A FOURIER INVERSION THEOREM FOR NORMAL FUNCTIONS

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Abstract. This paper proves an inversion theorem for the Fourier transform defined in [2], applied to the class of normal functions.

We recall the definition of the Fourier transform for quasi split normal functions, which includes normal functions, introduced in the paper [2], normalised by the factor $\frac{1}{2\pi}$ in dimension 2, and by $\frac{1}{(2\pi)^{\frac{3}{2}}}$ in dimension 3, which we denote by $\mathcal{F}$. The aim of this paper is to prove an inversion theorem for such functions. We first have the following;

Lemma 0.1. *Let $f:\mathcal{R}^2\rightarrow\mathcal{R}$ be smooth and quasi split normal, then $\mathcal{F}(f)\in L^1(\mathcal{R}^2)$ and is of rapid decay, in the sense that, for $|\overline{k}|>1$, $k_1\neq 0, k_2\neq 0$*

$|\mathcal{F}(f)(\overline{k})|\leq\frac{C_n}{|\overline{k}|^n}$

where $C_n\in\mathcal{R}$, $n\in\mathcal{N}$.

A similar result holds for smooth quasi split normal $f:\mathcal{R}^3\rightarrow\mathcal{R}$, with $\mathcal{F}(f)\in L^1(\mathcal{R}^3)$, and for $|\overline{k}|>1$, $k_1\neq 0, k_2\neq 0, k_3\neq 0$

$|\mathcal{F}(f)(\overline{k})|\leq\frac{C_n}{|\overline{k}|^n}$

where $C_n\in\mathcal{R}$, $n\in\mathcal{N}$.

Proof. In dimension 2, by [2], we have that integration by parts is justified, for $k_1\neq 0$, $k_2\neq 0$, and we obtain that;

$\mathcal{F}(\bigtriangledown^2(f))(\overline{k}=-k^2\mathcal{F}(f)(\overline{k})$

$\mathcal{F}((\bigtriangledown^2)^n f)=-k^{2n}\mathcal{F}(f)(\overline{k})$ $(*)$

By the definition of quasi split normality, $(\bigtriangledown^2)^n f$ is of moderate decrease $2n+1$ and smooth, so that for $n\geq 1$, $(\bigtriangledown^2)^n f\in L^1(\mathcal{R}^2)$, and we have the trivial bound;

$|\mathcal{F}((\bigtriangledown^2)^n f)|\leq\frac{||(\bigtriangledown^2)^n f||_{L^1(\mathcal{R}^2)}}{2\pi}=C_{2n}$

Rearranging $(*)$, we obtain that, for $|\overline{k}|>1$, $k_1\neq 0$, $k_2\neq 0$;

$|\mathcal{F}(f)(\overline{k})|\leq\frac{C_{2n}}{k^{2n}}\leq\frac{C_{2n}}{|k|^m}$, for $1\leq m\leq 2n$.

The proof for $f:\mathcal{R}^3\rightarrow\mathcal{R}$ is similar, noting that $(\bigtriangledown^2)^n f\in L^1(\mathcal{R}^3)$, for $n\geq 2$, and repeating the argument in three variables.

We have that, by the definition of quasi split normality, for $f:\mathcal{R}^2\rightarrow\mathcal{R}$, $\{\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\}$ are of moderate decrease 2, and smooth, so belong to $L^{\frac{3}{2}}(\mathcal{R}^2)$. By the Haussdorff-Young inequality, using the fact that $1\leq\frac{3}{2}\leq 2$, we have that $\{\mathcal{F}(\frac{\partial f}{\partial x}),\mathcal{F}(\frac{\partial f}{\partial y})\}\subset L^3(\mathcal{R}^2)$, in particularly $\{\mathcal{F}(\frac{\partial f}{\partial x}),\mathcal{F}(\frac{\partial f}{\partial y}),|\mathcal{F}(\frac{\partial f}{\partial x})|+|\mathcal{F}(\frac{\partial f}{\partial y})|\}\subset L^3(B(\overline{0},1))$. A simple integration using polar coordinates, shows that $\frac{1}{k}\in L^{\frac{3}{2}}(B(\overline{0},1))$. As above, we have that, for $k_1\neq 0,k_2\neq 0$;

$\mathcal{F}(f)(\overline{k})=\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{ik_1}=\frac{\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})}{ik_2}$ (A)

Observe that;

$\frac{1}{k}=\frac{1}{|k_1|}\frac{1}{(1+\frac{k_2^2}{k_1^2})^{\frac{1}{2}}}=\frac{1}{|k_2|}\frac{1}{(1+\frac{k_1^2}{k_2^2})^{\frac{1}{2}}}$

and;

$1\leq(1+\frac{k_1^2}{k_2^2})^{\frac{1}{2}}\leq\sqrt{2}$, for $|k_1|\leq|k_2|$

$1\leq(1+\frac{k_2^2}{k_1^2})^{\frac{1}{2}}\leq\sqrt{2}$, for $|k_2|\leq|k_1|$

so that $\frac{1}{|k_1|}\leq\frac{\sqrt{2}}{k}$, for $|k_2|\leq|k_1|$, $\frac{1}{|k_2|}\leq\frac{2}{k}$, for $|k_1|\leq|k_2|$, the cases being exhaustive, (B). Combining $(A),(B)$, we obtain that;

$|\mathcal{F}(f)(\overline{k})|\leq\sqrt{2}|\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|}{k}|$, for $|k_2|\leq|k_1|$

$|\mathcal{F}(f)(\overline{k})| \leq \sqrt{2}|\frac{\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})|}{k}|$, for $|k_1| \leq |k_2|$

$|\mathcal{F}(f)(\overline{k})| \leq \sqrt{2}\frac{max(|\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|,|\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})|)}{k}$

$\leq \frac{\sqrt{2}(|\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|+|\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})|)}{k}$

By Holder's inequality, we have that;

$\frac{\sqrt{2}(|\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|+|\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})|)}{k} \in L^1(B(\overline{0},1))$

so that $\mathcal{F}(f)(\overline{k}) \in L^1(B(\overline{0},1))$. By the rapid decrease of $\mathcal{F}(f)$, for $|\overline{k}| > 1$, we have that $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^2 \setminus B(\overline{0},1))$, so that $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^2)$.

For $f : \mathcal{R}^3 \rightarrow \mathcal{R}$, $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\}$ are of moderate decrease 2, and smooth, so belong to $L^2(\mathcal{R}^3)$, and by classical theory;

$\{\mathcal{F}(\frac{\partial f}{\partial x}), \mathcal{F}(\frac{\partial f}{\partial y}), \mathcal{F}(\frac{\partial f}{\partial z}), |\mathcal{F}(\frac{\partial f}{\partial x})| + |\mathcal{F}(\frac{\partial f}{\partial y})| + |\mathcal{F}(\frac{\partial f}{\partial z})|\} \subset L^2(\mathcal{R}^3)$

as well. In particular;

$\{\mathcal{F}(\frac{\partial f}{\partial x}), \mathcal{F}(\frac{\partial f}{\partial y}), \mathcal{F}(\frac{\partial f}{\partial z}), |\mathcal{F}(\frac{\partial f}{\partial x})| + |\mathcal{F}(\frac{\partial f}{\partial y})| + |\mathcal{F}(\frac{\partial f}{\partial z})|\} \subset L^2(B(\overline{0},1))$

A simple integration using polar coordinates, shows that $\frac{1}{k} \in L^2(B(\overline{0},1))$. As above, we have that, for $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0$;

$\mathcal{F}(f)(\overline{k}) = \frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{ik_1} = \frac{\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})}{ik_2} = \frac{\mathcal{F}(\frac{\partial f}{\partial z})(\overline{k})}{ik_3}$ (AA)

Observe that;

$\frac{1}{k} = \frac{1}{|k_1|}\frac{1}{(1+\frac{k_2^2}{k_1^2}+\frac{k_3^2}{k_1^2})^{\frac{1}{2}}} = \frac{1}{|k_2|}\frac{1}{(1+\frac{k_1^2}{k_2^2}+\frac{k_3^2}{k_2^2})^{\frac{1}{2}}} = \frac{1}{|k_3|}\frac{1}{(1+\frac{k_1^2}{k_3^2}+\frac{k_2^2}{k_3^2})^{\frac{1}{2}}}$

and;

$1 \leq (1 + \frac{k_1^2}{k_2^2} + \frac{k_3^2}{k_2^2})^{\frac{1}{2}} \leq \sqrt{3}$, for $max(|k_1|, |k_3|) \leq |k_2|$

$1 \leq (1 + \frac{k_2^2}{k_1^2} + \frac{k_3^2}{k_1^2})^{\frac{1}{2}} \leq \sqrt{3}$, for $max(|k_2|, |k_3|) \leq |k_1|$

$1 \leq (1 + \frac{k_1^2}{k_3^2} + \frac{k_2^2}{k_3^2})^{\frac{1}{2}} \leq \sqrt{3}$, for $max(|k_1|, |k_2|) \leq |k_3|$

so that $\frac{1}{|k_1|} \leq \frac{\sqrt{3}}{k}$, for $max(|k_2|, |k_3|) \leq |k_1|$, $\frac{1}{|k_2|} \leq \frac{\sqrt{3}}{k}$, for $max(|k_1|, |k_3|) \leq |k_2|$, $\frac{1}{|k_3|} \leq \frac{\sqrt{3}}{k}$, for $max(|k_1|, |k_2|) \leq |k_3|$ the cases being exhaustive, (BB). Combining $(AA), (BB)$, we obtain that;

$$|\mathcal{F}(f)(\overline{k})| \leq \sqrt{3}|\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|}{k}|, \text{ for } max(|k_2|, |k_3|) \leq |k_1|$$

$$|\mathcal{F}(f)(\overline{k})| \leq \sqrt{3}|\frac{\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})|}{k}|, \text{ for } max(|k_1|, |k_3|) \leq |k_2|$$

$$|\mathcal{F}(f)(\overline{k})| \leq \sqrt{3}|\frac{\mathcal{F}(\frac{\partial f}{\partial z})(\overline{k})|}{k}|, \text{ for } max(|k_1|, |k_2|) \leq |k_3|$$

$$|\mathcal{F}(f)(\overline{k})| \leq \sqrt{3}\frac{max(|\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|, |\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})|, |\mathcal{F}(\frac{\partial f}{\partial z})(\overline{k})|)}{k}$$

$$\leq \frac{\sqrt{3}(|\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})| + |\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})| + |\mathcal{F}(\frac{\partial f}{\partial z})(\overline{k})|)}{k}$$

By the Cauchy-Schwartz inequality, we have that;

$$\frac{\sqrt{3}(|\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})| + |\mathcal{F}(\frac{\partial f}{\partial y})(\overline{k})| + |\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})|)}{k} \in L^1(B(\overline{0}, 1))$$

so that $\mathcal{F}(f)(\overline{k}) \in L^1(B(\overline{0}, 1))$. By the rapid decrease of $\mathcal{F}(f)$, for $|\overline{k}| > 1$, we have that $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^3 \setminus B(\overline{0}, 1))$, so that $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^3)$.

