2+1-2yb+y^2b^2)^{\frac{1}{2}}}$

For $y<1$, we have that $y^2<y$, so that;

$|y^2(x_0-a)^2-2yb+y^2b^2|<y|(x_0-a)^2+2|b|+b^2|<1$

iff $y<\frac{1}{|(x_0-a)^2+2|b|+b^2|}$

and, applying Newton's theorem, with $b_n=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$;

$f_{x_0}(\frac{1}{y})=y\sum_{n=0}^{\infty}b_n(y^2(x_0-a)^2-2yb+y^2b^2)^n$

$=\sum_{n=1}^{\infty}a_ny^n$

is an absolutely convergent power series in y of order 1. A similar result holds for $y<0$, so that f_{x_0} is analytic at infinity. Similarly, fixing $y_0\in\mathcal{R}$, we can see that $f_{y_0}(x)$ is analytic at infinity. Let $\overline{r}=(x,y)$, $r=|x,y|$, $\overline{a}=(a,b)$, then, switching to polars, for $0\leq\theta<2\pi$;

$lim_{r\rightarrow\infty}rf(r,\theta)=lim_{r\rightarrow\infty}\frac{r}{|\overline{r}(r,\theta)-\overline{a}|}$

$=lim_{r\rightarrow\infty}\frac{1}{|\frac{\overline{r}(r,\theta)}{r}-\frac{\overline{a}}{r}|}$

$=lim_{r\rightarrow\infty}\frac{1}{|\hat{\overline{r}}(r,\theta)|}$

$=1$

so that fixing a closed ball $B(\overline{0},s)\supset W$, using the fact that f is smooth on $B(\overline{0},s)^c$, $r|f|\leq D$, where $D\in\mathcal{R}_{>0}$, on $B(\overline{0},s)^c$, so that $|f|\leq\frac{D}{r}$, for $r>s$. As f is continuous on $B(\overline{0},s)\setminus W^\circ$, it is bounded, by compactness of $B(\overline{0},s)\setminus W^\circ$, so that $|f|\leq M$ for $|x|\geq 1$. It follows that $|f|\leq\frac{C}{r}$, for $|x|\geq 1$, where $C=max(D,Ms)$. Therefore, $f|_{\mathcal{R}^2\setminus W}$ is of very moderate decrease.

We have that f_x has no zeros, similarly for f_y, and, by the chain rule;

$\frac{\partial f}{\partial x}=-\frac{1}{2}2(x-a)\frac{1}{((x-a)^2+(y-b)^2)^{\frac{3}{2}}}$

$=-\frac{x-a}{((x-a)^2+(y-b)^2)^{\frac{3}{2}}}$

so that $(f_y)'$ has a zero when $x=a$, similarly $(f_x)'$ has a zero when $y=b$. We have;

$lim_{r\rightarrow\infty}r^2|\frac{\partial f}{\partial x}|=lim_{r\rightarrow\infty}\frac{(x-a)r^2}{|\overline{r}-\overline{a}|^3}$

$\leq lim_{r\rightarrow\infty}\frac{r^2|\overline{r}-\overline{a}|}{|\overline{r}-\overline{a}|^3}$

$= lim_{r\rightarrow\infty}\frac{r^2}{|\overline{r}-\overline{a}|^2}$

$= lim_{r\rightarrow\infty}\frac{1}{|\frac{\overline{r}}{r}-\frac{\overline{a}}{r}|^2}$

$= lim_{r\rightarrow\infty}\frac{1}{|\hat{\overline{r}}|^2}$

$= 1$

so that fixing a closed ball $B(\overline{0},s)\supset W$, using the fact that $\frac{\partial f}{\partial x}$ is smooth on $B(\overline{0},s)^c$, $r^2|\frac{\partial f}{\partial x}|\leq D$, where $D\in\mathcal{R}_{>0}$, on $B(\overline{0},s)^c$, so that $|\frac{\partial f}{\partial x}|\leq\frac{D}{r^2}$, for $r>s$. As $\frac{\partial f}{\partial x}$ is continuous on $B(\overline{0},s)\setminus W^\circ$, it is bounded, by compactness of $B(\overline{0},s)\setminus W^\circ$ again, so that $|\frac{\partial f}{\partial x}|\leq M$ for $|x|\geq 1$. It follows that $|\frac{\partial f}{\partial x}|\leq\frac{C}{|\overline{x}|^2}$, for $|\overline{x}|\geq 1$, where $C=max(D,Ms^2)$. A similar proof works for $\frac{\partial f}{\partial y}$, and the higher derivatives $\frac{\partial f^{n+m}}{\partial x^m\partial y^n}$, $n+m\geq 1$, the details are left to the reader, so that $\{\frac{\partial^{n+m}f}{\partial x^n\partial y^n}_{\mathcal{R}^2\setminus W}: n+m\geq 1\}$ are of moderate decrease. We have that, by the product rule;

$\frac{\partial^2 f}{\partial x^2}=-\frac{1}{((x-a)^2+(y-b)^2)^{\frac{3}{2}}}+\frac{3(x-a)^2}{((x-a)^2+(y-b)^2)^{\frac{5}{2}}}=0$

iff $3(x-a)^2-[(x-a)^2+(y-b)^2]=0$

iff $2(x-a)^2=(y-b)^2$

iff $x-a=\frac{1}{\sqrt{2}}(y-b)$ or $x-a=-\frac{1}{\sqrt{2}}(y-b)$

so that $(f_y)''$ has at most 2 zeros for $y\in\mathcal{R}$ and we can take $val=2$. A similar result holds for $(f_x)''$.

For the second claim, note that if $(x,y)\in Supp(\rho)$, then;

$f(x,y)=\int_{\mathcal{R}^2}\frac{\rho(x',y')}{|(x,y)-(x',y')|}dx'dy'$

$=\int_{\mathcal{R}^2}\frac{\rho(x-x',y-y')}{|(x',y')|}dx'dy'$

$=\int_{0,2\pi}\int_{\mathcal{R}_{>0}}\frac{\rho_{x,y}(r,\theta)}{r}rsin(\theta)drd\theta$

$=\int_{0,2\pi}\int_{\mathcal{R}_{>0}}\rho_{x,y}(r,\theta)sin(\theta)drd\theta$

so that;

$|f(x,y)| \leq \int_{0,2\pi} \int_{\mathcal{R}_{>0}} |\rho_{x,y}(r,\theta)| dr d\theta$

$\leq 2M\pi R(x,y)$

where $Supp_{x,y}(\rho) \subset B(\overline{0}, R(x,y))$, $|\rho| \leq M$, so that f is defined everywhere. If ρ is smooth, we have that f is smooth, as;

$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x,y) = \frac{\partial^{i+j} \int_{\mathcal{R}^2} \frac{\rho(x-x',y-y')}{|(x',y')|} dx'dy'}{\partial x^i \partial y^j}$

$= \int_{\mathcal{R}^2} \frac{\frac{\partial^{i+j}\rho}{\partial x^i \partial y^j}(x-x',y-y')}{|(x',y')|} dx'dy'$

with $\frac{\partial^{i+j}\rho}{\partial x^i \partial y^j}$ having compact support again. If ρ is continuous, but not necessarily smooth, we have that, for $(x,y) \notin Supp(\rho)$;

$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x,y) = \int_{\mathcal{R}^2} \rho(x',y') \frac{\partial^{i+j}}{\partial x^i \partial y^j}(\frac{1}{|(x,y)-(x',y')|}) dx'dy'$

so that f is smooth, outside $Supp(\rho)$. For (i), in Definition 0.7, we have, for $x_0 \in \mathcal{R}$, $y > 0$;

$f_{x_0}(\frac{1}{y}) = \int_{\mathcal{R}^2} \frac{\rho(x',y')}{|(x_0,\frac{1}{y})-(x',y')|} dx'dy'$

$= y \int_{\mathcal{R}^2} \frac{\rho(x',y')}{(1+y^2(x_0-x')^2-2yy'+y^2y'^2)^{\frac{1}{2}}}$

so that, with $y < 1$, $y^2 < y$, letting;

$M_{x_0} = sup_{(x',y') \in Supp(\rho)} |(x_0 - x'|^2 + 2|y'| + y'^2|$, if $y < \frac{1}{M_{x_0}}$, then;

$|y^2(x_0-x')^2 - 2yy' + y^2y'^2| < y|(x_0-x')^2 + 2|y'| + y'^2| < 1$

so that, we can apply Newton's theorem uniformly in $(x',y') \in Supp(\rho)$, to obtain;

$f_{x_0}(\frac{1}{y}) = y \int_{Supp(\rho)} \rho(x',y') (\sum_{n=0}^{\infty} b_n (y^2(x_0-x')^2 - 2yy' + y^2y'^2)^n) dx'dy'$

where b_n is as above. With $y < 1$ again, $|x'| \leq M$, $|y'| \leq M$, for $(x',y') \subset Supp(\rho)$, $|\rho| \leq N$, $y < \frac{1}{(|x_0|+M)^2+2M+M^2}$, we have, applying the DCT;

$|f_{x_0}(\frac{1}{y})| \leq yN(2M)^2 \sum_{n=0}^{\infty} |b_n| (((|x_0|+M)^2 y + 2yM + yM^2)^n$

$\leq 4yNM^2\sum_{n=0}^{\infty}|b_n|y^n((|x_0|+M)^2+2M+M^2)^n$

$\leq 4yNM^2\sum_{n=0}^{\infty}y^n((|x_0|+M)^2+2M+M^2)^n$

defines an absolutely convergent series. A similar proof works for $y<0$. (ii) is similar. For (iii), with $Supp(\rho)\subset B(\bar{0},M)$, $|\rho|\leq N$, if $|(x,y)|\geq 2M$, and $(x',y')\in Supp(\rho)$, $|(x,y)-(x',y')|\geq\frac{|(x,y)}{2}$, so that $\frac{1}{|(x,y)-(x',y')|}\leq\frac{2}{|(x,y)|}$, and;

$|f(x,y)|=|\int_{Supp(\rho)}\frac{\rho(x',y')}{|(x,y)-(x',y')|}dx'dy'|$

$\leq\frac{2}{|(x,y)|}\int_{Supp(\rho)}|\rho(x',y')|dx'dy'$

$\leq\frac{2\pi M^2N}{|(x,y)|}$

For (iv), we have that, if $|(x,y)|\geq max(2M,1)$, and $(x',y')\in Supp(\rho)$, $\frac{1}{|(x,y)-(x',y')|^3}\leq\frac{8}{|(x,y)|^3}$, $|x-x'|\leq|(x,y)-(x',y')|\leq|(x,y)|+M$;

$|\frac{\partial f}{\partial x}|=|\int_{Supp(\rho)}\frac{-\rho(x',y')(x-x')}{|(x,y)-(x',y')|^3}dx'dy'|$

$\leq\frac{8(|(x,y)|+M)}{|(x,y)|^3}\int_{Supp(\rho)}|\rho(x',y')|dx'dy'$

$\leq\frac{8\pi M^2N(|(x,y)|+M)}{|(x,y)|^3}$

$\leq\frac{8\pi(M^2+M)N}{|(x,y)|^2}$

The proof for $\frac{\partial f}{\partial y}$ is similar. For $(v)'$, we have that if $\rho\geq 0$ is continuous with compact support, $\rho\neq 0$, that $f>0$. For $(x,y)\notin Supp(\rho)$, we have that;

$\frac{\partial f}{\partial x}=\int_{\mathcal{R}^2}-\frac{\rho(x',y')(x-x')}{|(x,y)-(x',y')|^3}dx'dy'$

and for $x>M$, $(x-x')>0$, for $x<M$, $(x-x')<0$, where $x'\in Supp(\rho)$, so that for $|y|>M$, $(f_y)'<0$, for $x>M$, and $(f_y)'>0$, for $x<M$, as $\rho\geq 0$. In particular, the zeros of f_y are contained in the interval $(-(M+1),(M+1))$, with the length $2(M+1)$ of the interval, uniformly bounded in y. We have that;

$\frac{\partial^2 f}{\partial x^2}=\int_{\mathcal{R}^2}\rho(x',y')[-\frac{1}{|(x,y)-(x',y')|^3}+\frac{3(x-x')^2}{|(x,y)-(x',y')|^5}]dx'dy'$

$= \int_{\mathcal{R}^2} \rho(x',y')[\frac{2(x-x')^2-(y-y')^2}{|(x,y)-(x',y')|^5}]dx'dy'$

so that if $2(x-x')^2-(y-y')^2 > 0$, for $(x'y') \in Supp(\rho)$, $(f_y)'' > 0$, and if $2(x-x')^2-(y-y')^2 < 0$, for $(x'y') \in Supp(\rho)$, $(f_y)'' < 0$. We have that;

$2(x-x')^2-(y-y')^2 > 0$

iff $|x-x'| > \frac{1}{\sqrt{2}}|y-y'|$ (i)

$2(x-x')^2-(y-y')^2 < 0$

iff $|x-x'| < \frac{1}{\sqrt{2}}|y-y'|$ (ii)

so that if $|x| > M + \frac{1}{\sqrt{2}}(|y|+M)$, (i) holds, and if $|x| < -M + \frac{1}{\sqrt{2}}(|y|-M)$, (ii) holds, for $|y| > 2M$. In particularly, the zeros of $(f_y)''$ are contained in the intervals $(-M+\frac{1}{\sqrt{2}}(|y|-M), M+\frac{1}{\sqrt{2}}(|y|+M)) \cup (-M-\frac{1}{\sqrt{2}}(|y|+M), M-\frac{1}{\sqrt{2}}(|y|-M))$, with the length of the intervals, $(2+\sqrt{2})M$ uniform in $|y| > 2M$. For $(v)''$, we can split ρ into ρ^+ and ρ^- which are continuous with compact support, and use the previous result, noting that quasi normal implies quasi split normal, as $f = \frac{f}{2}+\frac{f}{2}$.

$\square$

Remarks 0.9. *The above functions have stronger properties, for example it can probably be shown that* $\frac{\partial f^{n+m}}{\partial x^m \partial y^n}$ *has moderate decrease* $n+m+1$, *for* $n+m \geq 1$, *and there is a uniform bound on the zeros of all the higher derivatives* $f_x^{(n)}$ *and* $f_y^{(m)}$, *for all* $n \geq 1$, $m \geq 1$. *The details are left as an exercise. We include the case of* $(f_y)'''$ *for the first function;*

$\frac{\partial^3 f}{\partial x^3} = \frac{9(x-a)}{((x-a)^2+(y-b)^2)^{\frac{5}{2}}} - \frac{15(x-a)^3}{((x-a)^2+(y-b)^2)^{\frac{7}{2}}} = 0$

iff $9(x-a)[(x-a)^2+(y-b)^2] - 15(x-a)^3 = 0$

iff $9[(x-a)^2+(y-b)^2] - 15(x-a)^2 = 0$

iff $6(x-a)^2 = 9(y-b)^2$

iff $x-a = \sqrt{\frac{3}{2}}(y-b)$ *or* $x = -\sqrt{\frac{3}{2}}(y-b)$

so that $(f_y)'''$ has at most 2 zeros for $y \in \mathcal{R}$ and we can take $val = 2$. A similar result holds for $(f_x)'''$.

Lemma 0.10. *Let $f : \mathcal{R}^2 \rightarrow \mathcal{R}$ be smooth and normal, then, for $\{x, y\} \subset \mathcal{R}$, $k_1 \neq 0$, $k_2 \neq 0$;*

$F(k_1, y) = lim_{r\rightarrow\infty} \int_{-r}^{r} f(x, y)e^{-ik_1x}dx$

$G(x, k_2) = lim_{r\rightarrow\infty} \int_{-r}^{r} f(x, y)e^{-ik_2y}dy$

both exist and $F(k_1, y)$, $G(x, k_2)$ are of moderate decrease.

Proof. The first claim follows from [1] together with Lemma 0.5 and $(i), (ii)$ in Definition 0.7. In fact, the first integral is indefinite, in the sense that we could define it as;

$F(k_1, y) = lim_{r\rightarrow\infty, s\rightarrow\infty}(\int_{a}^{r} f(x, y)e^{-ik_1x}dx + \int_{-s}^{a} f(x, y)e^{-ik_1x}dx)$

for a choice of $a \in \mathcal{R}$, and similarly for $G(x, k_2)$. We then have, using integration by parts, for $y_0 \in \mathcal{R}$;

$\int_{-r}^{r} f(x, y_0)e^{-ik_1x}dx = [\frac{if(x,y_0)e^{-ik_1x}}{k_1}]_{-r}^{r} - \int_{-r}^{r} \frac{i\frac{\partial f}{\partial x}(x,y_0)e^{-ik_1x}}{k_1}dx$

so that, as f_{y_0} is of very moderate decrease, by (iii), integrating by parts;

$F(k_1, y_0) = lim_{r\rightarrow\infty} \int_{-r}^{r} f(x, y_0)e^{-ik_1x}dx$

$= lim_{r\rightarrow\infty}([\frac{if(x,y_0)e^{-ik_1x}}{k_1}]_{-r}^{r} - \int_{-r}^{r} \frac{i\frac{\partial f}{\partial x}(x,y_0)e^{-ik_1x}}{k_1}dx)$

$= lim_{r\rightarrow\infty}(-\int_{-r}^{r} \frac{i\frac{\partial f}{\partial x}(x,y_0)e^{-ik_1x}}{k_1}dx)$

$= -\frac{i}{k_1}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x, y_0)e^{-ik_1x}dx$

the last integral being definite, as $(\frac{\partial f}{\partial x})_{y_0}$ is of moderate decrease, using (iv). It follows that $F(k_1, y)$ is smooth, as differentiating under the integral sign is justified by the DCT, MVT and (iv), with;

$F^{(n)}(k_1, y) = -\frac{i}{k_1}\int_{-\infty}^{\infty} \frac{\partial^{1+n} f}{\partial x \partial y^n}(x, y)e^{-ik_1x}dx$

We have that $(\frac{\partial f}{\partial x})_{y_0}$ is analytic at infinity by (ii) and Lemma 0.4. By (v) in the definition of normality, we can assume that for sufficiently large y, $(\frac{\partial f}{\partial x})_y$ is monotone and positive/negative in the intervals $(-\infty, a_1(y)), \ldots, (a_{val}(y), \infty)$, where $a_1, \ldots, a_{val}$ vary continuously with y. Splitting the integral into $cos(k_1 x)$ and $sin(k_1 x)$ components, in a similar calculation to [2], for any interval of length at least $\frac{2\pi}{|k_1|}$, we obtain an alternating cancellation in the contribution to the integral of at most $\frac{2\pi||f'_y||}{|k_1|}$ and for any interval of length at most $\frac{2\pi}{|k_1|}$, we obtain a contribution to the integral of at most $\frac{4\pi||f'_y||}{|k_1|}$. It follows that, for sufficiently large y, using the fact that $\frac{\partial f}{\partial x}$ has moderate decrease;

$|-\frac{i}{k_1}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}(x,y)e^{-ik_1 x}dx|$

$\leq \frac{(val+1)}{|k_1|}\frac{4\pi||f'_y||}{|k_1|}$

$= \frac{4\pi(val+1)||f'_y||}{|k_1|^2}$

$\leq \frac{4\pi(val+1)C}{|k_1|^2|y|^2}$ $(*)$

so that;

$|F(k_1, y)| \leq \frac{D}{|y|^2}$

for sufficiently large y, with $D = \frac{4\pi(val+1)C}{|k_1|^2}$. As $F(k_1, y)$ is smooth, we obtain that $F(k_1, y)$ is of moderate decrease. Similarly, we can show that $G(x, k_2)$ is of moderate decrease, for $k_2 \neq 0$. $\square$

Lemma 0.11. *The same result as Lemma 0.10 holds, with the assumption that f is smooth and quasi normal or smooth and quasi split normal.*

Proof. We just have to replace the uses of (v) in Definition 0.7 with the use of $(v)'$ or $(v)''$. For $(v)'$, we have that, for sufficiently large y, $(\frac{\partial f}{\partial x})_y$ is monotone and positive/negative outside a finite union I_y of S intervals whose total length is uniformly bounded by a constant $R \in \mathcal{R}_{>0}$ which is independent of y. By the usual argument, we obtain a contribution of at most $||f'_y||R$ to the integral over this interval. As before, splitting the integral into $cos(k_1 x)$ and $sin(k_1 x)$ components, we can bound the contribution of the remaining integral by $\frac{(S+1)4\pi||f'_y||}{|k_1|}$ to give a total bound for the calculation in $(*)$ of Lemma 0.10 of $(\frac{(S+1)4\pi}{|k_1|} + R)||f'_y||$, and we can then, as before, use the fact that $\frac{\partial f}{\partial x}$ has moderate decrease.

For $(v)''$, we can split $f'_y = f'_{1,y} + f'_{2,y}$, and repeating the above argument twice obtain a total bound of $(\frac{(S+1)4\pi}{|k_1|} + R)(||f'_{1,y}|| + ||f'_{2,y}||)$, where the bounds S and R work for both $f'_{1,y}$ and $f'_{2,y}$. We can then use the the fact that $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_2}{\partial x}$ have moderate decrease.