□

Definition 0.2. *Let* $f \in C^\infty(\mathcal{R}^2)$ *be quasi split normal with* $\frac{\partial^{i_1+i_2} f}{\partial x^{i_1} \partial y^{i_2}}$ *bounded for* $0 \leq i_1 + i_2 \leq 27$. *Let* $C_m = \{(x, y) \in \mathcal{R}^2 : |x| \leq m, |y| \leq m\}$. *Let;*

$$Q_m = \mathcal{R}^2 \setminus (x = m \cup x = -m \cup y = m \cup y = -m)$$

$$C^{13,14,m}(\mathcal{R}^2) = \{h : \frac{\partial^{i+j} h}{\partial x^i \partial y^j}, 0 \leq i, j \leq 13, \ define\ continuous\ functions,$$

$$\frac{\partial^{i+14} h}{\partial x^i \partial y^{14}}, \frac{\partial^{i+14} h}{\partial x^{14} \partial y^i}, 0 \leq i \leq 13, \ define\ bounded\ functions\ on\ Q_m\}$$

Then we define an inflexionary approximation sequence $\{f_m : m \in \mathcal{N}\}$ *by the requirements;*

(i). $f_m \in C^{13,14,m}(\mathcal{R}^2)$

(ii). $f_m|_{C_m} = f|_{C_m}$

(iii) $f_m|_{(\mathcal{R}^2\setminus C_{m+\frac{1}{m^2}})} = 0$

Letting $g_m = f_m|_{[-m,m]\times[-m-\frac{1}{m^2},m+\frac{1}{m^2}]}$*;*

(iv). For $|x| \leq m$*, for* $0 \leq i \leq 13$*;*

$\frac{\partial^i g_m}{\partial y^i}|_{(x,m)} = \frac{\partial^i f}{\partial y^i}|_{(x,m)}$

$\frac{\partial^i g_m}{\partial y^i}|_{(x,-m)} = \frac{\partial^i f}{\partial y^i}|_{(x,-m)}$

$\frac{\partial^i g_m}{\partial y^i}|_{(x,m+\frac{1}{m})} = 0$

$\frac{\partial^i g_m}{\partial y^i}|_{(x,-m-\frac{1}{m})} = 0$

(v). For $|x| \leq m$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,m)} > 0$*,* $\frac{\partial^{14} g_m}{\partial y^{14}}|_{V_{x,m}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,m)} < 0$*,* $\frac{\partial^{14} g_m}{\partial y^{14}}|_{V_{x,m}} \leq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,-m)} > 0$*,* $\frac{\partial^{14} g_m}{\partial y^{14}}|_{V_{x,-m}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(x,-m)} < 0$*,* $\frac{\partial^{14} g_m}{\partial y^{14}}|_{V_{x,-m}} \leq 0$

The same property as $(iv), (v)$ *holding, replacing* f *and* g_m *with* $\frac{\partial^i f}{\partial x^i}$ *and* $\frac{\partial g_m}{\partial x^i}$*, for* $0 \leq i \leq 13$*.*

(vi). For $|y| \leq m + \frac{1}{m^2}$*,* $0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial x^i}|_{(m,y)} = \frac{\partial^i g_m}{\partial x^i}|_{(m,y)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m,y)} = \frac{\partial^i g_m}{\partial x^i}|_{(-m,y)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(m+\frac{1}{m},y)} = 0$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m-\frac{1}{m},y)} = 0$

(vii) For $|y| \leq m + \frac{1}{m^2}$

if $\frac{\partial^{14} g_m}{\partial x^{14}}|_{(m,y)} > 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y}} \geq 0$

if $\frac{\partial^{14} g_m}{\partial x^{14}}|_{(m,y)} < 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y}} \leq 0$

if $\frac{\partial^{14} g_m}{\partial x^{14}}|_{(-m,y)} > 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y}} \geq 0$

if $\frac{\partial^{14} g_m}{\partial x^{14}}|_{(-m,y)} < 0,\ \frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y}} \leq 0$

The same property as $(vi), (vii)$ *holding, replacing* f_m *and* g_m *with* $\frac{\partial^i f_m}{\partial y^i}$ *and* $\frac{\partial g_m}{\partial y^i}$, *for* $0 \leq i \leq 14$.

where;

$V_{x,m} = \{(x,y) \in \mathcal{R}^2 : y \in (m, m + \frac{1}{m^2})\}$

$V_{x,-m} = \{(x,y) \in \mathcal{R}^2 : y \in (-m - \frac{1}{m^2}, -m)\}$

$H_{m,y} = \{(x,y) \in \mathcal{R}^2 : x \in (m, m + \frac{1}{m^2})\}$

$H_{-m,y} = \{(x,y) \in \mathcal{R}^2 : x \in (-m - \frac{1}{m^2}, -m)\}$

Definition 0.3. *Let* $f \in C^\infty(\mathcal{R}^3)$ *be quasi split normal with* $\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}$ *bounded for* $0 \leq i_1 + i_2 + i_3 \leq 40$. *Let* $C_m = \{(x,y,z) \in \mathcal{R}^2 : |x| \leq m, |y| \leq m, |z| \leq m\}$. *Let;*

$Q_m = \mathcal{R}^3 \setminus (x = m \cup x = -m \cup y = m \cup y = -m \cup z = m \cup z = -m)$

$C^{13,13,14,m}(\mathcal{R}^3) = \{h : \frac{\partial^{i+j+k} h}{\partial x^i \partial y^j \partial z^k}, 0 \leq i,j,k \leq 13,\ define\ continuous\ functions,$

$\frac{\partial^{i+j+14} h}{\partial x^i \partial y^j \partial z^{14}}, \frac{\partial^{i+j+14} h}{\partial x^i \partial y^{14} \partial z^j}, \frac{\partial^{i+j+14} h}{\partial x^{14} \partial y^i \partial z^j}, 0 \leq i,j \leq 13,\ define\ bounded\ functions\ on\ Q_m\}$

Then we define an inflexionary approximation sequence $\{f_m : m \in \mathcal{N}\}$ *by the requirements;*

(i). $f_m \in C^{13,13,14}(\mathcal{R}^3)$

(ii). $f_m|_{C_m} = f|_{C_m}$

(iii) $f_m|_{(\mathcal{R}^3 \setminus C_{m+\frac{1}{m^3}})} = 0$

(iv). For $0 \leq |y| \leq m, 0 \leq |z| \leq m$, *for* $0 \leq i \leq 13$;

$\frac{\partial^i f_m}{\partial x^i}|_{(m,y,z)} = \frac{\partial^i f}{\partial x^i}|_{(m,y,z)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m,y,z)} = \frac{\partial^i f}{\partial x^i}|_{(-m,y,z)}$

$\frac{\partial^i f_m}{\partial x^i}|_{(m+\frac{1}{m},y,z)} = 0$

$\frac{\partial^i f_m}{\partial x^i}|_{(-m-\frac{1}{m},y,z)} = 0$

(v). For $0 \leq |y| \leq m, 0 \leq |z| \leq m$

if $\frac{\partial^{14} f}{\partial x^{14}}|_{(m,y,z)} > 0$, $\frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y,z}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(m,y,z)} < 0$, $\frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{m,y,z}} \leq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(-m,y,z)} > 0$, $\frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y,z}} \geq 0$

if $\frac{\partial^{14} f}{\partial y^{14}}|_{(-m,y,z)} < 0$, $\frac{\partial^{14} f_m}{\partial x^{14}}|_{H_{-m,y,z}} \leq 0$

(vi). For $0 \leq |x| \leq m + \frac{1}{m^3}$ $0 \leq |z| \leq m$, $0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial y^i}|_{(x,m,z)}$, $m \leq y \leq m + \frac{1}{m}$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial y^i}|_{(x,-m,z)}$, $-m - \frac{1}{m} \leq y \leq -m$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,m+\frac{1}{m^3},z)} = 0$

$\frac{\partial^i f_m}{\partial y^i}|_{(x,-m-\frac{1}{m^3},z)} = 0$

(vii) For $0 \leq |x| \leq m + \frac{1}{m^3}$, $0 \leq |z| \leq m$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,m,z)} > 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m,z}} \geq 0$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,m,z)} < 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,m,z}} \leq 0$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,-m,z)} > 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m,z}} \geq 0$

if $\frac{\partial^{14} f_m}{\partial y^{14}}|_{(x,-m,z)} < 0$, $\frac{\partial^{14} f_m}{\partial y^{14}}|_{V_{x,-m,z}} \leq 0$

(viii). For $0 \leq |x| \leq m + \frac{1}{m^3}$ $0 \leq |y| \leq m + \frac{1}{m^3}$, $0 \leq i \leq 13$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial z^i}|_{(x,y,m)}$, $m \leq z \leq m + \frac{1}{m^3}$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,z)} = \frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m)}$, $-m - \frac{1}{m^3} \leq z \leq -m$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m+\frac{1}{m^3})} = 0$

$\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m-\frac{1}{m^3})} = 0$

(ix) For $0 \leq |x| \leq m + \frac{1}{m^3}$, $0 \leq |y| \leq m + \frac{1}{m^3}$

if $\frac{\partial^{14} f_m}{\partial z^{14}}|_{(x,y,m)} > 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,m}} \geq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,m)} < 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,m}} \leq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,-m)} > 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,-m}} \geq 0$

if $\frac{\partial^{14} f}{\partial z^{14}}|_{(x,y,-m)} < 0$, $\frac{\partial^{14} f_m}{\partial z^{14}}|_{D_{x,y,-m}} \leq 0$

where;

$H_{m,y,z} = \{(x,y,z) \in \mathcal{R}^3 : x \in (m, m + \frac{1}{m^3})\}$

$H_{-m,y,z} = \{(x,y,z) \in \mathcal{R}^3 : x \in (-m - \frac{1}{m^3}, -m)\}$

$V_{x,m,z} = \{(x,y,z) \in \mathcal{R}^3 : y \in (m, m + \frac{1}{m^3})\}$

$V_{x,-m,z} = \{(x,y,z) \in \mathcal{R}^3 : y \in (-m - \frac{1}{m^3}, -m)\}$

$D_{x,y,m} = \{(x,y,z) \in \mathcal{R}^3 : z \in (m, m + \frac{1}{m^3})\}$

$D_{x,y,-m} = \{(x,y,z) \in \mathcal{R}^3 : z \in (-m - \frac{1}{m^3}, -m)\}$

We now address the issue of the construction of inflexionary approximation sequences in the 2 and 3 dimensional cases.

Lemma 0.4. *The results of Lemma 0.5 in* [3] *hold, replacing the intervals* $[m, m + \frac{1}{m}]$ *with* $[m, m + \frac{1}{m^2}]$ *and* $[m, m + \frac{1}{m^3}]$.

Proof. In the proof of Lemma 0.5 in [3], observe that the coefficients of the polynomial p, depend only on the $\frac{1}{m}$ term, so we can obtain the new coefficients for p by substituting m^2 or m^3 for m. We then calculate in the $\frac{1}{m^3}$ case, that;

$$h'''(x) = (-360a_0m^{15} + O(m^{12}))x^2 + (288a_0m^{18} + O(m^{16}))x$$

$$+(-36a_0m^{21} + O(m^{19}))$$

which has roots when;

$$x \simeq \frac{-288a_0 +/- 176a_0m^{18} + O(m^{16})}{-720a_0m^{15} + O(m^{12})} = O(m^3) + O(m) > 0$$

Clearly, we can then assume that for sufficiently large m, $h'''(x)$ has no roots in the interval $[-m-\frac{1}{m^3}] \cup [m, m+\frac{1}{m^3}]$. For the final calculation, with $|h|_{[m+\frac{1}{m^3}]}$, we can replace m by m^3 throughout the proof, to get the same result, that $|h|_{[m+\frac{1}{m^3}]} \leq C$, independently of $m > 1$. The case with m^2 replacing m is left to the reader. □

Lemma 0.5. *If $[a, b] \subset \mathcal{R}$, with a, b finite, and $\{g, g_1, g_2\} \subset C^\infty([a, b])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m^2}] \times [a, b])$, with the property that;*

$$h(m, y) = g(y),\ \frac{\partial h}{\partial x}|_{(m,y)} = g_1(y),\ \frac{\partial^2 h}{\partial x^2}|_{(m,y)} = g_2(y),\ y \in [a, b],\ (i)$$

$$h(m + \tfrac{1}{m^2}, y) = \frac{\partial h}{\partial x}(m + \tfrac{1}{m^2}, y) = \frac{\partial^2 h}{\partial x^2}(m + \tfrac{1}{m^2}, y) = 0,\ y \in [a, b],\ (ii)$$

$$|h|_{[m,m+\frac{1}{m^2}]\times[a,b]}| \leq C$$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^3 h}{\partial x^3}(m, y) > 0$, $\frac{\partial^3 h}{\partial x^3}(x, y) > 0$, for $x \in [m, m+\frac{1}{m^2}]$, and if $\frac{\partial^3 h}{\partial x^3}(m, y) < 0$, $\frac{\partial^3 h}{\partial x^3}(x, y) < 0$, for $x \in [m, m + \frac{1}{m^2}]$, $(*)$. In particularly;

$$\int_m^{m+\frac{1}{m^2}} |\frac{\partial^3 h}{\partial x^3}|_{(x,y)}|dx = |g_2(y)|$$

Moreover, for $i \in \mathcal{N}$, $\frac{\partial^i h}{\partial y^i}$ has the property that;

$$\frac{\partial^i h}{\partial y^i}(m, y) = g^{(i)}(y),\ \frac{\partial^{i+1} h}{\partial y^i \partial x}|_{(m,y)} = g_1^{(i)}(y),\ \frac{\partial^{i+2} h}{\partial y^i \partial x^2}|_{(m,y)} = g_2^{(i)}(y)$$

$$y \in [a, b],\ (i)'$$

$$\frac{\partial^i h}{\partial y^i}(m + \tfrac{1}{m^2}, y) = \frac{\partial^{i+1} h}{\partial y^i \partial x}(m + \tfrac{1}{m^2}, y) = \frac{\partial^{i+2} h}{\partial y^i \partial x^2}(m + \tfrac{1}{m^2}, y) = 0$$