$\square$

Lemma 0.12. *Let hypotheses and notation be as in Lemma 0.10, then we can define, for* $k_1 \neq 0$, $k_2 \neq 0$;

$$F(k_1, k_2) = \int_{-\infty}^{\infty} F(k_1, y) e^{-ik_2 y} dy$$

$$G(k_1, k_2) = \int_{-\infty}^{\infty} G(x, k_2) e^{-ik_1 x} dx$$

We have that;

$$F(k_1, k_2) = G(k_1, k_2) = lim_{m\to\infty, n\to\infty} \int_{-m}^{m} \int_{-n}^{n} f(x, y) e^{-ik_1 x} e^{-ik_2 y} dx dy$$

Proof. The definition follows from the previous lemma, as $F(k_1, y)$ and $G(x, k_2)$ are smooth and of moderate decrease. Integrating by parts again, using the fact that f is of very moderate decrease, we have that, for $n \in \mathcal{R}_{>0}$;

$$\int_{|x|>n} f(x, y) e^{-ik_1 x} dx$$

$$= ([\tfrac{if(x,y)e^{-ik_1 x}}{k_1}]_n^{\infty} + [\tfrac{if(x,y)e^{-ik_1 x}}{k_1}]_{-\infty}^{-n} - \int_{|x|>n} \tfrac{i\frac{\partial f}{\partial x}(x,y)e^{-ik_1 x}}{k_1} dx)$$

$$= (-\tfrac{if(n,y)e^{-ik_1 n}}{k_1} + \tfrac{if(-n,y)e^{ik_1 n}}{k_1} - \int_{|x|>n} \tfrac{i\frac{\partial f}{\partial x}(x,y)e^{-ik_1 x}}{k_1} dx)$$

We have that, using the above calculation $(*)$ in Lemma 0.10, f of very moderate decrease;

$$|\int_{|y|\leq m} (\tfrac{if(-n,y)e^{ik_1 n}}{k_1} - \tfrac{if(n,y)e^{-ik_1 n}}{k_1}) e^{-ik_2 y} dy|$$

$$\leq \tfrac{1}{k_1} |\int_{|y|\leq m} f(-n, y) e^{-ik_2 y} dy| + \tfrac{1}{k_1} |\int_{|y|\leq m} f(n, y) e^{-ik_2 y} dy|$$

$$\leq \tfrac{4\pi(val+1)||f_{-n}|_{|y|\leq m}||}{|k_1|^2} + \tfrac{4\pi(val+1)||f_n|_{|y|\leq m}||}{|k_1|^2}$$

$$\leq \tfrac{8\pi(val+1)C}{|k_1|^2 n}$$

and using the previous calculation $(*)$ again in Lemma 0.10, we have that;

$|-\frac{i}{k_1}\int_{|x|>n}\frac{\partial f}{\partial x}(x,y)e^{-ik_1x}dx|$

$\leq \frac{(val+1)}{|k_1|}\frac{4\pi||f_y'||_{|x|>n}}{|k_1|}$

$=\frac{4\pi(val+1)||f_y'||_{|x|>n}}{|k_1|^2}$

$\leq \frac{4\pi(val+1)C}{|k_1|^2(y^2+n^2)}$

so that;

$|-\frac{i}{k_1}\int_{|y|\leq m}\int_{|x|>n}\frac{\partial f}{\partial x}(x,y)e^{-ik_1x}dxe^{-ik_2y}dy|$

$\leq \int_{|y\leq m}\frac{4\pi(val+1)C}{|k_1|^2(y^2+n^2)}dy$

$\leq \int_{-\infty}^{\infty}\frac{4\pi(val+1)C}{|k_1|^2(y^2+n^2)}dy$

$=\int_{-\infty}^{\infty}\frac{1}{n^2}\frac{4\pi(val+1)C}{|k_1|^2(1+\frac{y^2}{n^2})}dy$

$\leq \frac{1}{n^2}\frac{4\pi(val+1)C}{|k_1|^2}ntan^{-1}(\frac{y}{n})|_{-\infty}^{\infty}$

$\leq \frac{\pi}{n}\frac{4\pi(val+1)C}{|k_1|^2}$

The above calculations combine, to give that;

$|\int_{|y|\leq m}lim_{n\rightarrow\infty}\int_{-n}^{n}f(x,y)e^{-ik_1x}e^{-ik_2y}dxdy-\int_{|y|\leq m}\int_{|x|\leq n}f(x,y)e^{-ik_1x}e^{-ik_2y}dxdy|$

$=|\int_{|y|\leq m}\int_{|x|>n}f(x,y)e^{-ik_1x}e^{-ik_2y}dxdy|$

$\leq \frac{8\pi(val+1)C}{|k_1|^2n}+\frac{\pi}{n}\frac{4\pi(val+1)C}{|k_1|^2}$

so that $lim_{n\rightarrow\infty}s_{n,m}=s_{\infty,m}$, uniformly in m, where;

$s_{n,m}=\int_{|y|\leq m}\int_{|x|\leq n}f(x,y)e^{-ik_1x}e^{-ik_2y}dxdy$

$s_{\infty,m}=\int_{|y|\leq m}F(k_1,y)e^{-ik_2y}dy$

By the Moore-Osgood Theorem, we obtain the result.

$\square$

Lemma 0.13. *Let hypotheses and notation be as in Lemma 0.12, then we have that there exist constants* $\{C_{k_1}, D_{k_1}\} \subset \mathcal{R}_{>0}$ *such that;*

$|s_{n,m} - s| \leq \frac{C_{k_1}}{m} + \frac{D_{k_1}}{n}$

In particular, there exists a constant $E_{k_1} \in \mathcal{R}_{>0}$ *such that;*

$|s_{m,m} - s| \leq \frac{E_{k_1}}{m}$

Similarly, there exist constants $\{C_{k_2}, D_{k_2}\} \subset \mathcal{R}_{>0}$ *such that;*

$|s_{n,m} - s| \leq \frac{C_{k_2}}{m} + \frac{D_{k_2}}{n}$

In particular, there exists a constant $E_{k_2} \in \mathcal{R}_{>0}$ *such that;*

$|s_{m,m} - s| \leq \frac{E_{k_2}}{m}$

Proof. We have that, by the proof of Lemma 0.12;

$|s_{n,m} - s| \leq |s_{n,m} - s_{\infty,m}| + |s_{\infty,m} - s|$

$\leq \frac{8\pi(val+1)C}{|k_1|^2 n} + \frac{\pi}{n}\frac{4\pi(val+1)C}{|k_1|^2} + |\int_{|y|\geq m} F(k_1, y)e^{-ik_2 y}dy|$

$\leq \frac{8\pi(val+1)C}{|k_1|^2 n} + \frac{\pi}{n}\frac{4\pi(val+1)C}{|k_1|^2} + \int_{|y|\geq m} |F(k_1, y)|dy$

where, by the result of Lemma 0.10, $F(k_1, y)$ is of moderate decrease;

$|F(k_1, y)| \leq \frac{D}{|y|^2}$

for sufficiently large y, with $D = \frac{4\pi(val+1)C}{|k_1|^2}$. It follows that, for sufficiently large m;

$|s_{n,m} - s| \leq \frac{8\pi(val+1)C}{|k_1|^2 n} + \frac{\pi}{n}\frac{4\pi(val+1)C}{|k_1|^2} + \int_{|y|\geq m} \frac{D}{y^2}dy$

$\leq \frac{8\pi(val+1)C}{|k_1|^2 n} + \frac{\pi}{n}\frac{4\pi(val+1)C}{|k_1|^2} + \frac{2D}{m}$

$= \frac{C_{k_1}}{m} + \frac{D_{k_1}}{n}$

where;

$C_{k_1} = \frac{8\pi(val+1)C}{|k_1|^2}$

$D_{k_1} = \frac{8\pi(val+1)C}{|k_1|^2} + \pi\frac{4\pi(val+1)C}{|k_1|^2}$

For the next claim, we can take $E_{k_1} = C_{k_1} + D_{k_1}$

The claim with k_2 follows by symmetry, using a corresponding estimate for $|s_{n,m} - s_{n,\infty}|$ in terms of m and the fact from Lemma 0.10 that $G(x, k_2)$ is of moderate decrease, for $k_2 \neq 0$.

□

Lemma 0.14. *The same results as Lemma 0.12 and Lemma 0.13 hold with the assumption that f is smooth and quasi normal or smooth and quasi split normal.*

Proof. Again, we can replace the estimates from (v) in Definition 0.7, used in the proof of Lemma 0.12, with the estimate used in Lemma 0.11. The proof of Lemma 0.13 then goes through.

□

Definition 0.15. *Let $f : \mathcal{R}^3 \setminus W \rightarrow \mathcal{R}$ be smooth, with W closed and bounded, we say that f is of very moderate decrease if there exists a constant $C \in \mathcal{R}_{>0}$ such that $|f(x,y,z)| \leq \frac{C}{|(x,y,z)|}$, for $|(x,y,z)| > 1$. We say that f is of moderate decrease if there exists a constant $C \in \mathcal{R}_{>0}$ such that $|f(x,y,z)| \leq \frac{C}{|(x,y,z)|^2}$, for $|(x,y,z)| > 1$. We say that f is of moderate decrease n, if there exists a constant $C \in \mathcal{R}_{>0}$ such that $|f(x,y,z)| \leq \frac{C}{|(x,y,z)|^n}$, for $|(x,y,z)| > 1$, with $n \geq 2$. We say that f is normal, if;*

(i). For $x \in \mathcal{R}$, $f_x(y,z)$ is normal.

(ii). For $y \in \mathcal{R}$, $f_y(x,z)$ is normal.

(iii). For $z \in \mathcal{R}$, $f_z(x,y)$ is normal.

(iv). f is of very moderate decrease.

(v). The higher derivatives $\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}$ are of moderate decrease $i+j+k+1$, for $i+j+k \geq 1$.

(vi). There is a uniform bound $val(x,y)$ in the number of zeros of;

$f_{x,y}, (f_{x,y})', (f_{x,y})'', (f_{x,y})''', (f_{x,y})''''$

and similarly, for $f_{x,z}, f_{y,z}$.

We say that f is quasi normal, if;

(i)'. For $x \in \mathcal{R}$, $f_x(y,z)$ is quasi normal, and, similarly for $(ii), (iii), (iv), (v)$ hold and;

(vi)'. For sufficiently large (x,y), the zeros of;

$f_{x,y}, (f_{x,y})', (f_{x,y})'', (f_{x,y})''', (f_{x,y})''''$

are contained in a finite union of S intervals, with total length R, uniform in (x,y).

and similarly, for $f_{x,z}, f_{y,z}$.

We say that f is quasi split normal, if;

(i)''. For $x \in \mathcal{R}$, $f_x(y,z)$ is quasi split normal, and, similarly for $(ii), (iii), (iv), (v)$ hold and;

(vi)''. For sufficiently large (x,y), $f = f_1 + f_2$, with f, f_1, f_2 smooth and with f_1, f_2 having the property (vi)'.

Lemma 0.16. *If $\{a,b,c\} \subset \mathcal{R}$, and W is a closed ball $B((a,b,c),s)$, $s > 0$, containing (a,b,c), the function $f(x,y,z)|_{\mathcal{R}^3 \setminus W}$*

where $f(x,y,z) = \frac{1}{|(x,y,z)-(a,b,c)|}$, $(x,y,z) \neq (a,b,c)$

is normal.

If $\rho \geq 0$ is continuous with compact support, $\rho \neq 0$, the function;

$f(x,y,z) = \int_{\mathcal{R}^3} \frac{\rho(x',y',z')}{|(x,y,z)-(x',y',z')|} dx'dy'dz'$

is quasi normal.

If ρ is smooth with compact support, $\rho \neq 0$, the function;

$f(x,y,z) = \int_{\mathcal{R}^3} \frac{\rho(x',y',z')}{|(x,y,z)-(x',y',z')|} dx'dy'dz'$

is quasi split normal.

Proof. For the first claim, we have to show, for $x_0 \in \mathcal{R}$, that $f_{x_0}(y,z)$ is normal. Fix $z_0 \in \mathcal{R}$, then;

$f_{x_0,z_0}(y) = \frac{1}{((x_0-a)^2+(z_0-c)^2+(y-b)^2)^{\frac{1}{2}}}$

Without loss of generality, assuming that $x_0 \neq a, z_0 \neq c$, we have that, for $y > 0$;

$f_{x_0,z_0}(\frac{1}{y}) = \frac{1}{((x_0-a)^2+(z_0-c)^2+(\frac{1}{y}-b)^2)^{\frac{1}{2}}}$

$= \frac{y}{(y^2(x_0-a)^2+y^2(z_0-c)^2+1-2yb+y^2b^2)^{\frac{1}{2}}}$

For $y < 1$, we have that $y^2 < y$, so that;

$|y^2(x_0-a)^2 + y^2(z_0-c)^2 - 2yb + y^2b^2| < y|(x_0-a)^2 + (z_0-c)^2 + 2|b| + b^2| < 1$

iff $y < \frac{1}{|(x_0-a)^2+(z_0-c)^2+2|b|+b^2|}$

and, applying Newton's theorem, with $b_n = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$;

$f_{x_0,z_0}(\frac{1}{y}) = y\sum_{n=0}^{\infty} b_n(y^2(x_0-a)^2 + y^2(z_0-c)^2 - 2yb + y^2b^2)^n$

$= \sum_{n=1}^{\infty} a_n y^n$

is an absolutely convergent power series in y of order 1. A similar result holds for $y < 0$, so that f_{x_0,z_0} is analytic at infinity. Similarly, fixing $y_0 \in \mathcal{R}$, we can see that $f_{x_0,y_0}(z)$ is analytic at infinity, and $(i), (ii)$ in Definition 0.7 hold for $f_{x_0}(y,z)$. That f_{x_0} is of very moderate decrease will follow from the proof below that f is of very moderate decrease. As if there exists a constant $C \in \mathcal{R}_{>0}$ such that $|f|(x,y,z) \leq \frac{C}{|x,y,z|}$, for $|(x,y,z)| \geq 1$, then if $|(y,z)| \geq 1$, we have that $|x_0,y,z| \geq |(y,z)| \geq 1$, and;

$|f_{x_0}|(y,z) \leq \frac{C}{|(x_0,y,z)|} \leq \frac{C}{|(y,z)|}$

Similarly, the proof that $\frac{\partial f_{x_0}}{\partial y}$ and $\frac{\partial f_{x_0}}{\partial z}$ are of moderate decrease, will follow from the proof below that the higher derivatives $\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}$ are of moderate decrease $i+j+k+1$, As if $\frac{\partial f}{\partial y}$ is of moderate decrease 2, so of moderate decrease, than, for $|(x,y,z)| > 1$;

$|\frac{\partial f}{\partial y}| \leq \frac{C}{|(x,y,z)|^2}$

so that, for $|(y,z)| \geq 1$, as above;

$|\frac{\partial f_{x_0}}{\partial y}| \leq \frac{C}{|(x_0,y,z)|^2} \leq \frac{C}{|y,z|^2}$

The proof that for $y \in \mathcal{R}$, the zeros of $\{f_{x_0,y}(z), f'_{x_0,y}(z), f''_{x_0,y}(z)\}$ are uniformly bounded in y follows from the proof below that the zeros of $\{f_{x,y}(z), f'_{x,y}(z), f''_{x,y}(z), f'''_{x,y}(z), f''''_{x,y}(z)\}$ are uniformly bounded in (x,y). It follows that for $x_0 \in \mathcal{R}$, $f_{x_0}(y,z)$ is normal, similarly for $f_{y_0}(x,z)$ and $f_{z_0}(x,y)$, where $y_0 \in \mathcal{R}$, $z_0 \in \mathcal{R}$. We have then verified $(i)-(iii)$ of Definition 0.15. For (iv), let $\overline{r} = (x,y,z)$, $r = |(x,y,z)|$, $\overline{a} = (a,b,c)$, then, switching to polars, $x = rsin(\theta)cos(\phi)$, $y = rsin(\theta)sin(\phi)$, $z = rcos(\theta)$, for $0 \leq \theta \leq \pi$, $-\pi \leq \phi < \pi$;

$lim_{r\to\infty} rf(r,\theta,\phi) = lim_{r\to\infty}\frac{r}{|\overline{r}(r,\theta,\phi)-\overline{a}|}$

$= lim_{r\to\infty}\frac{1}{|\frac{\overline{r}(r,\theta,\phi)}{r}-\frac{\overline{a}}{r}|}$

$= lim_{r\to\infty}\frac{1}{|\hat{\overline{r}}(r,\theta,\phi)|}$

$= 1$

so that fixing a closed ball $B(\overline{0},s) \supset W$, using the fact that f is smooth on $B(\overline{0},s)^c$, $r|f| \leq D$, where $D \in \mathcal{R}_{>0}$, on $B(\overline{0},s)^c$, so that $|f| \leq \frac{D}{r}$, for $r > s$. As f is continuous on $B(\overline{0},s) \setminus W^{\circ}$, it is bounded, by compactness of $B(\overline{0},s) \setminus W^{\circ}$, so that $|f| \leq M$ for $|\overline{x}| \geq 1$. It follows that $|f| \leq \frac{C}{|\overline{x}|}$, for $|\overline{x}| \geq 1$, where $C = max(D, Ms)$. Therefore, $f|_{\mathcal{R}^3\setminus W}$ is of very moderate decrease.

Suppose inductively, that, for $(i+j+k \geq 0)$, $\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k}$ is of the form;

$\frac{p(x-a,y-b,z-c)}{|(x,y,z)-(a,b,c)|^{1+2(i+j+k)}}$, where p is homogeneous of degree $i+j+k$, then, using the product rule;

$\frac{\partial f^{i+j+k+1}}{\partial x^i \partial y^j \partial z^{k+1}} = \frac{|(x,y,z)-(a,b,c)|^2 \frac{\partial p}{\partial z}(x-a,y-b,z-c)-(z-c)(1+2(i+j+k))p(x-a,y-b,z-c)}{|(x,y,z)-(a,b,c)|^{1+2(i+j+k+1)}}$

which is of the form;

$\frac{q(x-a,y-b,z-c)}{|(x,y,z)-(a,b,c)|^{1+2(i+j+k+1)}}$, where q is homogeneous of degree $i+j+k+1$;

as $(1+2(i+j+k)) > i+j+k$, for $i+j+k \geq 0$

Similar inductions work for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, so that we can assume that;

$\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k}$ is of the form;

$\frac{p(x-a,y-b,z-c)}{|(x,y,z)-(a,b,c)|^{1+2(i+j+k)}}$, where p is homogeneous of degree $i+j+k$

We then have that;

$lim_{r\rightarrow\infty} r^{1+i+j+k} |\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k}| = lim_{r\rightarrow\infty} \frac{r^{1+i+j+k}\sum_{i'+j'+k'=i+j+k} c_{i'j'k'}(x-a)^{i'}(y-b)^{j'}(z-c)^{k'}}{|\overline{r}-\overline{a}|^{1+2(i+j+k)}}$

$\leq lim_{r\rightarrow\infty} \frac{r^{1+i+j+k}(\sum_{i'+j'+k'}|c_{i'j'k'}|)|\overline{r}-\overline{a}|^{i+j+k}}{|\overline{r}-\overline{a}|^{1+2(i+j+k)}}$

$= E lim_{r\rightarrow\infty} \frac{r^{1+i+j+k}}{|\overline{r}-\overline{a}|^{1+i+j+k}}$

$= E lim_{r\rightarrow\infty} \frac{1}{|\frac{\overline{r}}{r}-\frac{\overline{a}}{r}|^{1+i+j+k}}$

$= E lim_{r\rightarrow\infty} \frac{1}{|\hat{\overline{r}}|^{1+i+j+k}}$

$= E$

where $E = \sum_{i'+j'+k'=i+j+k} |c_{i'j'k'}|$, so that fixing a closed ball $B(\overline{0}, s) \supset W$, using the fact that $\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}$ is smooth on $B(\overline{0}, s)^c$, $r^{i+j+k+1}|\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k}| \leq D$, where $D \in \mathcal{R}_{>0}$, on $B(\overline{0}, s)^c$, so that $|\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k}| \leq \frac{D}{r^{i+j+k+1}}$, for $r > s$. As $\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}$ is continuous on $B(\overline{0}, s) \setminus W^\circ$, it is bounded, by compactness of $B(\overline{0}, s) \setminus W^\circ$ again, so that $|\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}| \leq M$ for $|x| \geq 1$. It follows that $|\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}| \leq \frac{C}{|\overline{x}|^{i+j+k+1}}$, for $|\overline{x}| \geq 1$, where $C = max(D, Ms^{i+j+k+1})$. It follows that $\{\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}_{\mathcal{R}^3\setminus W} : i+j+k \geq 1\}$ are of moderate decrease

$i+j+k+1$.

Fixing $\{y,z\}\subset\mathcal{R}$, we have that $f_{y,z}$ has no zeros, and, by the chain rule;

$\frac{\partial f}{\partial x}|_{y,z}=-\frac{1}{2}2(x-a)\frac{1}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{3}{2}}}$

$=-\frac{x-a}{((x-a)^2+(y-b)^2+(z-b)^2)^{\frac{3}{2}}}$

so that $(f_{y,z})'$ has a zero when $x=a$. We have that, by the product rule;

$\frac{\partial^2 f}{\partial x^2}|_{y,z}=-\frac{1}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{3}{2}}}+\frac{3(x-a)^2}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{5}{2}}}=0$

iff $3(x-a)^2-[(x-a)^2+(y-b)^2+(z-c)^2]=0$

iff $2(x-a)^2=(y-b)^2+(z-c)^2$

iff $x-a=\frac{1}{\sqrt{2}}[(y-b)^2+(z-c)^2]^{\frac{1}{2}}$ or $x-a=-\frac{1}{\sqrt{2}}[(y-b)^2+(z-c)^2]^{\frac{1}{2}}$

so that $(f_{y,z})''$ has at most 2 zeros for $(y,z)\in\mathcal{R}^2$

Similarly;

$\frac{\partial^3 f}{\partial x^3}|_{y,z}=\frac{9(x-a)}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{5}{2}}}-\frac{15(x-a)^3}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{7}{2}}}=0$

iff $9(x-a)[(x-a)^2+(y-b)^2+(z-c)^2]-15(x-a)^3=0$

iff $9[(x-a)^2+(y-b)^2+(z-c)^2]-15(x-a)^2=0$

iff $6(x-a)^2=9((y-b)^2+(z-c)^2)$

iff $x-a=\sqrt{\frac{3}{2}}[(y-b)^2+(z-c)^2]^{\frac{1}{2}}$ or $x-a=-\sqrt{\frac{3}{2}}[(y-b)^2+(z-c)^2]^{\frac{1}{2}}$

so that $(f_{y,z})'''$ has at most 2 zeros for $(y,z)\in\mathcal{R}^2$.