$y \in [a,b]$, $(ii)'$

$|\frac{\partial^i h}{\partial y^i}|_{[m,m+\frac{1}{m^2}]\times[a,b]}| \leq C_i$

for some $C_i \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^{i+3}h}{\partial y^i \partial x^3}(m,y) > 0$, $\frac{\partial^{i+3}h}{\partial y^i \partial x^3}(x,y) > 0$, for $x \in [m, m+\frac{1}{m^2}]$, and if $\frac{\partial^{i+3}h}{\partial y^i \partial x^3}(m,y) < 0$, $\frac{\partial^{i+3}h}{\partial y^i \partial x^3}(x,y) < 0$, for $x \in [m, m+\frac{1}{m^2}]$, $(**)$. In particularly;

$\int_m^{m+\frac{1}{m^2}} |\frac{\partial^{i+3}h}{\partial y^i \partial x^3}|_{(x,y)}|dx = |g_2^{(i)}(y)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.4 and Lemma 0.5 in [3], replacing the constant coefficients $\{a_0, a_1, a_2\} \subset \mathcal{R}$ with the data $\{g(y), g_1(y), g_2(y)\}$. The properties $(i), (ii)$ are then clear. Noting that $[a,b]$ is a finite interval and $\{g, g_1, g_2\} \subset C^\infty([a,b])$, by continuity, there exists a constant D, with $max(|g(y)|, |g_1(y)|, |g_2(y)| : y \in [a,b]) \leq D$, so, as in the proof of Lemma 0.4 and Lemma 0.5 in [3], we can use the bound $C = 16D + 7D + D = 24D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.4 and Lemma 0.5 in [3], for sufficiently large m, again using the fact that the data $\{g(y), g_1(y), g_2(y) : y \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^i h}{\partial y^i}$, for $i \in \mathcal{N}$, we are just differentiating the coefficients which are linear in the data $\{g(y), g_1(y), g_2(y)\}$, so we obtain a function which fits the data $\{g^{(i)}(y), g_1^{(i)}(y), g_2^{(i)}(y)\}$ and $(i)', (ii)'$ follow. Noting that, for $i \in \mathcal{N}$, $\{g^{(i)}, g_1^{(i)}, g_2^{(i)}\} \subset C^\infty([a,b])$, again by continuity, there exists constants D_i, with $max(|g^{(i)}(y)|, |g_1^{(i)}(y)|, |g_2^{(i)}(y)| : y \in [a,b]) \leq D_i$, so, again, as in the proof of Lemma **??**, we can use the bound $C_i = 16D_i + 7D_i + D_i = 24D_i$, for $m > 1$. The proof of $(**)$ follows uniformly in y, for each $i \in \mathcal{N}$, as in the proof of Lemma 0.4 and Lemma 0.5 in [3], for sufficiently large m, again using the fact that the data $\{g^{(i)}(y), g_1^{(i)}(y), g_2^{(i)}(y) : y \in [a,b]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.6. *Conjecture*

Fix $n \in \mathcal{N}$, with $n \geq 3$. If $m \in \mathcal{R}_{>0}$ is sufficiently large, $\{a_i : 0 \leq i \leq n-1\} \subset \mathcal{R}$, there exists $h \in \mathcal{R}[x]$ of degree $2n-1$, with the property that;

$h^{(i)}(m) = a_i$, $0 \leq i \leq n-1$ (i)

$h^{(i)}(m+\frac{1}{m})=0,\ 0\leq i\leq n-1\ (ii)$

$|h|_{[m,m+\frac{1}{m}]}|\leq C$

for some $C\in\mathcal{R}_{>0}$*, independent of* m *sufficiently large, and, if* $h^{(n)}(m)>0$*,* $h^{(n)}(x)|_{[m,m+\frac{1}{m}]}>0$*, if* $h^{(n)}(m)<0$*,* $h^{(n)}|_{[m,m+\frac{1}{m}]}<0$*. In particularly;*

$\int_m^{m+\frac{1}{m}}|h^{(n)}(x)|dx=|a_{n-1}|$, ([1])

The same conjecture applies with $\frac{1}{m^2}$ *and* $\frac{1}{m^3}$ *replacing* $\frac{1}{m}$*.*

Proof. We sketch a proof based on the special case $n=3$, which was shown in Lemma 0.5 of [3], leaving the details to the reader, ([2]). We have that $h(x)=(x-(m+\frac{1}{m}))^n p(x)$ where $p(x)$ is a polynomial satisfies condition (ii). Computing the derivatives $h^{(i)}(m)$, for $0\leq i\leq n-1$, we obtain n linear equations involving the unknowns $p^{(i)}(m)$, $0\leq i\leq n-1$, of the form;

$\sum_{k=0}^{i}\frac{d_{ik}p^{(k)}(m)}{m^{n-i+k}}=a_i,\ (0\leq i\leq n-1)\ (*)$

which we can solve for $p^{(i)}(m)$, $0\leq i\leq n-1$, using the fact that the matrix $(d_{ik})_{0\leq i\leq n-1,0\leq k\leq i}$ is lower triangular and $|d_{ii}|=1$,

[1] If $a_0>0$, $a_1>0$, there does not exist a smooth function h on the interval $(m,m+\frac{1}{m})$, with $h(m)=a_0$, $h'(m)=a_1$, $h(m+\frac{1}{m})=0$, $h'(m+\frac{1}{m})=0$, such that $h''>0$ or $h''<0$. To see this, if $h''>0$, using the MVT, we have that $h'(x)>h'(m)>0$, for $x\in(m,m+\frac{1}{m})$, contradicting the fact that $h'(m+\frac{1}{m})=0$. If $h''<0$, and $h'(x)$ has no roots in the interval $(m,m+\frac{1}{m})$, then as $h'(m)>0$, $h'(x)>0$ on $(m,m+\frac{1}{m})$, and h is increasing on $(m,m+\frac{1}{m})$, so that $h(m+\frac{1}{m})>h(m)=a_0>0$, contradicting the fact that $h(m+\frac{1}{m})=0$. Otherwise, if $h'(x)$ has a root in the interval $(m,m+\frac{1}{m})$, as $h''<0$, it attains a maximum at $x_0\in(m,m+\frac{1}{m})$. Using the MVT again, we must have that for $y\in(x_0,m+\frac{1}{m})$, $h'(y)<h'(x_0)=0$, so that $h'(m+\frac{1}{m})<0$, contradicting the fact that $h'(m+\frac{1}{m})=0$.

[2] One step requires the verification that for a computable polynomial r_n of degree $n-1$, $r_n(1)\neq 0$, which is highly unlikely on generic grounds and the fact that $r_3(1)\neq 1$, although $r_2(1)=1$, see footnote 1. The geometric idea is that allowing for inflexionary type curves, where we can have points $x_{0,i}\in(m,m+\frac{1}{m})$ for which $h^{(i)}(x_{0,i})=0$, where $2\leq i\leq n-1$, the end conditions can be satisfied while still having $h^{(n)}|_{(m,m+\frac{1}{m})}>0$ or $h^{(n)}|_{(m,m+\frac{1}{m})}<0$. However, you still need to do a concrete calculation, which in the case of verifying the conjecture for all $n\in\mathcal{N}$, $n\geq 3$, would involve finding the exact pattern in the coefficients obtained in the proof of Lemma 0.5 of [3]. We actually only need the result for some $n\geq 14$ in the rest of this paper.

for $0\leq i\leq n-1$. Then we can take;

$p(x)=\sum_{i=0}^{n-1}p^{(i)}(m)(x-m)^{i}$

so that h has degree $n+(n-1)=2n-1$. It is clear from $(*)$, that we have;

$p^{(i)}(m)=\sum_{k=0}^{i}c_{ik}a_{i-k}m^{n+k}$, $(0\leq i\leq n-1)$

where $(c_{ik})_{0\leq i\leq n-1,0\leq k\leq i}$ is a real matrix, so that $p(x)$ has the form;

$p(x)=\sum_{i=0}^{n-1}v_{i}x^{i}$ $(**)$

where;

$v_{n-1-i}=\sum_{k=0}^{n-1}r_{ik}m^{n+k}+\sum_{l=0}^{i}s_{il}m^{2n-1+l}$, $(0\leq i\leq n-1)$

for real matrices $(r_{ik})_{0\leq i\leq n-1,0\leq k\leq n-1}$ and $(s_{il})_{0\leq i\leq n-1,0\leq l\leq i}$.

It is then clear, using the product rule and $(**)$, that;

$h^{(n)}(x)=\sum_{k=0}^{n-1}w_{k}x^{k}$

where $w_{k}=z_{k}a_{0}m^{3n-2-k}+O(m^{3n-3-k})$, $(0\leq k\leq n-1)$

By homogeneity, it is then clear that the real roots of $h^{(n)}(x)$ are of the form $t_{s_0}m+O(1)$, where $t_{s_0}\in\mathcal{R}$, $1\leq s_0\leq n-1$, and t_{s_0} satisfies a polynomial $r(x)$ of degree $n-1$, which is effectively computable for given n. We can exclude any roots in the interval $[m,m+\frac{1}{m}]$, for sufficiently large m, provided $t_{s_0}\neq 1$, for $1\leq s_0\leq n-1$, which we can check by showing that $r(1)\neq 0$. We have that;

$|h|_{(m,m+\frac{1}{m})}|=|(x-(m+\frac{1}{m}))^{n}p(x)|$

$\leq\frac{1}{m^{n}}|\sum_{i=0}^{n-1}p^{(i)}(m)(x-m)^{i}|$

$\leq\frac{1}{m^{n}}\sum_{i=0}^{n-1}\frac{|p^{(i)}(m)|}{m^{i}}$

$\leq\sum_{i=0}^{n-1}\sum_{k=0}^{i}|c_{ik}|a_{i-k}|\frac{m^{n+k}}{m^{n+i}}$

$\leq \sum_{i=0}^{n-1}\sum_{k=0}^{i}|c_{ik}|a_{i-k}| = C$, $(m > 1)$

The last claim is just the FTC.

□

Lemma 0.7. *If $[a,b] \subset \mathcal{R}$, with a,b finite, $n \geq 3$, and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m^2}] \times [a,b])$, with the property that;*

$\frac{\partial^{(j)}h}{\partial x^j}|_{(m,y)} = g_j(y)$, $y \in [a,b]$, (i)

$\frac{\partial h^j}{\partial x^j}(m+\frac{1}{m^2}, y) = 0$, $y \in [a,b]$, (ii)

$|h|_{[m,m+\frac{1}{m^2}]\times[a,b]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^n h}{\partial x^n}(m,y) > 0$, $\frac{\partial^n h}{\partial x^n}(x,y) > 0$, for $x \in [m, m+\frac{1}{m^2}]$, and if $\frac{\partial^n h}{\partial x^n}(m,y) < 0$, $\frac{\partial^n h}{\partial x^n}(x,y) < 0$, for $x \in [m, m+\frac{1}{m^2}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m^2}} |\frac{\partial^n h}{\partial x^n}|_{(x,y)}|dx = |g_{n-1}(y)|$

Moreover, for $i \in \mathcal{N}$, $\frac{\partial^i h}{\partial y^i}$ has the property that;

$\frac{\partial^{i+j}h}{\partial x^j \partial y^i}(m,y) = g_j^{(i)}(y)$, $y \in [a,b]$, $(i)'$

$\frac{\partial^{i+j}h}{\partial x^j \partial y^i}(m+\frac{1}{m^2}, y) = 0$, $y \in [a,b]$, $(ii)'$

$|\frac{\partial^i h}{\partial y^i}|_{[m,m+\frac{1}{m^2}]\times[a,b]}| \leq C_i$

for some $C_i \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(m,y) > 0$, $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(x,y) > 0$, for $x \in [m, m+\frac{1}{m^2}]$, and if $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(m,y) < 0$, $\frac{\partial^{i+n}h}{\partial y^i \partial x^n}(x,y) < 0$, for $x \in [m, m+\frac{1}{m^2}]$, $(**)$. In particularly;

$\int_m^{m+\frac{1}{m^2}} |\frac{\partial^{i+n}h}{\partial y^i \partial x^n}|_{(x,y)}|dx = |g_{n-1}^{(i)}(y)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.6, replacing the constant coefficients $\{a_j : 0 \leq j \leq n-1\} \subset \mathcal{R}$ with the data $\{g_j(y) : 0 \leq j \leq n-1\}$. The properties $(i), (ii)$ are then clear. Noting that $[a,b]$ is a finite interval and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, by continuity, there exists a constant D, with