Finally;

$\frac{\partial^4 f}{\partial x^4}|_{y,z}=\frac{9}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{5}{2}}}-\frac{90(x-a)^2}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{7}{2}}}+\frac{105(x-a)^4}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{9}{2}}}$

$= 0$

iff $9[(x-a)^2+(y-b)^2+(z-c)^2]^2 - 90(x-a)^2[(x-a)^2+(y-b)^2$

$+(z-c)^2] + 105(x-a)^4 = 0$

iff $9(u^2+s^2)^2 - 90u^2(u^2+s^2) + 105u^4 = 0$

iff $24u^4 - 72u^2s^2 + 9s^4 = 0$

iff $8u^4 - 24u^2s^2 + 3s^4 = 0$

iff $u^2 = \frac{24s^2 +/- \sqrt{24^2s^4 - 4.24s^4}}{16}$

iff $u^2 = \frac{24s^2 +/- \sqrt{480s^4}}{16}$

iff $u^2 = \frac{6+/-\sqrt{30}}{4}s^2$

iff $u = \sqrt{6+\sqrt{30}}s$ or $u = \sqrt{6-\sqrt{30}}s$ or $u = -(\sqrt{6+\sqrt{30}})s$

or $u = -(\sqrt{6-\sqrt{30}})s$

where $u = x-a$, $s = [(y-b)^2+(z-c)^2]^{\frac{1}{2}}$, so that $(f_{y,z})''''$ has at most 4 zeros for $(y,z) \in \mathcal{R}^2$, and we can take $val = 4$. A similar result holds for $f_{x,y}$ and $f_{x,z}$. It follows that f is normal.

For the second claim, note that if $(x,y,z) \in Supp(\rho)$, then, switching to polars;

$f(x,y,z) = \int_{\mathcal{R}^2} \frac{\rho(x',y',z')}{|(x,y,z)-(x',y',z')|} dx'dy'dz'$

$= \int_{\mathcal{R}^3} \frac{\rho(x-x',y-y',z-z')}{|(x',y',z')|} dx'dy'dz'$

$= \int_{0,\pi} \int_{-\pi}^{\pi} \int_{\mathcal{R}_{>0}} \frac{\rho_{x,y,z}(r,\theta,\phi)}{r} r^2 sin(\theta) dr d\theta d\phi$

$= \int_{0,\pi} \int_{-\pi}^{\pi} \int_{\mathcal{R}_{>0}} \rho_{x,y,z}(r,\theta,\phi) r sin(\theta) dr d\theta d\phi$

so that;

$|f(x,y,z)| \leq \int_{0,\pi} \int_{-\pi}^{\pi} \int_{\mathcal{R}_{>0}} |\rho_{x,y,z}(r,\theta,\phi)| dr d\theta d\phi$

$\leq 2M\pi^2 R(x,y,z)$

where $Supp_{x,y,z}(\rho) \subset B(\overline{0}, R(x,y,z))$, $\rho| \leq M$, so that f is defined everywhere. If ρ is smooth, we have that f is smooth, as;

$\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}(x,y,z) = \frac{\partial^{i+j+k} \int_{\mathcal{R}^3} \frac{\rho(x-x',y-y',z-z')}{|(x',y',z')|} dx'dy'dz'}{\partial x^i \partial y^j \partial z^k}$

$= \int_{\mathcal{R}^3} \frac{\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k}(x-x',y-y',z-z')}{|(x',y',z')|} dx'dy'dz'$

with $\frac{\partial^{i+j+k}\rho}{\partial x^i \partial y^j \partial z^k}$ having compact support again. If ρ is continuous, but not necessarily smooth, we have that, for $(x,y,z) \notin Supp(\rho)$;

$\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}(x,y,z) = \int_{\mathcal{R}^3} \rho(x',y',z') \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k}\left(\frac{1}{|(x,y,z)-(x',y',z')|}\right) dx'dy'dz'$

so that f is smooth, outside $Supp(\rho)$. When $\rho \geq 0$ with compact support, we have to show that $f_{x_0}(y,z)$ is quasi normal. For (i), in Definition 0.7, we have, for $x_0 \in \mathcal{R}$, $z_0 \in \mathcal{R}$, $y > 0$;

$f_{x_0,z_0}(\frac{1}{y}) = \int_{\mathcal{R}^3} \frac{\rho(x',y',z')}{|(x_0,\frac{1}{y},z)-(x',y',z')|} dx'dy'dz'$

$= y \int_{\mathcal{R}^3} \frac{\rho(x',y',z')}{(1+y^2(x_0-x')^2+y^2(z_0-z')^2-2yy'+y^2y'^2)^{\frac{1}{2}}}$

so that, with $y < 1$, $y^2 < y$, letting;

$M_{x_0,z_0} = sup_{(x',y',z') \in Supp(\rho)} |(x_0-x')^2 + (z_0-z')^2 + 2|y'| + y'^2|$, if $y < \frac{1}{M_{x_0,z_0}}$, then;

$|y^2(x_0-x')^2 + y^2(z_0-z')^2 - 2yy' + y^2y'^2| < y|(x_0-x')^2 + (z_0-z)^2 + 2|y'| + y'^2| < 1$

so that, we can apply Newton's theorem uniformly in $(x',y',z') \in Supp(\rho)$, to obtain;

$f_{x_0,z_0}(\frac{1}{y}) = y \int_{Supp(\rho)} \rho(x',y',z')(\sum_{n=0}^{\infty} b_n(y^2(x_0-x')^2 + y^2(z_0-z')^2 - 2yy' + y^2y'^2)^n) dx'dy'dz'$

where b_n is as above. With $y < 1$ again, $|x'| \leq M$, $|y'| \leq M$, $|z'| \leq M$ for $(x',y',z') \subset Supp(\rho)$, $|\rho| \leq N$, $y < \frac{1}{(|x_0|+M)^2+(|z_0+M)^2+2M+M^2}$, we have, applying the DCT;

$|f_{x_0,z_0}(\frac{1}{y})| \leq yN(2M)^3\sum_{n=0}^{\infty}|b_n|(((|x_0|+M)^2y+(|z_0|+M)^2+2yM+yM^2)^n$

$\leq 8yNM^3\sum_{n=0}^{\infty}|b_n|y^n((|x_0|+M)^2+(|z_0+M)^2+2M+M^2)^n$

$\leq 8yNM^3\sum_{n=0}^{\infty}y^n((|x_0|+M)^2+(|z_0|+M)^2+2M+M^2)^n$

defines an absolutely convergent series. A similar proof works for $y<0$. (ii) is similar. For (iii), in Definition 0.7, this will follow, as above, from the main result that f itself is of very moderate decrease, similarly, for (iv), as above, the moderate decrease in the fibre $\frac{\partial f}{\partial y}_{x_0}$ and $\frac{\partial f}{\partial z}_{x_0}$, will follow from the moderate decrease $i+j+k+1$ in the derivatives $\frac{\partial^{i+j+k}f}{\partial x^i\partial y^j\partial z^k}$, $i+j+k\geq 1$. For (v), again, as above, the claim on the intervals and zeros follows from the main proof.

For $(iv)'$ in Definition 0.15, with $Supp(\rho)\subset B(\overline{0},M)$, $M>1$, $|\rho|\leq N$, if $|(x,y,z)|\geq 2M$, and $(x',y',z')\in Supp(\rho)$, $|(x,y,z)-(x',y',z)|\geq \frac{|(x,y,z)}{2}$, so that $\frac{1}{|(x,y,z)-(x',y',z')|}\leq\frac{2}{|(x,y,z)|}$, and;

$|f(x,y,z)|=|\int_{Supp(\rho)}\frac{\rho(x',y',z')}{|(x,y,z)-(x',y',z')|}dx'dy'dz'|$

$\leq\frac{2}{|(x,y,z)|}\int_{Supp(\rho)}|\rho(x',y',z')|dx'dy'dz'$

$\leq\frac{8\pi M^3N}{3|(x,y)|}$

For $(v)'$, we have that, combining results above, with $i+j+k\geq 1$, $(x,y,z)\notin Supp(\rho)$;

$\frac{\partial^{i+j+k}f}{\partial x^i\partial y^j\partial z^k}(x,y,z)=\int_{\mathcal{R}^3}\frac{\rho(x',y',z')p(x-x',y-y',z-z')}{|(x,y,z)-(x',y',z')|^{1+2(i+j+k)}}dx'dy'dz'$

where p is homogeneous of degree $i+j+k$. Then, if $|(x,y,z)|\geq max(2M,1)$, and $(x',y',z')\in Supp(\rho)$;

$\frac{1}{|(x,y,z)-(x',y',z')|^{1+2(i+j+k)}}\leq\frac{2^{1+2(i+j+k)}}{|(x,y,z)|^{1+2(i+j+k)}}$

$|x-x'|\leq|(x,y,z)-(x',y',z')|\leq|(x,y,z)|+M$, and similarly $|y-y'|\leq|(x,y,z)|+M$, $|z-z'|\leq|(x,y,z)|+M$, so that;

$|p(x-x',y-y',z-z')|\leq T(|(x,y,z)|+M)^{i+j+k}$

where $T=\sum_{i'+j'+k'=i+j+k}|a_{i'j'k'}|$ and $p=\sum_{i'+j'+k'=i+j+k}a_{i'j'k'}x_1^{i'}x_2^{j'}x_3^{k'}$. It follows that;

$$|\frac{\partial^{i+j+k}f}{\partial x^i\partial y^j\partial z^k}|=|\int_{Supp(\rho)}\frac{\rho(x',y',z')p(x-x',y-y',z-z')}{|(x,y,z)-(x',y',z')|^{1+2(i+j+k)}}dx'dy'dz'|$$

$$\leq\frac{2^{1+2(i+j+k)}T(|(x,y,z)|+M)^{i+j+k}}{|(x,y,z)|^{1+2(i+j+k)}}\int_{Supp(\rho)}|\rho(x',y',z')|dx'dy'dz'$$

$$\leq\frac{2^{1+2(i+j+k)}TN\frac{4\pi M^3}{3}(|(x,y,z)|+M)^{i+j+k}}{|(x,y,z)|^{1+2(i+j+k)}}$$

$$\leq\frac{2^{1+2(i+j+k)}TN\frac{4\pi M^3}{3}(i+j+k+1)!M^{i+j+k}}{|(x,y,z)|^{1+(i+j+k)}}$$

so that $\frac{\partial^{i+j+k}f}{\partial x^i\partial y^j\partial z^k}(x,y,z)$ is of moderate decrease $i+j+k+1$.