$max(|g_j(y)| : 0 \leq j \leq n-1, y \in [a,b]) \leq D$, so, as in the proof of Lemma 0.5 in [3], we can use the bound $C = \sum_{0\leq j\leq n-1} L_j D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of Lemma 0.5 in [3], for sufficiently large m, again using the fact that the data $\{g_j(y) : 0 \leq j \leq n-1, y \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^i h}{\partial y^i}$, for $i \in \mathcal{N}$, we are just differentiating the coefficients which are linear in the data $\{g_j(y) : 0 \leq j \leq n-1\}$, so we obtain a function which fits the data $\{g_j^{(i)}(y) : 0 \leq j \leq n-1\}$ and $(i)', (ii)'$ follow. Noting that, for $i \in \mathcal{N}$, $\{g_j^{(i)} : 0 \leq j \leq n-1\} \subset C^\infty([a,b])$, again by continuity, there exist constants D_i, with $max(|g_j^{(i)}(y)| : 0 \leq j \leq n-1, y \in [a,b]) \leq D_i$, so, again, as in the proof of Lemma 0.5 in [3], we can use the bound $C_i = \sum_{0\leq j\leq n-1} L_j D_i$, for $m > 1$. The proof of $(**)$ follows uniformly in y, for each $i \in \mathcal{N}$, as in the proof of Lemma 0.5 in [3], for sufficiently large m, again using the fact that the data $\{g_j^{(i)}(y) : 0 \leq j \leq n-1, y \in [a,b]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.8. *If $[a,b] \subset \mathcal{R}$, $[c,d] \subset \mathcal{R}$, with a,b,c,d finite, $n \geq 3$, and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b]\times[c,d])$, then, if $m \in \mathcal{R}_{>0}$ is sufficiently large, there exists $h \in C^\infty([m, m+\frac{1}{m^3}]\times[a,b]\times[c,d])$, with the property that;*

$\frac{\partial^{(j)} h}{\partial x^j}|_{(m,y,z)} = g_j(y,z)$, $(y,z) \in [a,b]\times[c,d]$, (i)

$\frac{\partial h^j}{\partial x^j}(m+\frac{1}{m^3}, y, z) = 0$, $(y,z) \in [a,b]\times[c,d]$, (ii)

$|h|_{[m,m+\frac{1}{m^3}]\times[a,b]\times[c,d]}| \leq C$

for some $C \in \mathcal{R}_{>0}$, independent of m sufficiently large, and, if $\frac{\partial^n h}{\partial x^n}(m,y,z) > 0$, $\frac{\partial^n h}{\partial x^n}(x,y,z) > 0$, for $x \in [m, m+\frac{1}{m^3}]$, and if $\frac{\partial^n h}{\partial x^n}(m,y,z) < 0$, $\frac{\partial^n h}{\partial x^n}(x,y,z) < 0$, for $x \in [m, m+\frac{1}{m^3}]$, $(*)$. In particularly;

$\int_m^{m+\frac{1}{m^3}} |\frac{\partial^n h}{\partial x^n}|_{(x,y,z)}| dx = |g_{n-1}(y,z)|$

Moreover, for $(i,k) \subset \mathcal{N}^2$, $0 \leq j \leq n-1$, $\frac{\partial^{i+k} h}{\partial y^i \partial z^k}$, has the property that;

$\frac{\partial^{i+j+k} h}{\partial x^j \partial y^i \partial z^k}(m,y,z) = \frac{\partial^{i+k} g_j}{\partial y^i \partial z^k}(y,z)$, $(y,z) \in [a,b]\times[c,d]$, $(i)'$

$\frac{\partial^{i+j+k} h}{\partial x^j \partial y^i \partial z^k}(m+\frac{1}{m^3}, y, z) = 0$, $(y,z) \in [a,b]\times[c,d]$, $(ii)'$

$|\frac{\partial^{i+k}h}{\partial y^i \partial z^k}|_{[m,m+\frac{1}{m^3}]\times[a,b]\times[c,d]}| \leq C_{i,k}$

for some $C_{i,k} \in \mathcal{R}_{>0}$*, independent of* m *sufficiently large, and, if* $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(m,y,z) > 0$, $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(x,y,z) > 0$*, for* $x \in [m, m+\frac{1}{m^3}]$*, and if* $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(m,y) < 0$, $\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}(x,y,z) < 0$*, for* $x \in [m, m+\frac{1}{m^3}]$*,* $(**)$*. In particularly;*

$\int_m^{m+\frac{1}{m^3}} |\frac{\partial^{i+k+n}h}{\partial y^i \partial z^k \partial x^n}|_{(x,y,z)}|dx = |\frac{\partial^{i+k}g_{n-1}}{\partial y^i \partial z^k}(y,z)|$

Proof. For the construction of h in the first part, just use the proof of Lemma 0.6, replacing the constant coefficients $\{a_j : 0 \leq j \leq n-1\} \subset \mathcal{R}$ with the data $\{g_j(y,z) : 0 \leq j \leq n-1\}$. The properties $(i), (ii)$ are then clear. Noting that $[a,b] \times [c,d]$ is compact and $\{g_j : 0 \leq j \leq n-1\} \subset C^\infty([a,b] \times [c,d])$, by continuity, there exists a constant D, with $max(|g_j(y,z)| : 0 \leq j \leq n-1, (y,z) \in [a,b] \times [c,d]) \leq D$, so, as in the proof of Lemma 0.6, we can use the bound $C = \sum_{0 \leq j \leq n-1} L_j D$, for $m > 1$. The proof of $(*)$ follows uniformly in y, as in the proof of 0.6, for sufficiently large m, again using the fact that the data $\{g_j(y,z) : 0 \leq j \leq n-1, (y,z) \in [a,b]\}$ is bounded. The next claim is just the FTC again. For the second part, when we calculate $\frac{\partial^{i+k}h}{\partial y^i \partial z^k}$, for $(i,j \in \mathcal{N}^2$, we are just differentiating the coefficients which are linear in the data $\{g_j(y,z) : 0 \leq j \leq n-1\}$, so we obtain a function which fits the data $\{\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k}(y,z) : 0 \leq j \leq n-1\}$ and $(i)', (ii)'$ follow. Noting that, for $(i,k) \in \mathcal{N}^2$, $\{\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k} : 0 \leq j \leq n-1\} \subset C^\infty([a,b] \times [c,d])$, again by continuity, there exist constants $D_{i,k}$, with $max(|\frac{\partial^{i+k}g_j}{\partial y^i \partial z^k}(y,z)| : 0 \leq j \leq n-1, y \in [a,b] \times [c,d]) \leq D_{i,k}$, so, again, as in the proof of Lemma 0.6, we can use the bound $C_{i,k} = \sum_{0 \leq j \leq n-1} L_j D_{i,k}$, for $m > 1$. The proof of $(**)$ follows uniformly in (y,z), for each $(i,k) \in \mathcal{N}^2$, as in the proof of Lemma 0.6, for sufficiently large m, again using the fact that the data $\{\frac{\partial^{i+k}g_j}{\partial Y^i \partial z^k}(y) : 0 \leq j \leq n-1, (y,z) \in [a,b] \times [c,d]\}$ is bounded. The last claim is again just the FTC. □

Lemma 0.9. *For* $f \in C^\infty(\mathcal{R}^2)$ *with* $\frac{\partial^{i_1+i_2}f}{\partial x^{i_1} \partial y^{i_2}}$ *bounded by some constant* $F \in \mathcal{R}_{>0}$*, for* $0 \leq i_1 + i_2 \leq 27$*. Then for sufficiently large* m*, there exists an inflexionary approximation sequence* $\{f_m : m \in \mathcal{N}\}$*, with the property that;*

$max(\int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial x^{14}}|dxdy, \int_{\mathcal{R}^2} |\frac{\partial f_m}{\partial y^{14}}|dxdy) \leq Gm^2$

for some $G\in\mathcal{R}_{>0}$, for sufficiently large m.

Proof. Define $f_m=f$ on C_m, so that (ii) of Definition 0.2 is satisfied. Using two applications of Lemma 0.7 with $n=14$, changing to a vertical rather than horizontal orientation, and the fact that, for $0\leq i\leq 13$, $|x|\leq m$, $\frac{\partial^i f}{\partial y^i}|_{(x,m)}$ and $\frac{\partial^i f}{\partial y^i}|_{(x,-m)}$ define smooth functions on $[-m,m]$, we can extend f_m to $R=\{(x,y):|x|\leq m, m\leq |y|\leq m+\frac{1}{m^2}\}$, such that $f_m|R_1$ satisfies conditions $(iv),(v)$ of Definition 0.2, where $R_1=\{(x,y):|x|\leq m, 0\leq |y|\leq m+\frac{1}{m^2}\}$. Again, using two applications of Lemma 0.7 with $n=14$, and the original horizontal orientation, and the fact that, for $0\leq i\leq 13$, $0\leq |y|\leq m+\frac{1}{m^2}$, $\frac{\partial^i f_m}{\partial x^i}|_{(m,y)}$ and $\frac{\partial^i f}{\partial x^i}|_{(-m,y)}$ define smooth functions on $[-m-\frac{1}{m^2},m+\frac{1}{m^2}]$, we can extend f_m to $S=\{(x,y):m\leq |x|\leq m+\frac{1}{m^2}, 0\leq |y|\leq m+\frac{1}{m^2}\}$, such that $f_m|C_{m+\frac{1}{m^2}}$ satisfies conditions $(vi),(vii)$ of Definition 0.2. Conditions $(i),(iii)$ are then clear. We then have, using (iii), that;

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial x^{14}}|dxdy=\int_{C_{m+\frac{1}{m^2}}}|\frac{\partial f_m}{\partial x^{14}}|dxdy$

$=\int_{|x|\leq m,|y|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdy+\int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m^2}}|\frac{\partial f_m}{\partial x^{14}}|dxdy+\int_{m\leq |x|\leq m+\frac{1}{m^2},|y|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdy$

$+\int_{m\leq |x|\leq m+\frac{1}{m^2}, m\leq |y|\leq m+\frac{1}{m^2}}|\frac{\partial f_m}{\partial x^{14}}|dxdy$

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial y^{14}}|dxdy=\int_{C_{m+\frac{1}{m^2}}}|\frac{\partial f_m}{\partial y^{14}}|dxdy$

$=\int_{|x|\leq m,|y|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdy+\int_{|x|\leq m, m\leq |y|\leq m+\frac{1}{m^2}}|\frac{\partial f_m}{\partial y^{14}}|dxdy+\int_{m\leq |x|\leq m+\frac{1}{m^2},|y|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdy$

$+\int_{m\leq |x|\leq m+\frac{1}{m^2}, m\leq |y|\leq m+\frac{1}{m^2}}|\frac{\partial f_m}{\partial y^{14}}|dxdy\ (*)$

We then have the following cases, using the second clause in Lemma 0.7 repeatedly with the appropriate orientations;

Case 1;

$\int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14} f_m}{\partial x^{14}}|dxdy$

$=\int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14} f}{\partial x^{14}}|dxdy\leq Fm^2$

$\int_{|x|\leq m,|y|\leq m}|\frac{\partial^{14} f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m,|y|\leq m} |\frac{\partial^{14}f}{\partial y^{14}}|dxdy \leq Fm^2$

Case 2;

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^2}} |\frac{\partial^{14}f_m}{\partial x^{14}}|dxdy$

$= \int_{|x|\leq m}(\int_{|y|\leq m+\frac{1}{m^2}} |\frac{\partial^{14}f_m}{\partial x^{14}}|dy)dx$

$\leq \frac{2}{m^2}\int_{|x|\leq m} C_{14}dx$

$\leq 2m\frac{2}{m^2}C_{14}$

$= 4\frac{C_{14}}{m}$

Case 3;

$\int_{m\leq|x|\leq m+\frac{1}{m^2},|y|\leq m} |\frac{\partial^{14}f_m}{\partial x^{14}}|dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^2}} |\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dy$

$= \int_{|y|\leq m}(|\frac{\partial^{13}f}{\partial x^{13}}|_{(m,y)} + |\frac{\partial^{13}f}{\partial x^{13}}|_{(-m,y)})dy$

$\leq 4mF$

Case 4.