For $(vi)'$, we have that if $\rho\geq 0$ is continuous with compact support, $\rho\neq 0$, that $f>0$. Repeating the argument above, and the calculation of the derivatives, fixing $(y,z)\in\mathcal{R}^2$, and letting $s=(x^2+y^2)^{\frac{1}{2}}$, we see that the zeros of $f'_{x,y}$ are contained in the interval $(-(M+1),M+1)$, with the length $2(M+1)$ of the interval, uniformly bounded in (y,z), the zeros of $(f_{x,y})''$ are contained in the intervals $(-M+\frac{1}{\sqrt{2}}(s-M),M+\frac{1}{\sqrt{2}}(s+M))\cup(-M-\frac{1}{\sqrt{2}}(s+M),M-\frac{1}{\sqrt{2}}(s-M))$, with the length of the intervals, $(2+\sqrt{2})M$ uniform in $s>2M$, the zeros of $(f_{x,y})''$ are contained in the intervals $(-M+\frac{\sqrt{3}}{\sqrt{2}}(s-M),M+\frac{\sqrt{3}}{\sqrt{2}}(s+M))\cup(-M-\frac{\sqrt{3}}{\sqrt{2}}(s+M),M-\frac{\sqrt{3}}{\sqrt{2}}(s-M))$, with the length of the intervals, $(2+\sqrt{6})M$ uniform in $s>2M$, the zeros of $(f_{x,y})'''$ are contained in the intervals $(-M+\sqrt{6+\sqrt{30}}(s-M),M+\sqrt{6+\sqrt{30}}(s+M))\cup(-M-\sqrt{6+\sqrt{30}}(s+M),M-\sqrt{6+\sqrt{30}}(s-M))$, $(-M+\sqrt{6-\sqrt{30}}(s-M),M+\sqrt{6-\sqrt{30}}(s+M))\cup(-M-\sqrt{6-\sqrt{30}}(s+M),M-\sqrt{6-\sqrt{30}}(s-M))$ with the length of the intervals, $(2+2\sqrt{6+\sqrt{30}})M$ and $(2+2\sqrt{6-\sqrt{30}})M$ uniform in $s>2M$.

Again the proof that if $\rho\neq 0$ is smooth with compact support, then f is quasi split normal, follows easily by observing that $\rho=\rho^++\rho^-$, with $\{\rho^+,\rho^-\}$ being continuous, $\rho^+\geq 0$, $\rho^-\leq 0$, and using the proof for quasi normality, along with the previous observation in dimension 2, that quasi normality implies quasi split normality.

$\square$

Lemma 0.17. *Let $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ be normal, then, for $\{x, y, z\} \subset \mathcal{R}$, $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$;*

$A(k_1, y, z) = lim_{r\rightarrow\infty} \int_{-r}^{r} f(x, y, z)e^{-ik_1x}dx$

$B(x, k_2, z) = lim_{r\rightarrow\infty} \int_{-r}^{r} f(x, y, z)e^{-ik_2y}dy$

$C(x, y, k_3) = lim_{r\rightarrow\infty} \int_{-r}^{r} f(x, y, z)e^{-ik_3z}dz$

all exist and $A(k_1, y, z), B(x, k_2, z), C(x, y, k_3)$ are of moderate decrease 3*.*

and, for $\{x, y, z\} \subset \mathcal{R}$, $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$;

$F(k_1, k_2, z) = lim_{r,s\rightarrow\infty} \int_{-r}^{r} \int_{-s}^{s} f(x, y, z)e^{-ik_1x}e^{-ik_2y}dxdy$

$G(k_1, y, k_3) = lim_{r,s\rightarrow\infty} \int_{-r}^{r} \int_{-s}^{s} f(x, y, z)e^{-ik_1x}e^{-ik_3z}dxdz$

$H(x, k_2, k_3) = lim_{r,s\rightarrow\infty} \int_{-r}^{r} \int_{-s}^{s} f(x, y, z)e^{-ik_2y}e^{-ik_3z}dydz$

all exist and $F(k_1, k_2, z), G(k_1, y, k_3), H(x, k_2, k_3)$ are of moderate decrease.

Moreover;

$F(k_1, k_2, z) = \int_{-\infty}^{\infty} A(k_1, y, z)e^{-ik_2y}dy$

and corresponding results hold for $\{A, B, C, F, G, H\}$, integrating out the variables in a similar way.

Proof. The first claim follows from [1] together with Lemma 0.5 and $(i), (ii), (iii)$ in Definition 0.15, using the fact that, by normality, $f_{x,y}(z), f_{x,z}(y), f_{y,z}(x)$ are analytic at infinity, for $\{(x, y), (x, z), (y, z)\} \subset \mathcal{R}^2$, $(*)$. We then have, using integration by parts, for $(y_0, z_0) \in \mathcal{R}^2$;

$\int_{-r}^{r} f(x, y_0, z_0)e^{-ik_1x}dx = [\frac{if(x,y_0)e^{-ik_1x}}{k_1}]_{-r}^{r} - \int_{-r}^{r} \frac{i\frac{\partial f}{\partial x}(x,y_0,z_0)e^{-ik_1x}}{k_1}dx$

so that, as f_{y_0,z_0} is of very moderate decrease, by (iv) in Definition 0.15, integrating by parts;

$A(k_1,y_0,z_0)=lim_{r\to\infty}\int_{-r}^{r}f(x,y_0,z_0)e^{-ik_1x}dx$

$=lim_{r\to\infty}([\frac{if(x,y_0,z_0)e^{-ik_1x}}{k_1}]_{-r}^{r}-\int_{-r}^{r}\frac{i\frac{\partial f}{\partial x}(x,y_0,z_0)e^{-ik_1x}}{k_1}dx)$

$=lim_{r\to\infty}(-\int_{-r}^{r}\frac{i\frac{\partial f}{\partial x}(x,y_0,z_0)e^{-ik_1x}}{k_1}dx)$

$=-\frac{i}{k_1}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}(x,y_0,z_0)e^{-ik_1x}dx$

the last integral being definite, as $(\frac{\partial f}{\partial x})_{y_0,z_0}$ is of moderate decrease, using (v) in Definition 0.15. It follows that $A(k_1,y,z)$ is smooth, as differentiating under the integral sign is justified by the DCT, MVT and (v) again, with;

$\frac{\partial^{j+k}A}{\partial y^j\partial z^k}(k_1,y,z)=-\frac{i}{k_1}\int_{-\infty}^{\infty}\frac{\partial^{1+j+k}f}{\partial x\partial y^j\partial z^k}(x,y,z)e^{-ik_1x}dx$

Integrating by parts again;

$A(k_1,y_0,z_0)=-\frac{1}{k_1^2}\int_{-\infty}^{\infty}\frac{\partial^2 f}{\partial x^2}(x,y_0,z_0)e^{-ik_1x}dx$

with $\frac{\partial^2 f}{\partial x^2}$ of moderate decrease 3, by (v) again . We have that $(\frac{\partial^2 f}{\partial x^2})_{y_0,z_0}$ is analytic at infinity by $(*)$ and Lemma 0.4. By (vi) in the Definition 0.15, we can assume that for sufficiently large (y,z), $(\frac{\partial^2 f}{\partial x^2})_{y,z}$ is monotone and positive/negative in the intervals $(-\infty,a_1(y,z)),\ldots,(a_{val}(y,z),\infty)$, where $a_1,\ldots,a_{val}$ vary continuously with y,z. Splitting the integral into $cos(k_1x)$ and $sin(k_1x)$ components, in a similar calculation to [2], for any interval of length at least $\frac{2\pi}{|k_1|}$, we obtain an alternating cancellation in the contribution to the integral of at most $\frac{2\pi||f_y'||}{|k_1|}$ and for any interval of length at most $\frac{2\pi}{|k_1|}$, we obtain a contribution to the integral of at most $\frac{4\pi||f_y'||}{|k_1|}$. It follows that, for sufficiently large (y,z), using the fact that $\frac{\partial^2 f}{\partial x^2}$ has moderate decrease 3;

$|-\frac{i}{k_1}\int_{-\infty}^{\infty}\frac{\partial^2 f}{\partial x^2}(x,y,z)e^{-ik_1x}dx|$

$\leq\frac{(val+1)}{|k_1|}\frac{4\pi||f_{y,z}''||}{|k_1|}$

$=\frac{4\pi(val+1)||f_{y,z}''||}{|k_1|^2}$

$\leq\frac{4\pi(val+1)C}{|k_1|^2|(y,z)|^3}$ $(*)$

so that;

$|A(k_1,y,z)| \leq \frac{D}{|y,z|^3}$ $(**)$

for sufficiently large (y,z), with $D = \frac{4\pi(val+1)C}{|k_1|^2}$. As $A(k_1,y,z)$ is smooth, we obtain that $A(k_1,y,z)$ is of moderate decrease 3. Similarly, we can show that $B(x,k_2,z)$ and $C(x,y,k_3)$ are of moderate decrease 3, for $k_2 \neq 0$, $k_3 \neq 0$.

The second claim follows from Lemma 0.12, using normality of the fibres, $(i),(ii),(iii)$ in Definition 0.15. In fact, the first integral is indefinite, in the sense that we could define it as;

$F(k_1,k_2,z) = lim_{r_1\rightarrow\infty,r_2\rightarrow\infty,s_1\rightarrow\infty,s_2\rightarrow\infty}(\int_{-r_1}^{a} + \int_{a}^{r_2})(\int_{-s_1}^{b} + \int_{a}^{s_2})f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy$

for a choice of $a,b \subset \mathcal{R}$, and similarly for $G(k_1,y,k_3), H(x,k_2,k_3)$. Now observe that;

$F(k_1,k_2,z) = \int_{-\infty}^{\infty} A(k_1,y,z)e^{-ik_2y}dy$

where A_{k_1} is from the first part of the lemma, and of moderate decrease 3. This follows from the result of Lemma 0.12, and the fact that the fibre f_z is normal. We claim that $F(k_1,k_2,z)$ is of moderate decrease 2. We have that $|A_{k_1}(y,z)| \leq \frac{C_{k_1}}{|y,z|^3}$ for $|(y,z) > 1$, by $(**)$. It follows that, for sufficiently large z;

$|F(k_1,k_2,z)| \leq |\int_{-\infty}^{\infty} A(k_1,y,z)e^{-ik_2y}dy|$

$\leq \int_{-\infty}^{\infty} |\frac{C_{k_1}}{(y^2+z^2)^{\frac{3}{2}}}|dy$

$= \frac{C_{k_1}}{|z|^3}\int_{-\infty}^{\infty} |\frac{C_{k_1}}{(1+\frac{y^2}{z^2})^{\frac{3}{2}}}|dy$

$= \frac{C_{k_1}}{|z|^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{sec^3(\theta)}zsec^2(\theta)d\theta$, $((tan(\theta) = \frac{y}{z}), dy = zsec^2(\theta)d\theta)$

$= \frac{C_{k_1}}{|z|^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{sec(\theta)}d\theta$

$\leq \frac{C_{k_1}}{|z|^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(\theta)d\theta$

$\leq \frac{\pi C_{k_1}}{|z|^2}$

so that $F(k_1,k_2,z)$ is of moderate decrease. Similar results hold for $G(x,k_2,z)$ and $H(x,y,k_3)$, with $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$.