$\int_{m\leq|x|\leq m+\frac{1}{m^2},m\leq|y|\leq m+\frac{1}{m^2}} |\frac{\partial^{14}f_m}{\partial x^{14}}|dxdy$

$= \int_{m\leq|y|\leq m+\frac{1}{m^2}}(\int_{m\leq|x|\leq m+\frac{1}{m^2}} |\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dy$

$= \int_{m\leq|y|\leq m+\frac{1}{m^2}}(|\frac{\partial^{13}f_m}{\partial x^{13}}|_{(m,y)} + |\frac{\partial^{13}f_m}{\partial x^{13}}|_{(-m,y)}dy$

$\leq \int_{m\leq y\leq m+\frac{1}{m^2}} C_{13,1}dy + \int_{-m-\frac{1}{m^2}\leq -m} C_{13,2}dy$

$\leq \frac{max(C_{13,1},C_{13,2})}{m^2}$ (the constants $\{C_{13,1}, C_{13,2}\}$ coming from the two applications of Lemma 0.7 at the two boundaries)

Case 5;

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^2}} |\frac{\partial^{14}f_m}{\partial y^{14}}|dxdy$

$= \int_{|x|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m^2}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dy)dx$

$= \int_{|x|\leq m}(|\frac{\partial f}{\partial y^{13}}|_{(x,m)}+|\frac{\partial f(x,y)}{\partial y^{13}}|_{(x,-m)}dx)$

$\leq 4mF$

Case 6;

$\int_{|y|\leq m, m\leq|x|\leq m+\frac{1}{m^2}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^2}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dx)dy$

$\leq \frac{1}{m^2}\int_{|y|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(m,y)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(-m,y))dy$

$\leq \frac{2}{m^2}(2m)F(\sum_{i=0}^{13}D_i)$

$= 4F\frac{(\sum_{i=0}^{13}D_i)}{m}$

Case 7.

$\int_{m\leq|x|\leq m+\frac{1}{m^2}, m\leq|y|\leq m+\frac{1}{m^2}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdy$

$= \int_{m\leq|y|\leq m+\frac{1}{m^2}}(\int_{m\leq|x|\leq m+\frac{1}{m^2}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dx)dy$

$\leq \frac{1}{m^2}\int_{m\leq|y|\leq m+\frac{1}{m^2}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial x^i\partial y^{14}}|_{(m,y)}+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial x^i\partial y^{14}}|_{(-m,y)})dy$

$= \frac{1}{m^2}\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(m,m)}|+|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(m,-m)}|+|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(-m,m)}|+|\frac{\partial^{i+13}f}{\partial x^i\partial y^{13}}|_{(-m,-m)}|)$

$\leq \frac{4F(\sum_{i=0}^{13}L_{i,14})}{m^2}$ (the constants $L_{i,14}$,$0\leq i\leq 13$ coming from the proof of Lemma 0.7)

Combining the seven cases and $(*)$, we obtain, for sufficiently large m, that;

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial x^{14}}|dxdy \leq Fm^2+4\frac{C_{14}}{m}+4mF+\frac{max(C_{13,1},C_{13,2})}{m^2}\leq Gm^2$

$\int_{\mathcal{R}^2}|\frac{\partial f_m}{\partial y^{14}}|dxdy \leq Fm^2+4mF+4F\frac{(\sum_{i=0}^{13}D_i)}{m}+\frac{4F(\sum_{i=0}^{13}L_{i,14})}{m^2}\leq Gm^2$

□

Lemma 0.10. *For $f\in C^{40}(\mathcal{R}^3)$ with $\frac{\partial^{i_1+i_2+i_3}f}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}$ bounded by some constant $F\in\mathcal{R}_{>0}$, for $0\leq i_1+i_2+i_3\leq 40$. Then for sufficiently large m, there exists an inflexionary approximation sequence $\{f_m : m\in\mathcal{N}\}$, with the property that;*

$$max\left(\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial x^{14}}|dxdydz,\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz,\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz\right)\leq Gm^3$$

for some $G\in\mathcal{R}_{>0}$, for sufficiently large m.

Proof. Define $f_m=f$ on W_m, so that (ii) of Definition 0.3 is satisfied. Using two applications of Lemma 0.8 with $n=14$, with a horizontal orientation, and the fact that, for $0\leq i\leq 13$, $0\leq|y|\leq m$, $0\leq|z|\leq m$ $\frac{\partial^i f}{\partial x^i}|_{(m,y,z)}$ and $\frac{\partial^i f}{\partial x^i}|_{(-m,y,z)}$ define smooth functions on $[-m,m]^2$, we can extend f_m to $A_1=\{(x,y,z):m\leq|x|\leq m+\frac{1}{m^3},0\leq|y|\leq m,0\leq|z|\leq m\}$, such that $f_m|A_2$ satisfies conditions $(iv),(v)$ of Definition 0.3, where $A_2=\{(x,y,z):0\leq|x|\leq m+\frac{1}{m^3},0\leq|y|\leq m,0\leq|z|\leq m\}$. Again, using two applications of Lemma 0.8 with $n=14$ again, this time with a vertical orientation, and the fact that, for $0\leq i\leq 13$, $0\leq|x|\leq m+\frac{1}{m^3}$, $0\leq|z|\leq m$, $\frac{\partial^i f_m}{\partial y^i}|_{(x,m,z)}$ and $\frac{\partial^i f_m}{\partial y^i}|_{(x,-m,z)}$ define smooth functions on $[-m-\frac{1}{m^3},m+\frac{1}{m^3}]\times[-m,m]$, we can extend f_m to $A_3=\{(x,y,z):0\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},0\leq|z|\leq m\}$, such that $f_m|A_4$ satisfies conditions $(vi),(vii)$ of Definition 0.3, where $A_4=\{(x,y,z):0\leq|x|\leq m+\frac{1}{m^3},0\leq|y|\leq m+\frac{1}{m^3},0\leq|z|\leq m\}$. Again, using two applications of Lemma 0.8 with $n=14$ again, this time with a lateral orientation, and the fact that, for $0\leq i\leq 13$, $0\leq|x|\leq m+\frac{1}{m^3}$, $0\leq|y|\leq m+\frac{1}{m^3}$, $\frac{\partial^i f_m}{\partial z^i}|_{(x,y,m)}$ and $\frac{\partial^i f_m}{\partial z^i}|_{(x,y,-m)}$ define smooth functions on $[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^2$, we can extend f_m to $W_{m+\frac{1}{m^3}}$ such that $f_m|_{W_{m+\frac{1}{m^3}}}$ satisfies conditions $(viii),(ix)$ of Definition 0.3.

Conditions $(i),(iii)$ are then clear. We then have, using (iii), that;

(a). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial x^{14}}|dxdydz=\int_{W_{m+\frac{1}{m^3}}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial x^{14}}|dxdydz$

(b). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz=\int_{W_{m+\frac{1}{m^3}}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial y^{14}}|dxdydz$

(c). $\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz=\int_{W_{m+\frac{1}{m^3}}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$+\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz+\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$(*)$

We then have the following cases, using the second clause in Lemma 0.8 repeatedly with the appropriate orientations;

Case 1;

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f}{\partial x^{14}}|dxdydz\leq Fm^3$

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f}{\partial y^{14}}|dxdydz\leq Fm^3$

$\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{|x|\leq m,|y|\leq m,|z|\leq m}|\frac{\partial^{14}f}{\partial z^{14}}|dxdydz\leq Fm^3$

Case 2;

$\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dx)dydz$

$=\int_{|y|\leq m,|z|\leq m}(|\frac{\partial^{13}f}{\partial x^{13}}|_{(m,y,z)}+|\frac{\partial^{13}f}{\partial x^{13}}|_{(-m,y,z)})dydz$

$\leq 2(2m)^2F$

$=8m^2F$

Case 3;

$\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dx)dydz$

$\leq\frac{1}{m^3}\int_{|y|\leq m,|z|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(m,y,z)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial y^{14}\partial x^i}|(-m,y,z))dydz$

$\leq\frac{2}{m^3}(2m)^2F(\sum_{i=0}^{13}D_i)$

$=\frac{8F(\sum_{i=0}^{13}D_i)}{m}$

Case 4;

$\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{|y|\leq m,|z|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dx)dydz$

$\leq\frac{1}{m^3}\int_{|y|\leq m,|z|\leq m}(|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial z^{14}\partial x^i}|(m,y,z)+|\sum_{i=0}^{13}D_i|\frac{\partial^i\partial^{14}f}{\partial z^{14}\partial x^i}|(-m,y,z))dydz$

$\leq\frac{2}{m^3}(2m)^2F(\sum_{i=0}^{13}D_i)$

$=\frac{8F(\sum_{i=0}^{13}D_i)}{m}$

Case 5.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$= \int_{|x|\leq m,|z|\leq m}(\int_{|y|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dy)dxdz$

$\leq \frac{2}{m^3}\int_{|x|\leq m,|z|\leq m}C_{14}dx$

$= (2m)^2\frac{2}{m^3}C_{14,0}$

$= \frac{8C_{14,0}}{m}$

Csse 6.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$= \int_{|x|\leq m,|z|\leq m}(\int_{|y|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dy)dxdz$

$\leq \frac{2}{m^3}\int_{|x|\leq m,|z|\leq m}C_{0,14}dx$

$= (2m)^2\frac{2}{m^3}C_{0,14}$

$= \frac{8C_{0,14}}{m}$

Case 7.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$= \int_{|x|\leq m,|z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dy)dxdz$

$= \int_{|x|\leq m,|z|\leq m}(|\frac{\partial f}{\partial y^{13}}|_{(x,m,z)}+|\frac{\partial f}{\partial y^{13}}|_{(x,-m,z)})dxdz)$

$\leq 2(2m)^2F$

$= 8m^2F$

Case 8.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dy)dxdz$

$\leq \frac{1}{m^3}\int_{m\leq|x|\leq m+\frac{1}{m^3},|z|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}\partial^{14}f_m}{\partial y^i\partial x^{14}}|_{(x,m,z)}+L_{i,14}|\frac{\partial^{i+14}\partial^{14}f_m}{\partial y^i\partial x^{14}}|_{(x,-m,z)})dxdz$

$= \frac{1}{m^3}\int_{|z|\leq m}(\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+13}\partial^{14}f}{\partial y^i\partial x^{13}}|_{(m,m,z)}|+|\frac{\partial^{i+13}\partial^{14}f}{\partial y^i\partial x^{13}}|_{(m,-m,z)}|+|\frac{\partial^{i+13}\partial^{14}f}{\partial y^i\partial x^{13}}|_{(-m,m,z)}|$

$+|\frac{\partial^{i+13}\partial^{14}f}{\partial y^i\partial x^{13}}|_{(-m,-m,z)}|))dz$

$\leq (2m)\frac{4F(\sum_{i=0}^{13} L_{i,14})}{m^3}$

$= \frac{8F(\sum_{i=0}^{13} L_{i,14})}{m^2}$

(the constants $L_{i,14}$,$0\leq i\leq 13$ coming from the proof of Lemma 0.7)

Case 9.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dy)dxdz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|z|\leq m}(|\frac{\partial^{13}f_m}{\partial y^{13}}|_{(x,m,z)}+|\frac{\partial^{13}f_m}{\partial y^{13}}|_{(x,-m,z)})dxdz$

$\leq \frac{1}{m^3}(\int_{|z|\leq m} C_{13,1}dz+\int_{|z|\leq m} C_{13,2}dz)$

$\leq (2m)\frac{max(C_{13,1},C_{13,2})}{m^3}$

$= \frac{2max(C_{13,1},C_{13,2})}{m^2}$

(the constants $\{C_{13,1},C_{13,2}\}$ coming from the two applications of Lemma 0.7 at the two boundaries)

Case 10.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},|z|\leq m}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|z|\leq m}(\int_{m\leq|y|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dy)dxdz$

$\leq \frac{1}{m^3}\int_{m\leq|x|\leq m+\frac{1}{m^3},|z|\leq m}(\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14}f_m}{\partial y^i\partial z^{14}}|_{(x,m,z)}+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial y^i\partial z^{14}}|_{(x,-m,z)})dxdz$

$\leq \frac{1}{m^6}\int_{|z|\leq m}(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{j,i,14}(|\frac{\partial^{i+j+14}f}{\partial x^j\partial y^i\partial z^{14}}|_{(m,m,z)}|+|\frac{\partial^{i+j+14}f}{\partial x^j\partial y^i\partial z^{14}}|_{(m,-m,z)}|$

$+|\frac{\partial^{i+j+14}f}{\partial x^j\partial y^i\partial z^{14}}|_{(-m,m,z)}|+|\frac{\partial^{i+j+14}f}{\partial x^j\partial y^i\partial z^{14}}|_{(-m,-m,z)}|))dz$

$\leq (2m)\frac{4F(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{j,i,14})}{m^6}$

$= \frac{8F(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{j,i,14})}{m^5}$

(the constants $L_{i,14}, L_{j,i,14}$,$0 \leq i \leq 13, 0 \leq j \leq 13$ coming from two applications of the proof of Lemma 0.8)

Case 11.

$\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m^3}} |\frac{\partial f_m}{\partial x^{14}}|dxdydz$

$= \int_{|x|\leq m,|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m^3}} |\frac{\partial f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{2}{m^3}\int_{|x|\leq m,|y|\leq m}(E_{14,0})$

$= (2m)^2\frac{2}{m^3}E_{14,0}$

$= \frac{8E_{14,0}}{m}$

Case 12.

$\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m^3}} |\frac{\partial f_m}{\partial y^{14}}|dxdydz$

$= \int_{|x|\leq m,|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m^3}} |\frac{\partial f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{2}{m^3}\int_{|x|\leq m,|y|\leq m}(E_{0,14})$

$= (2m)^2\frac{2}{m^3}E_{0,14}$

$= \frac{8E_{0,14}}{m}$

(the constants $E_{0,14}, E_{14,0}$ coming from an application of Lemma 0.8 with a different orientation)

Case 13.
$\int_{|x|\leq m,|y|\leq m,m\leq |z|\leq m+\frac{1}{m^3}} |\frac{\partial f_m}{\partial z^{14}}|dxdydz$

$= \int_{|x|\leq m,|y|\leq m}(\int_{m\leq |z|\leq m+\frac{1}{m^3}} |\frac{\partial f_m}{\partial z^{14}}|dz)dxdy$

$= \int_{|x|\leq m,|y|\leq m}(|\frac{\partial f}{\partial z^{13}}|(x,y,m) + |\frac{\partial f}{\partial z^{13}}|(x,y,m))dxdy$

$\leq 2(2m)^2F$

$= 8m^2F$

Case 14.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{1}{m^3}\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,m)+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dxdy$

$= \frac{1}{m^3}\int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^3}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,m)+L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial x^{14}}|(x,y,-m))dx)dy$

$= \frac{1}{m^3}\int_{|y|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(m,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(-m,y,m)$

$+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(m,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f}{\partial z^i\partial x^{13}}|(-m,y,-m))dy$

$\leq (2m)\frac{1}{m^3}(4F)(\sum_{i=0}^{13}L_{i,14})$

$= \frac{8F(\sum_{i=0}^{13}L_{i,14})}{m^2}$

Case 15.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{1}{m^3}\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+L_{i,14}|\frac{\partial^{i+14}f_m}{\partial y^i\partial x^{14}}|(x,y,-m))dxdy$

$= \frac{1}{m^3}\int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^3}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dx)dy$

$\leq \frac{1}{m^6}\int_{|y|\leq m}(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(m,y,m)$

$+\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(-m,y,m)$

$+\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(m,y,-m)$

$+\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14}|\frac{\partial^{i+j+14}f}{\partial x^j\partial z^i\partial y^{14}}|(-m,y,-m))dy$

$\leq (2m)\frac{1}{m^6}(4F)(\sum_{i=0}^{13}\sum_{j=0}^{13}L_{i,14}L_{i,j,14})$

$= \frac{8F(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14})}{m^5}$

Case 16.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m,m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m}(\int_{m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dz)dxdy$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},|y|\leq m}(|\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,m) + |\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,-m))dxdy$

$= \int_{|y|\leq m}(\int_{m\leq|x|\leq m+\frac{1}{m^3}}(|\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,m) + |\frac{\partial^{13} f_m}{\partial z^{13}}|(x,y,-m))dx)dy$

$\leq \frac{1}{m^3}\int_{|y|\leq m}(\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f}{\partial x^i\partial z^{13}}|(m,y,m)+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f}{\partial x^i\partial z^{13}}|(-m,y,m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f}{\partial x^i\partial z^{13}}|(m,y,-m) + \sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13} f}{\partial x^i\partial z^{13}}|(-m,y,-m))$

$\leq (2m)\frac{1}{m^3}(4F)(\sum_{i=0}^{13} L_{i,13})$

$= \frac{8F(\sum_{i=0}^{13} L_{i,13})}{m^2}$

Cases 17-19 are similar to cases 14-16, interchanging the orders of integration, with case 17 corresponding to case 15, case 18 corresponding to case 14 and case 19 corresponding to case 16, so that;

Case 17.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dxdydz$

$\leq \frac{8F(\sum_{i=0}^{13}\sum_{j=0}^{13} L_{i,14}L_{i,j,14})}{m^5}$

Case 18.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdydz$

$\leq \frac{8F(\sum_{i=0}^{13} L_{i,14})}{m^2}$

Case 19.

$\int_{|x|\leq m,m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial z^{14}}|dxdydz$

$\leq \frac{8F(\sum_{i=0}^{13} L_{i,13})}{m^2}$

Case 20.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3}} (\int_{m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial x^{14}}|dz)dxdy$

$\leq \frac{1}{m^3} \int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3}} (\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14} f_m}{\partial z^i \partial x^{14}}|(x,y,m)+\sum_{i=0}^{13} L_{i,14}|\frac{\partial^{i+14} f_m}{\partial z^i \partial x^{14}}|(x,y,-m))dxdy$

$= \frac{1}{m^3} \int_{m\leq|x|\leq m+\frac{1}{m^3}} (\int_{m\leq|y|\leq m+\frac{1}{m^3}} (\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+14} f_m}{\partial z^i \partial x^{14}}|(x,y,m)$

$+\sum_{i=0}^{13} L_{i,14}(|\frac{\partial^{i+14} f_m}{\partial z^i \partial x^{14}}|(x,y,-m))dy)dx$

$\leq \frac{1}{m^6} \int_{m\leq|x|\leq m+\frac{1}{m^3}} (\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j \partial z^i \partial x^{14}}|(x,m,m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j \partial z^i \partial x^{14}}|(x,-m,m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j \partial z^i \partial x^{14}}|(x,m,-m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+14} f_m}{\partial y^j \partial z^i \partial x^{14}}|(x,-m,-m))dx$

$= \frac{1}{m^6}(\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(m,m,m)+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(m,-m,m)+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(m,m,-m)+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14}|\frac{\partial^{i+j+13} f}{\partial y^j \partial z^i \partial x^{13}}|(-m,-m,-m))$

$\leq \frac{8F}{m^6}(\sum_{j=0}^{13} \sum_{i=0}^{13} L_{i,14} L_{i,j,14})$

Case 21.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dxdydz$

$= \int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3}} (\int_{m\leq|z|\leq m+\frac{1}{m^3}} |\frac{\partial^{14} f_m}{\partial y^{14}}|dz)dxdy$

$\leq \frac{1}{m^3}\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dxdy$

$=\frac{1}{m^3}\int_{|x|\leq m+\frac{1}{m^3}}(\int_{m\leq|y|\leq m+\frac{1}{m^3}}(\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,m)+\sum_{i=0}^{13}L_{i,14}(|\frac{\partial^{i+14}f_m}{\partial z^i\partial y^{14}}|(x,y,-m))dy)d$

$=\frac{1}{m^3}\int_{m\leq|x|\leq m+\frac{1}{m^3}}(\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,m,m)$

$+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,-m,m)$

$+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,m,-m)$

$+\sum_{i=0}^{13}L_{i,14}|\frac{\partial^{i+13}f_m}{\partial z^i\partial y^{13}}|(x,-m,-m))dx$

$\leq\frac{1}{m^6}(\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial z^i\partial y^{13}}|(-m,-m,-m))$

$\leq\frac{8F(\sum_{j=0}^{13}\sum_{i=0}^{13}L_{i,14}L_{i,j,13})}{m^6}$

Case 22.

$\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3},m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dxdydz$

$=\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3}}(\int_{m\leq|z|\leq m+\frac{1}{m^3}}|\frac{\partial^{14}f_m}{\partial z^{14}}|dz)dxdy$

$=\int_{m\leq|x|\leq m+\frac{1}{m^3},m\leq|y|\leq m+\frac{1}{m^3}}(|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dxdy$

$=\int_{m\leq|x|\leq m+\frac{1}{m^3}}(\int_{m\leq|y|\leq m+\frac{1}{m^3}}((|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,m)+|\frac{\partial^{13}f_m}{\partial z^{13}}|(x,y,-m))dy)dx$

$\leq\frac{1}{m^3}\int_{|x|\leq m+\frac{1}{m^3}}(\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,m,m)$

$+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,-m,m)$

$+\sum_{i=0}^{13}L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,m,-m)$

$+\sum_{i=0}^{13} L_{i,13}|\frac{\partial^{i+13}f_m}{\partial y^i\partial z^{13}}|(x,-m,-m))dx$

$\leq \frac{1}{m^6}(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,-m,m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,-m,m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,m,-m)+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(m,-m,-m)$

$+\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13}|\frac{\partial^{i+j+13}f}{\partial x^j\partial y^i\partial z^{13}}|(-m,-m,-m))$

$\leq \frac{8F(\sum_{j=0}^{13}\sum_{i=0}^{13} L_{i,13}L_{i,j,13})}{m^6}$

It is then clear from $(*)$, summing the bounds from the individual cases 1-19, as at the end of the proof of Lemma 0.9, that there exists a constant $G \in \mathcal{R}_{>0}$ with;

$max(\int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial x^{14}}|dxdydz, \int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial y^{14}}|dxdydz, \int_{\mathcal{R}^3}|\frac{\partial f_m}{\partial z^{14}}|dxdydz) \leq Gm^3$

for sufficiently large m.

□

Lemma 0.11. *Let $\{f_m : m \in \mathcal{N}\}$ be the inflexionary sequence constructed in Lemma 0.10, then for $\overline{k} \in \mathcal{R}^3$, $\overline{k} \neq \overline{0}$, sufficiently large m, we have that there exists $D \in \mathcal{R}_{>0}$, independent of m, with;*

$|\mathcal{F}(f_m)(\overline{k})| \leq \frac{Dm^3}{|\overline{k}|^{14}}$

Moreover, for sufficiently large m, $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$.

A similar result holds for the inflexionary sequence $\{f_m : m \in \mathcal{N}\}$, constructed in Lemma 0.9, for $\overline{k} \neq 0$, sufficiently large m, we have that there exists $D \in \mathcal{R}_{>0}$, independent of m, with;

$|\mathcal{F}(f_m)(\overline{k})| \leq \frac{Dm^2}{|\overline{k}|^{14}}$

Moreover, for sufficiently large m, $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$.

Proof. For $(k_1,k_2,k_3)\in\mathcal{R}^3$, using repeated integration by parts, and the fact that;

$\{\frac{\partial f_m}{\partial x^{14}},\frac{\partial f_m}{\partial y^{14}},\frac{\partial f_m}{\partial z^{14}}\}\subset L^1(\mathcal{R}^3)$

$\{\frac{\partial f_m}{\partial x^{i}},\frac{\partial f_m}{\partial y^{i}},\frac{\partial f_m}{\partial z^{i}}\}\subset C_c(\mathcal{R}^3)$, for $1\leq i\leq 13$

where $C_c(\mathcal{R}^3)$ is the space of continuous functions with compact support, we have, for $m\in\mathcal{N}$;

$\mathcal{F}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}g}{\partial y^{14}}+\frac{\partial^{14}g}{\partial z^{14}})(\overline{k})$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}f_m}{\partial y^{14}}+\frac{\partial^{14}f_m}{\partial z^{14}})e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$=((ik_1)^{14}+(ik_2)^{14}+(ik_3)^{14})\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_m(x,y,z)e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz$

$=(-k_1^{14}-k_2^{14}-k_3^{14})\mathcal{F}(f_m)(\overline{k})$

so that, for $\overline{k}\neq\overline{0}$;

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{|\mathcal{F}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}f_m}{\partial y^{14}}+\frac{\partial^{14}f_m}{\partial z^{14}})(\overline{k})|}{(k_1^{14}+k_2^{14}+k_3^{14})}$ $(\dagger)$

We have, using the result of Lemma 0.10, for sufficiently large m, that;

$|\mathcal{F}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}f_m}{\partial y^{14}}+\frac{\partial^{14}f_m}{\partial z^{14}})(\overline{k})|$

$\frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{\mathcal{R}^3}(\frac{\partial^{14}f_m}{\partial x^{14}}+\frac{\partial^{14}f_m}{\partial y^{14}}+\frac{\partial^{14}f_m}{\partial z^{14}})e^{-ik_1x}e^{-ik_2y}e^{-ik_3z}dxdydz|$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}(|\frac{\partial f_m}{\partial x^{14}}|+|\frac{\partial f_m}{\partial y^{14}}|+|\frac{\partial f_m}{\partial z^{14}}|)dxdydz$

$\leq\frac{3G}{(2\pi)^{\frac{3}{2}}}m^3$ $(\dagger\dagger)$

so that, combining $(\dagger)$ and $(\dagger\dagger)$, we have, for $\overline{k}\neq\overline{0}$, sufficiently large m;