□

Lemma 0.18. *Let $f:\mathcal{R}^3\rightarrow\mathcal{R}$, the same result as Lemma 0.17 holds, if f is quasi normal or f is quasi split normal.*

Proof. Again, we just have to replace the uses of $(i)-(vi)$ in Definition 0.15, within Lemma 0.17, with the use of $(i)-(vi)'$ or $(i)-(vi)''$. The method of replacing (vi) by $(vi)'$ is given in Lemma 0.11. The fact that we can replace $(i),(ii),(iii)$ by $(i)',(ii),(iii)'$ at the beginning of the proof of the second claim follows from Lemma 0.14, and at the beginning of the proof from Definition 0.7. The use of $(iv),(v)$ and $(iv)'(v')$ is the same. A similar argument works in the quasi split normal case, using the argument at the end of Lemma 0.11.

□

Lemma 0.19. *Let hypotheses and notation be as in the previous lemma, then we can define, for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$;*

$$A(k_1,k_2,k_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(k_1,y,z)e^{-ik_2y}e^{-ik_3z}dydz$$

$$B(k_1,k_2,k_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}B(x,k_2,z)e^{-ik_1x}e^{-ik_3z}dxdz$$

$$C(k_1,k_2,k_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}B(x,y,k_3)e^{-ik_1x}e^{-ik_2y}dxdy$$

$$F(k_1,k_2,k_3)=\int_{-\infty}^{\infty}F(k_1,k_2,z)e^{-ik_3z}dz$$

$$G(k_1,k_2,k_3)=\int_{-\infty}^{\infty}G(k_1,y,k_3)e^{-ik_2y}dy$$

$$H(k_1,k_2,k_3)=\int_{-\infty}^{\infty}H(x,k_2,k_3)e^{-ik_1x}dx$$

We have that;

$A(k_1,k_2,k_3)=B(k_1,k_2,k_3)=C(k_1,k_2,k_3)=F(k_1,k_2,k_3)=G(k_1,k_2,k_3)=H(k_1,k_2,k_3)$ †)

$$=lim_{r\rightarrow\infty,s\rightarrow\infty,t\rightarrow\infty}\int_{-r}^{r}\int_{-s}^{s}\int_{-t}^{t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz\ (\dagger\dagger)$$

Proof. The definitions follows from Lemma 0.17, using the fact that $A(k_1,y,z)$ is of moderate decrease 3 and smooth, so it belongs to $L^1(\mathcal{R}^2)$. We can then use the usual Fourier transform. Similarly, for

$B(x,k_2,z)$ and $C(x,y,k_3)$.

Similarly, as $F(k_1,k_2,z)$ is smooth and of moderate decrease, we can then define the usual Fourier transform, for $k_3\neq 0$;

$F(k_1,k_2,k_3)=\int_{-\infty}^{\infty}F(k_1,k_2,z)e^{-ik_3z}dz$

It is clear that (†) holds, from the last claim in Lemma 0.17, once we have shown that $F=G=H$ and (††).

We have that;

$|\int_{-r}^{r}(F(k_1,k_2,z)-\int_{-s}^{s}\int_{-t}^{t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$=|\int_{-r}^{r}(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy-\int_{-s}^{s}\int_{-t}^{t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$=|\int_{-r}^{r}(\int_{(|x|\leq s,|y\leq t)^c}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$=|\int_{-r}^{r}[\int_{|y|>t}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy$

$+\int_{|x|>s}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy$

$-\int_{|x|>s}\int_{|y|>t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy]e^{-ik_3z}dz|$

$\leq|\int_{-r}^{r}(\int_{|y|>t}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$ (i)

$+|\int_{-r}^{r}(\int_{|x|>s}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$ (ii)

$+|\int_{-r}^{r}(\int_{|x|>s}\int_{|y|>t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$ (iii)

We estimate the three terms separately, using integration by parts, for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$. For (i), as f is of very moderate decrease, $\frac{\partial^3 f}{\partial x^3}$ is of moderate decrease 4;

$|\int_{-r}^{r}(\int_{|y|>t}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$=|\frac{-i}{k_1}\int_{-r}^{r}(\int_{|y|>t}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$=|-\frac{1}{k_1^2}\int_{-r}^{r}(\int_{|y|>t}\int_{-\infty}^{\infty}\frac{\partial^2 f}{\partial x^2}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$= |\frac{i}{k_1^3}\int_{-r}^{r}(\int_{|y|>t}\int_{-\infty}^{\infty}\frac{\partial^3 f}{\partial x^3}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{1}{|k_1|^3}\int_{|y|>t}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}|\frac{\partial^3 f}{\partial x^3}|dxdzdy$

$\leq \frac{1}{|k_1|^3}\int_{|y|>t}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\frac{C}{(x^2+y^2+z^2)^2}dxdzdy$

$= \frac{1}{|k_1|^3}\int_{|y|>t}\int_{-\infty}^{\infty}\frac{1}{(y^2+z^2)^2}\int_{\infty}^{\infty}\frac{C}{(1+\frac{x^2}{y^2+z^2})^2}dxdzdy$

$= \frac{1}{|k_1|^3}\int_{|y|>t}\int_{-\infty}^{\infty}\frac{1}{(y^2+z^2)^2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{C}{sec^4(\theta)}sec^2(\theta)(y^2+z^2)^{\frac{1}{2}}d\theta dzdy$

$\leq \frac{C}{k_1^3}\int_{|y|>t}\int_{-\infty}^{\infty}\frac{1}{(y^2+z^2)^{\frac{3}{2}}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}Ccos^2(\theta)d\theta dzdy$

$\leq \frac{C\pi}{|k_1|^3}\int_{|y|>t}\int_{-\infty}^{\infty}\frac{1}{(y^2+z^2)^{\frac{3}{2}}}dzdy$

$= \frac{C\pi}{|k_1|^3}\int_{|y|>t}\frac{1}{|y|^3}\int_{-\infty}^{\infty}\frac{1}{(1+\frac{z^2}{y^2})^{\frac{3}{2}}}dzdy$

$\leq \frac{C\pi}{|k_1|^3}\int_{|y|>t}\frac{1}{|y|^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}cos^2(\theta)d\theta dy$

$\leq \frac{C\pi^2}{|k_1|^3}\int_{|y|>t}\frac{1}{|y|^2}dy$

$\leq \frac{2C\pi^2}{|k_1|^3 t}$ $(\dagger)$

For (ii), as f is of very moderate decrease, $\frac{\partial^3 f}{\partial y^3}$ is of moderate decrease 4, using the same argument;

$|\int_{-r}^{r}(\int_{|x|>s}\int_{-\infty}^{\infty}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz| \leq \frac{2C\pi^2}{|k_2|^3 s}$

For (iii), we have that, using integration by parts again;

$|\int_{-r}^{r}(\int_{|x|>s}\int_{|y|>t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$= |\int_{-r}^{r}(\int_{|x|>s}\frac{i}{k_2}(-f(x,t,z)+f(x,-t,z))e^{-ik_1x}dx$

$-\frac{i}{k_2}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq |\int_{-r}^{r}\int_{|x|>s}\frac{i}{k_2}(-f(x,t,z)+f(x,-t,z))e^{-ik_1x}e^{-ik_3z}dxdz|$

$+|\int_{-r}^{r}\frac{i}{k_2}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$= |\int_{-r}^{r}\frac{-1}{k_1k_2}(-f(s,-t,z)+f(-s,-t,z)+f(s,t,z)-f(-s,-t,z))e^{-ik_3z}dz$

$+\int_{-r}^{r}\int_{|x|>s}\frac{1}{k_1k_2}(-\frac{\partial f}{\partial x}(x,t,z)+\frac{\partial f}{\partial x}(x,-t,z))e^{-ik_1x}e^{-ik_3z}dxdz|$

$+|\int_{-r}^{r}\frac{i}{k_2}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq |\int_{-r}^{r}\frac{-1}{k_1k_2}(-f(s,-t,z)+f(-s,-t,z)+f(s,t,z)-f(-s,-t,z))e^{-ik_3z}dz|$

$+|\int_{-r}^{r}\int_{|x|>s}\frac{1}{k_1k_2}(-\frac{\partial f}{\partial x}(x,t,z)+\frac{\partial f}{\partial x}(x,-t,z))e^{-ik_1x}e^{-ik_3z}dxdz|$

$+|\int_{-r}^{r}\frac{i}{k_2}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$=|\int_{-r}^{r}\frac{-1}{k_1k_2}(-f(s,-t,z)+f(-s,-t,z)+f(s,t,z)-f(-s,-t,z))e^{-ik_3z}dz|$

$+|\int_{-r}^{r}\frac{i}{k_1^2k_2}(\frac{\partial f}{\partial x}(s,t,z)-\frac{\partial f}{\partial x}(-s,t,z)-\frac{\partial f}{\partial x}(s,-t,z)+\frac{\partial f}{\partial x}(-s,-t,z))e^{-ik_3z}dz$

$-\int_{-r}^{r}\frac{i}{k_1^2k_2}\int_{|x|>s}(-\frac{\partial^2 f}{\partial x^2}(x,t,z)+\frac{\partial^2 f}{\partial x^2}(x,-t,z))e^{-ik_1x}e^{-ik_3z}dxdz|$

$+|\int_{-r}^{r}\frac{i}{k_2}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{1}{|k_1k_2|}|\int_{-r}^{r}(-f(s,-t,z)+f(-s,-t,z)+f(s,t,z)-f(-s,-t,z))e^{-ik_3z}dz|$

(a)

$+\frac{1}{|k_1^2k_2|}|\int_{-r}^{r}(\frac{\partial f}{\partial x}(s,t,z)-\frac{\partial f}{\partial x}(-s,t,z)-\frac{\partial f}{\partial x}(s,-t,z)+\frac{\partial f}{\partial x}(-s,-t,z))e^{-ik_3z}dz|$

(b)