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{3G}{(2\pi)^{\frac{3}{2}}}\frac{m^3}{(k_1^{14}+k_2^{14}+k_3^{14})}$ $(*)$

Using polar coordinates $k_1=rsin(\theta)cos(\phi)$, $k_2=rsin(\theta)sin(\phi)$, $k_3=rcos(\theta)$, $0\leq\theta\leq\pi$, $-\pi<\phi\leq\pi$, we have that;

$\frac{1}{(k_1^{14}+k_2^{14}+k_3^{14})} = \frac{1}{r^{14}}\frac{1}{\alpha(\theta,\phi)}$

where $\alpha(\theta,\phi) = sin^{14}(\theta)(cos^{14}(\phi) + sin^{14}(\phi)) + cos^{14}(\theta)$

We have that, in the range $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$, with $\theta \neq \frac{\pi}{2}$, $|\phi| \neq \frac{\pi}{2}$;

$\alpha(\theta,\phi) = 0$

iff $tan^{14}(\theta)(1 + tan^{14}(\phi)) + \frac{1}{cos^{14}(\phi)} = 0$

iff $tan^{14}(\theta)(1 + tan^{14}(\phi)) = -\frac{1}{cos^{14}(\phi)}$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta = \frac{\pi}{2}$, , $|\phi| \neq \frac{\pi}{2}$

$\alpha(\theta,\phi) = 0$

iff $cos^{14}(\phi) + sin^{14}(\phi) = 0$

iff $tan^{14}(\phi) = -1$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta \neq \frac{\pi}{2}$, , $|\phi| = \frac{\pi}{2}$

$\alpha(\theta,\phi) = 0$

iff $cos^{14}(\theta) + sin^{14}(\theta) = 0$

iff $tan^{14}(\theta) = -1$

which has no solution, as the two sides of the equation have opposite signs.

and, with $\theta = \frac{\pi}{2}$, , $|\phi| = \frac{\pi}{2}$

$\alpha(\theta,\phi)=0$

iff $1=0$

which is not the case. It follows that $\alpha(\theta,\phi)=0$ has no solution in the range $0\leq\theta\leq\pi$, $-\pi\leq\phi\leq\pi$. By continuity, compactness of $[0\,\pi]\times[-\pi,\pi]$ and the fact that $\alpha(\frac{\pi}{2},\frac{\pi}{2})=1$, restricting the interval $[-\pi,\pi]$, there exists $\epsilon>0$, with $\alpha(\theta,\phi)\geq\epsilon$, for $0\leq\theta\leq\pi$, $-\pi<\phi\leq\pi$. In particularly;

$\frac{1}{(k_1^{14}+k_2^{14}+k_3^{14})}\leq\frac{1}{\epsilon r^{14}}$

$=\frac{1}{\epsilon|\overline{k}|^{14}}$

so that, from $(*)$;

$|\mathcal{F}(f_m)(\overline{k})|\leq\frac{3G}{(2\pi)^{\frac{3}{2}}}\frac{m^3}{\epsilon|\overline{k}|^{14}}$

$=\frac{Dm^3}{|\overline{k}|^{14}}$

where $D=\frac{3G}{\epsilon(2\pi)^{\frac{3}{2}}}$

For the final claim, we have, for $1\leq i\leq 3$, $m\in\mathcal{N}$, as f_m is supported on $W_{m+\frac{1}{m}}$ and continuous, that $x_if_m\in L^1(\mathcal{R}^3)$ and, differentiating under the integral sign;

$|\frac{\partial\mathcal{F}(f_m)(\overline{k})}{\partial k^i}|=|\frac{\partial}{\partial k^i}(\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x})|$

$=|\frac{-i}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}x_if_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x})|$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3}|x_if_m(\overline{x})|d\overline{x}$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}||x_if_m(\overline{x})||_1$

so that $\frac{\partial\mathcal{F}(f_m)(\overline{k})}{\partial k^i}$ is bounded, and, in particularly, $\mathcal{F}(f_m)$ is continuous, for $m\in\mathcal{N}$. It follows, using the first result, and polar coordinates, that, for $n>1$, sufficiently large m;

$|\int_{\mathcal{R}^3}\mathcal{F}(f_m)(\overline{k})d\overline{k}|\leq\int_{B(\overline{0},n)}|\mathcal{F}(f_m)(\overline{k})|d\overline{k}+\int_{\mathcal{R}^3\setminus B(\overline{0},n)}|\mathcal{F}(f_m)(\overline{k})|d\overline{k}$

$\leq \frac{4C_n\pi^3}{3} + \int_{\mathcal{R}^3\setminus B(\bar{0},n)} \frac{Dm^3}{|\bar{k}|^{14}}$

$\leq \frac{4C_n\pi^3}{3} + \int_0^\pi \int_{-\pi}^\pi \int_n^\infty \frac{Dm^3}{r^{14}}|r^2 sin(\theta)|drd\theta d\phi$

$\leq \frac{4C_n\pi^3}{3} + 2D\pi^2 m^3 \int_n^\infty \frac{dr}{r^{12}}$

$\leq \frac{4C_n\pi^3}{3} + 2D\pi^2 m^3 [\frac{-1}{11r^{11}}]_n^\infty$

$= \frac{4C_n\pi^3}{3} + \frac{2D\pi^2 m^3}{11n^{11}}$

where $C_n = ||\mathcal{F}(f_m)|_{B(\bar{0},n)}||_\infty$, so that $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$.

A similar proof works in the two dimensional case.

□

Lemma 0.12. *Let $\{f_m : m \in \mathcal{N}\}$ be the inflexionary sequences constructed in Lemmas 0.9 and 0.10, then;*

$\int_{[-m-\frac{1}{m^2},m+\frac{1}{m^2}]^2\setminus[-m,m]^2} |f_m|dxdy \leq \frac{E}{m}$

for sufficiently large $m \in \mathcal{N}$, where $E \in \mathcal{R}_{>0}$.

$\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3} |f_m|dxdydz \leq \frac{E}{m}$

for sufficiently large $m \in \mathcal{N}$, where $E \in \mathcal{R}_{>0}$.

Proof. By the construction, we obtain the result that for an inflexionary approximation sequence f_m in $\mathcal{R}^2$ or $\mathcal{R}^3$;

$|f_m|_{[-m-\frac{1}{m^2},m+\frac{1}{m^2}]^2\setminus[-m,m]^2} \leq D$

$|f_m|_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3} \leq D$ $(*)$

independently of m. We give the proof of $(*)$ in the 3-dimensional case. We have that, for $m \leq x \leq m + \frac{1}{m^3}$, $m \leq y \leq m + \frac{1}{m^3}$, $m \leq z \leq m + \frac{1}{m^3}$;

$|f_m|(x, y, z) \leq \sum_{i=0}^{13} D_i |\frac{\partial^i f_m}{\partial z^i}|(x, y, m)$

$\leq \sum_{i=0}^{13} D_i \sum_{j=0}^{13} D_{ij} \frac{\partial^{i+j} f_m}{\partial y^j \partial z^i}|(x, m, m)$

$\leq \sum_{i=0}^{13} D_i \sum_{j=0}^{13} D_{ij} \sum_{k=0}^{13} D_{ijk} \frac{\partial^{i+j+k} f_m}{\partial x^k \partial y^j \partial z^i}|(m,m,m)$

$= \sum_{i=0}^{13} D_i \sum_{j=0}^{13} D_{ij} \sum_{k=0}^{13} D_{ijk} \frac{\partial^{i+j+k} f}{\partial x^k \partial y^j \partial z^i}|(m,m,m)$

$\leq C \sum_{i,j,k=0}^{13} D_i D_{ij} D_{ijk}$

$= C \sum_{i,j,k=0}^{13} D_i D_j D_k = D$

The proof of the bound for the other regions is similar and left to the reader, as is the two dimensional case. It follows that, using the binomial theorem;

$\int_{[-m-\frac{1}{m^2},m+\frac{1}{m^2}]^2 \setminus [-m,m]^2} |f_m| dxdy$

$\leq Darea([-m-\frac{1}{m^2}, m+\frac{1}{m^2}]^2 \setminus [-m,m]^2)$

$= 4D((m+\frac{1}{m^2})^2 - m^2)$

$4D(m^2 + \frac{2m}{m^2} + \frac{1}{m^4} - m^2)$

$\leq \frac{E}{m}$

and;

$\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3 \setminus [-m,m]^3} |f_m| dxdydz$

$\leq Dvol([-m-\frac{1}{m^3}, m+\frac{1}{m^3}]^3 \setminus [-m,m]^3)$

$= 8D((m+\frac{1}{m^3})^3 - m^3)$

$8D(m^3 + \frac{3m^2}{m^3} + \frac{3m}{m^6} + \frac{1}{m^9} - m^3)$

$\leq \frac{E}{m}$

for m sufficiently large, where $E \in \mathcal{R}_{>0}$.

$\square$

Lemma 0.13. *Let $f \in C^\infty(\mathcal{R}^3)$ be quasi split normal, with the Fourier transform $\mathcal{F}$ defined in [2]. Let $\{f_m : m \in \mathcal{N}\}$ be the inflexionary sequence constructed in Lemma 0.10. Let $\mathcal{F}$ be the ordinary Fourier transform, defined for each f_m, then, for any (k_{01}, k_{02}, k_{03}), with $k_{01} \neq$*

0, $k_{02}\neq 0$, $k_{03}\neq 0$, the sequence $\{\mathcal{F}(f_m):m\in\mathcal{N}\}$ converges pointwise and uniformly to $\mathcal{F}(f)$ on $\mathcal{R}^3\setminus(|k_1|<k_{01})\cup(|k_2|<k_{02})\cup(|k_3|<k_{03})$. In particularly, $\mathcal{F}(f)\in C(\mathcal{R}^3\setminus\{k_1=0\cup k_2=0\cup k_3=0\})$. A corresponding result holds in dimension 2.

Proof. For $g\in C_c(\mathcal{R}^3)$ or g quasi split normal, and $m\in\mathcal{N}$, define;

$\mathcal{F}_m(g)(\overline{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{C_m}g(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}$

For $\overline{k}\in\mathcal{R}^3\setminus(|k_1|<k_{01})\cup(|k_2|<k_{02})\cup(|k_3|<k_{03})$, $m\in\mathcal{N}$, $\epsilon>0$, we have, using Lemma 0.12;

$|\mathcal{F}(f)(\overline{k})-\mathcal{F}(f_m)(\overline{k})|\leq|\mathcal{F}(f)(\overline{k})-\mathcal{F}_m(f)(\overline{k})|+|\mathcal{F}_m(f)(\overline{k})-\mathcal{F}_m(f_m)(\overline{k})|$

$+|\mathcal{F}_m(f_m)(\overline{k})-\mathcal{F}(f_m)(\overline{k})|$

$=|\mathcal{F}(f)(\overline{k})-\mathcal{F}_m(f)(\overline{k})|+|\mathcal{F}_m(f_m)(\overline{k})-\mathcal{F}(f_m)(\overline{k})|$

$\leq|\mathcal{F}(f)(\overline{k})-\mathcal{F}_m(f)(\overline{k})|+|\int_{\mathcal{R}^3\setminus C_m}f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$

$\leq|\mathcal{F}(f)(\overline{k})-\mathcal{F}_m(f)(\overline{k})|+\int_{C_{m+\frac{1}{m^3}}\setminus C_m}|f_m(\overline{x})|d\overline{x}$

$\leq|\mathcal{F}(f)(\overline{k})-\mathcal{F}_m(f)(\overline{k})|+\frac{E}{m}\ (BB)$

By the result in [2], we have that, for sufficiently large m;

$|\mathcal{F}(f)(\overline{k})-\mathcal{F}_m(f)(\overline{k})|\leq\frac{C_{k_{01},k_{02},k_{03}}}{m}\ (B)$

Combining (B) and (BB), we obtain that;

$|\mathcal{F}(f)(\overline{k})-\mathcal{F}(f_m)(\overline{k})|\leq\frac{C_{k_{01},k_{02},k_{03}}+E}{m}$

$\leq\epsilon$

for $m\geq\frac{C_{k_{01},k_{02},k_{03}}+E}{\epsilon}$. As $\epsilon>0$ was arbitrary, we obtain the first result. The fact that each $\mathcal{F}(f_m)$ is continuous, follows from the differentiability $\mathcal{F}(f_m)$, which is a consequence of the fact that $x_if_m(\overline{x}$ has compact support, for $1\leq i\leq 3$. The last result then follows immediately from the fact that $k_{01}\neq 0$, $k_{02}\neq 0$, $k_{03}\neq 0$ were arbitrary and the uniform limit of continuous functions is continuous. The last claim is similar.