$+\frac{1}{|k_1^2k_2|}|\int_{-r}^{r}\int_{|x|>s}(-\frac{\partial^2 f}{\partial x^2}(x,t,z)+\frac{\partial^2 f}{\partial x^2}(x,-t,z))e^{-ik_1x}e^{-ik_3z}dxdz|\ (c)$

$+\frac{1}{|k_2|}|\int_{-r}^{r}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|\ (d)$

For $(a),(b)$, by the definition of normality, the fibres $\{f_{s,t},f_{-s,t},f_{s,-t},f_{-s,-t}\}$ are analytic at infinity, and of very moderate decrease, and, similarly, the fibres $\{\frac{\partial f}{\partial x}_{s,t},\frac{\partial f}{\partial x}_{-s,t},\frac{\partial f}{\partial x}_{s,-t},\frac{\partial f}{\partial x}_{-s,-t}\}$ are analytic at infinity, and of moderate decrease. Using the bound val on the zeros, uniform in (s,t), we can then repeat the calculation above 8 times, to obtain that, uniformly in r;

$(a)+(b)\leq \frac{4}{|k_1k_2k_3|}(val+1)4\pi max(||f_{s,t}||,||f_{-s,t}||,||f_{s,-t}||,||f_{-s,-t}||)$

$+\frac{4}{|k_1^2k_2k_3|}(val+1)4\pi max(||\frac{\partial f}{\partial x}_{s,t}||,||\frac{\partial f}{\partial x}_{-s,t}||,||\frac{\partial f}{\partial x}_{s,-t}||,||\frac{\partial f}{\partial x}_{-s,-t}||)$

$\leq \frac{4C}{|k_1k_2k_3||(s,t)|}(val+1)4\pi + \frac{4}{|k_1^2k_2k_3|(s,t)|^2}(val+1)4\pi$

For (c), we have that $\frac{\partial^2 f}{\partial x^2}$ is of moderate decrease 3, hence belongs to $L^1(\mathcal{R}^2)$, so we can repeat the calculation above, to obtain that, uniformly in r;

$(c) \leq \frac{1}{|k_1^2k_2|}|\int_{-\infty}^{\infty}\int_{|x|>s}| - \frac{\partial^2 f}{\partial x^2}(x,t,z) + \frac{\partial^2 f}{\partial x^2}(x,-t,z)|dxdz$

$\leq \frac{4C}{|k_1^2k_2|}\int_{|x|>s}\int_{-\infty}^{\infty}\frac{1}{(x^2+z^2+t^2)^{\frac{3}{2}}}dzdx$

$\leq \frac{4C\pi}{|k_1^2k_2|}\int_{|x|>s}\frac{1}{(x^2+t^2)}dx$

$\leq \frac{4C\pi}{|k_1^2k_2|}\int_{|x|>s}\frac{1}{x^2}dx$

$\leq \frac{8C\pi}{|k_1^2k_2|s}$

We also have that;

$|\int_{|x|>s}\frac{1}{(x^2+t^2)}dx \leq \frac{1}{t^2}\int_{-\infty}^{\infty}\frac{1}{1+\frac{x^2}{t^2}}dx$

$= \frac{1}{t}[tan^{-1}(\frac{x}{t})]_{-\infty}^{\infty}$

$= \frac{\pi}{t}$

so that;

$(c) \leq min(\frac{8C\pi}{|k_1^2k_2|s}, \frac{4C\pi^2}{|k_1^2k_2|t})$

$\leq \frac{4\sqrt{2}C\pi^2}{|k_1^2k_2||s,t|}$

For (d), we can combine $(a),(b),(c)$ to obtain that;

$|\int_{-r}^{r}(\int_{|x|>s}\int_{|y|>t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{4C}{|k_1k_2k_3||(s,t)|}(val+1)4\pi + \frac{4}{|k_1^2k_2k_3|(s,t)|^2}(val+1)4\pi + \frac{4\sqrt{2}C\pi^2}{|k_1^2k_2||s,t|}$

$+\frac{1}{|k_2|}|\int_{-r}^{r}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{E}{|(s,t)|} + \frac{1}{|k_2|}|\int_{-r}^{r}\int_{|x|>s}\int_{|y|>t}\frac{\partial f}{\partial y}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$|(s,t)|>1$

so that repeating the above argument with $\frac{\partial f}{\partial y}$ replacing f;

$(d)\leq \frac{F}{|(s,t)||k_2|}+\frac{1}{|k_2|^2}|\int_{-r}^{r}\int_{|x|>s}\int_{|y|>t}\frac{\partial^2 f}{\partial y^2}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

and, nesting the arguments, uniformly in r;

$|\int_{-r}^{r}(\int_{|x|>s}\int_{|y|>t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{E}{|(s,t)|}+\frac{F}{|(s,t)||k_2|}+\frac{G}{|s,t||k_2|^2}+\frac{1}{|k_2|^3}|\int_{-r}^{r}\int_{|x|>s}\int_{|y|>t}\frac{\partial^3 f}{\partial y^3}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$(\dagger\dagger)$

Now, we can use the fact that $\frac{\partial^3 f}{\partial y^3}$ is of moderate decrease 4, to see that $\frac{\partial^3 f}{\partial y^3}\in L^1(\mathcal{R}^3)$, so that, uniformly in r, repeating the argument $(\dagger)$;

$|\int_{-r}^{r}\int_{|x|>s}\int_{|y|>t}\frac{\partial^3 f}{\partial y^3}(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \int_{-\infty}^{\infty}\int_{|x|>s}\int_{|y|>t}|\frac{\partial^3 f}{\partial y^3}|(x,y,z)dxdydz$

$\leq min(\int_{|x|>s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\frac{\partial^3 f}{\partial y^3}|(x,y,z)dydzdx,\int_{|y|>t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\frac{\partial^3 f}{\partial y^3}|(x,y,z)dxdzdy)$

$\leq min(\frac{2C\pi^2}{s},\frac{2C\pi^2}{t})$

$\leq \frac{2\sqrt{2}C\pi^2}{|(s,t)|}$

Now, from $(\dagger\dagger)$, we obtain that, uniformly in r;

$|\int_{-r}^{r}(\int_{|x|>s}\int_{|y|>t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{H}{|(s,t)|}$

which is (iii). Combining with $(i),(ii)$, we obtain that, uniformly in r;

$|\int_{-r}^{r}(F(k_1,k_2,z)-\int_{-s}^{s}\int_{-t}^{t}f(x,y,z)e^{-ik_1x}e^{-ik_2y}dxdy)e^{-ik_3z}dz|$

$\leq \frac{A_{k_1k_2k_3}}{s}+\frac{B_{k_1k_2k_3}}{t}+\frac{C_{k_1k_2k_3}}{|(s,t)|}$

where the constants $\{A_{k_1k_2k_3}, B_{k_1k_2k_3}, C_{k_1k_2k_3}\} \subset \mathcal{R}_{>0}$ can be read from the proof.

Applying the Moore-Osgood Theorem, it is then clear that;

$F(k_1, k_2, k_3) = C(k_1, k_2, k_3)$

$= lim_{r\to\infty,s\to\infty,t\to\infty} \int_{-r}^{r} \int_{-s}^{s} \int_{-t}^{t} f(x, y, z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

and similarly;

$G(k_1, k_2, k_3) = B(k_1, k_2, k_3)$

$= lim_{r\to\infty,s\to\infty,t\to\infty} \int_{-r}^{r} \int_{-s}^{s} \int_{-t}^{t} f(x, y, z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$H(k_1, k_2, k_3) = A(k_1, k_2, k_3)$

$= lim_{r\to\infty,s\to\infty,t\to\infty} \int_{-r}^{r} \int_{-s}^{s} \int_{-t}^{t} f(x, y, z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

□

Lemma 0.20. *Let hypotheses and notation be as in Lemma 0.19, then we have that there exist constants* $\{D_{k_1k_2k_3}, E_{k_1k_2k_3}, F_{k_1k_2k_3}\} \subset \mathcal{R}_{>0}$ *such that;*

$|s_{s,t,r} - s| \leq \frac{D_{k_1k_2k_3}}{s} + \frac{E_{k_1k_2k_3}}{t} + \frac{F_{k_1k_2k_3}}{r}$

In particular, there exists a constant $G_{k_1k_2k_3} \in \mathcal{R}_{>0}$ *such that;*

$|s_{m,m,m} - s| \leq \frac{G_{k_1k_2k_3}}{m}$

Proof. We have that, by the proof of Lemma 0.12;

$|s_{s,t,r} - s| \leq |s_{s,t,r} - s_{\infty,\infty,r}| + |s_{\infty,\infty,r} - s|$

$\leq \frac{A_{k_1k_2k_3}}{s} + \frac{B_{k_1k_2k_3}}{t} + \frac{C_{k_1k_2k_3}}{|(s,t)|} + |\int_{|z|\geq r} F(k_1, k_2, z)e^{-ik_3z}dz|$

$\leq \frac{A_{k_1k_2k_3}}{s} + \frac{B_{k_1k_2k_3}+C_{k_1k_2k_3}}{t} + \int_{|z|\geq r} |F(k_1, k_2, z)|dz$

where, by the result of Lemma 0.17, $F(k_1, k_2, z)$ is of moderate decrease;

$|F(k_1,k_2,z)|\leq \frac{D}{|z|^2}$

for sufficiently large z, with $D=\pi C_{k_1}$. It follows that, for sufficiently large r;

$|s_{s,t,r}-s|\leq \frac{A_{k_1k_2k_3}}{s}+\frac{B_{k_1k_2k_3}+C_{k_1k_2k_3}}{t}+\int_{|y|\geq r}\frac{D}{z^2}dz$

$\leq \frac{A_{k_1k_2k_3}}{s}+\frac{B_{k_1k_2k_3}+C_{k_1k_2k_3}}{t}+\frac{2D}{r}$

$=\frac{D_{k_1k_2k_3}}{s}+\frac{E_{k_1k_2k_3}}{t}+\frac{F_{k_1k_2k_3}}{r}$

where;

$D_{k_1k_2k_3}=A_{k_1k_2k_3}$

$E_{k_1k_2k_3}=B_{k_1k_2k_3}+C_{k_1k_2k_3}$

$F_{k_1k_2k_3}=2\pi C_{k_1}$

For the next claim, we can take $G_{k_1k_2k_3}=D_{k_1k_2k_3}+E_{k_1k_2k_3}+F_{k_1k_2k_3}$

$\square$

Lemma 0.21. *If $f:\mathcal{R}^3\rightarrow\mathcal{R}$, the same results as Lemma 0.19 and Lemma 0.20 hold, with the assumption that f is quasi normal or quasi split normal.*

Proof. Again, we can replace the use of $(i)-(vi)$ in Definition 0.15, within the proof of Lemma 0.19, by $(i)'-(vi)'$ or $(i)''-(vi)''$, with the argument used in Lemma 0.18. The argument of Lemma 0.20 then goes through.

$\square$

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