□

Lemma 0.14. *Let $f \in C^\infty(\mathcal{R}^3)$, with $\frac{\partial^{i_1+i_2+i_3}}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}$ bounded for $0 \leq i_1+i_2+i_3 \leq 40$, f quasi split normal, and of moderate decrease. Then;*

$f(\overline{x}) = \mathcal{F}^{-1}(\mathcal{F}(f))(\overline{x})$, $(\overline{x} \in \mathcal{R}^3)$

where, for $g \in L^1(\mathcal{R}^3)$;

$\mathcal{F}^{-1}(g)(\overline{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathcal{R}^3} g(\overline{k})e^{i\overline{k}.\overline{x}}d\overline{k}$

The same claim holds in dimension 2.

Proof. By Lemma 0.1, we have that $\mathcal{F}(f) \in L^1(\mathcal{R}^3)$. Let $\{f_m : m \in \mathcal{N}\}$ be the inflexionary approximating sequence, given by Lemma 0.9, then, for sufficiently large m, $f_m \in L^1(\mathcal{R}^3)$ and $\mathcal{F}(f_m) \in L^1(\mathcal{R}^3)$ by Lemma 0.11. It follows, see [1] or the method of [4], that for such m, $f_m = \mathcal{F}^{-1}(\mathcal{F}(f_m))$, $(***)$, By the proof of Lemma 0.13, we have that, for $\overline{k}$ with $min(|k_1|,|k_2|,|k_3|) > \epsilon > 0$, $|\mathcal{F}(f)(k) - \mathcal{F}(f_m)(k)| \leq \frac{E_\epsilon}{m}$, (B). By the fact that f is of very moderate decrease, we have that $\mathcal{F}(f) - \mathcal{F}(f_m) \in L^2(\mathcal{R}^3)$, and by the classical theory, and by the proof of Lemma 0.12, we have that;

$||\mathcal{F}(f) - \mathcal{F}(f_m)||^2_{L^2(\mathcal{R}^3)}$

$= ||f - f_m||^2_{L^2(\mathcal{R}^3)}$

$\leq \int_{\mathcal{R}^3\setminus C_m} |f|^2 d\overline{x} + \int_{C_{m+\frac{1}{m^3}}\setminus C_m} |f_m|^2 d\overline{x}$

$\leq \int_{\mathcal{R}^3\setminus B(\overline{0},m)} |f|^2 d\overline{x} + \frac{G}{m}$

$\leq \int_{\mathcal{R}^3\setminus B(\overline{0},m)} \frac{C}{|\overline{x}|^4} d\overline{x} + \frac{G}{m}$

$\leq 2\pi^2 \int_m^\infty \frac{C}{r^2} dr + \frac{G}{m}$

$\leq \frac{C}{m} + \frac{G}{m}$

$\leq \frac{F}{m}$

where $\{C,F,G\}\subset\mathcal{R}_{>0}$. It follows that $||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^2(\mathcal{R}^3)}\rightarrow 0$ as $m\rightarrow\infty$. In particularly, there exists a constant $H\in\mathcal{R}_{>0}$ with $||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^2(\mathcal{R}^3)}\leq H$, for sufficiently large m. By the Cauchy Schwarz inequality, we have that, for m sufficiently large;

$||\mathcal{F}(f)-\mathcal{F}(f_m)||_{L^1(B(\bar{0},n))}$

$\leq ||(\mathcal{F}(f)-\mathcal{F}(f_m))|_{B(\bar{0},n)}||_{L^2(B(\bar{0},n))}||1_{B(\bar{0},n)}||_{L^2(B(\bar{0},n))}$

$\leq \frac{\sqrt{F}}{\sqrt{m}}||1_{B(\bar{0},n)}||_{L^2(B(\bar{0},n))}$

$= \frac{2\sqrt{F\pi}n^{\frac{3}{2}}}{\sqrt{3m}}$

$= \frac{Kn^{\frac{3}{2}}}{m^{\frac{1}{2}}}, (A)$

Using the fact from Lemma 0.1, that $\mathcal{F}(f)\in L^1(\mathcal{R})$, and of rapid decrease, for $\delta>0$ arbitrary, we have that;

$\int_{\mathcal{R}^3\setminus B(\bar{0},n)}|\mathcal{F}(f)(\bar{k}|d\bar{k}<\delta$

for $n\in\mathcal{N}$, sufficiently large, $n\geq n_0$. Choosing $n\in\mathcal{N}$, with $m=[n^{\frac{10}{3}}]$, and using (A), Lemma 0.11, we have, for $\bar{x}\in\mathcal{R}^3$, that;

$|\mathcal{F}^{-1}(\mathcal{F}(f))(\bar{x})-\mathcal{F}^{-1}(\mathcal{F}(f_m))(\bar{x})|=|\mathcal{F}^{-1}(\mathcal{F}(f)(\bar{k})-\mathcal{F}(f_m)(\bar{k}))|$

$=\frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{B(\bar{0},n)}(\mathcal{F}(f)(\bar{k})-\mathcal{F}(f_m)(\bar{k}))e^{i\bar{k}\cdot\bar{x}}d\bar{k}$

$+\int_{\mathcal{R}^3\setminus B(\bar{0},n)}(\mathcal{F}(f)(\bar{k})-\mathcal{F}(f_m)(\bar{k}))e^{i\bar{k}\cdot\bar{x}}d\bar{k}|$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{B(\bar{0},n)}|\mathcal{F}(f)(\bar{k})-\mathcal{F}(f_m)(\bar{k})|d\bar{k}$

$+\int_{\mathcal{R}^3\setminus B(\bar{0},n)}|\mathcal{F}(f)(\bar{k})|d\bar{k}+\int_{\mathcal{R}^3\setminus B(\bar{0},n)}|\mathcal{F}(f_m)(\bar{k})|d\bar{k})$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{B(\bar{0},n)}|\mathcal{F}(f)(\bar{k})-\mathcal{F}(f_m)(\bar{k})|d\bar{k}+\delta+\int_{\mathcal{R}^3\setminus B(\bar{0},n)}\frac{Dm^3}{|k|^{14}}d\bar{k})$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}(\frac{Kn^{\frac{3}{2}}}{3m^{\frac{1}{2}}}+\delta+\int_{\mathcal{R}^3\setminus B(\bar{0},n)}\frac{Dm^3}{|k|^{14}}d\bar{k})$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}(\frac{Kn^{\frac{3}{2}}}{[n^{\frac{10}{3}}]^{\frac{1}{2}}}+\delta+\int_{\mathcal{R}^3\setminus B(\bar{0},n)}\frac{Dn^{10}}{|k|^{14}}d\bar{k})$

$\leq\frac{1}{(2\pi)^{\frac{3}{2}}}(\frac{K}{n^{\frac{1}{6}}}+\delta+2\pi^2\int_{r>n}\frac{Dn^{10}}{r^{14}}dr)$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}(\frac{K}{n^{\frac{1}{6}}} + \delta + 2D\pi^2 n^{10}[\frac{-1}{13r^{13}}]_n^\infty)$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}(\frac{K}{n^{\frac{1}{6}}} + \delta + \frac{2D\pi^2}{13n^3})$

$< \frac{2\delta}{(2\pi)^{\frac{3}{2}}}$

for sufficiently large $n \geq n_0$, or $m \geq m_0$, so that, as $\epsilon > 0$ and $\delta > 0$ were arbitrary, for $\overline{x} \in \mathcal{R}^3$;

$lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x}) = \mathcal{F}^{-1}\mathcal{F}(f)(\overline{x})$, $(****)$

and, by Definition 0.3, $(***)$, $(****)$;

$f(\overline{x}) = lim_{m\rightarrow\infty}f_m(\overline{x}) = lim_{m\rightarrow\infty}\mathcal{F}^{-1}(\mathcal{F}(f_m))(\overline{x}) = \mathcal{F}^{-1}\mathcal{F}(f)(\overline{x})$

The proof of the final claim in dimension 2 is identical.

□

The following results are not required for the proof of the inversion theorem but are required in [4].

Definition 0.15. *We say that $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ is of very moderate decrease if $|f(\overline{x})| \leq \frac{C}{|\overline{x}|}$ for $|\overline{x}| > C$, $C \in \mathcal{R}_{>0}$. We say that $f : \mathcal{R}^3 \rightarrow \mathcal{R}$ is of moderate decrease n if $|f(\overline{x})| \leq \frac{C}{|\overline{x}|^n}$ for $|\overline{x}| > C$, $C \in \mathcal{R}_{>0}$, $n \geq 2$. We just say that f is of moderate decrease if f is of moderate decrease 2. We call $\{\theta, \phi\}$ generic if $sin(\theta)cos(\phi) \neq 0$, $sin(\theta)sin(\phi) \neq 0$, $cos(\theta) \neq 0$*

Lemma 0.16. *Let f be of very moderate decrease and quasi split normal, $f \in C^{41}(\mathcal{R}^3)$, such that the partial derivatives $\{\frac{\partial f^{i+j+k}}{\partial x^i \partial y^j \partial z^k} : 1 \leq i+j+k \leq 41\}$ are of moderate decrease, and of moderate decrease $i+j+k+1$, then for $1 \leq i \leq 3$;*

$k_i\mathcal{F}(f)(\overline{k}) \in C^1(\mathcal{R}^3 \setminus (k_1 = 0 \cup k_2 = 0 \cup k_3 = 0))$

$lim_{\overline{k}\rightarrow 0, \overline{k}\notin(k_1=0\cup k_2=0\cup k_3=0)}k_i\mathcal{F}(f)(\overline{k}) = 0$

The same results hold for $k_i\mathcal{F}(\frac{\partial f}{\partial x_j})$, $1 \leq i \leq j \leq 3$, when $f \in C^{42}(\mathcal{R}^3)$.

Making a polar coordinate change, for $\{\theta,\phi\}$ *generic,* $r\mathcal{F}(f)_{\theta,\phi}(r)\in C^1(\mathcal{R}_{>0})$, $lim_{r\rightarrow 0}r\mathcal{F}(f)_{\theta,\phi}(r)=0$, *and similarly for* $r\mathcal{F}(\frac{\partial f}{\partial x_j})$, $1\leq j\leq 3$.

We have that $\mathcal{F}(f)(\bar{k})\in L^1(\mathcal{R}^3)$, $\{\frac{\mathcal{F}(\frac{\partial f}{\partial x_j})(\bar{k})}{|\bar{k}|}:1\leq j\leq 3\}\subset L^1(\mathcal{R}^3)$

and $\{\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i\partial x_j})(\bar{k})}{|\bar{k}|^2}:1\leq i,j\leq 3\}\subset L^1(\mathcal{R}^3)$

For any given $\epsilon>0$, *there exists* $\delta>0$, *for* $1\leq j\leq 3$, *such that for a generic translation* $\bar{l}$ *with* $l_1\neq 0$, $l_2\neq 0$, $l_3\neq 0$;

$$max(|\int_0^\delta r\mathcal{F}_{\theta,\phi,\bar{l}}(\frac{\partial f}{\partial x_j})(r)dr|,|\int_0^\delta \frac{d}{dr}(r\mathcal{F}_{\theta,\phi,\bar{l}}(\frac{\partial f}{\partial x_j})(r))dr|)<\epsilon$$

uniformly in $\{\theta,\phi\}$.

Proof. As $\frac{\partial f}{\partial x}$ is of moderate decrease and quasi split normal, for fixed y,z, $f_{y,z}$ is of very moderate decrease and analytic at infinity, we have for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$;

$$\mathcal{F}(\frac{\partial f}{\partial x})=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_1\rightarrow\infty}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}\frac{\partial f}{\partial x}(\bar{x})e^{-i\bar{k}.\bar{x}}dx_1dx_2dx_3$$

$$=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{r_1\rightarrow\infty}\int_{-r_1}^{r_1}\frac{\partial f}{\partial x}(\bar{x})e^{-ik_1x_1}dx_1)e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$$

$$=\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{r_1\rightarrow\infty}([fe^{-ikx_1}]_{-r_1}^{r_1}+ik_1\int_{-r_1}^{r_1}f(\bar{x})e^{-ikx_1}dx_1)$$

$$e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$$

$$=ik_1\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}(lim_{r_1\rightarrow\infty}\int_{-r_1}^{r_1}f(\bar{x})e^{-ikx_1}dx_1)e^{-i(k_2x_2+k_3x_3)}dx_2dx_3$$

$$=ik_1\frac{1}{(2\pi)^{\frac{3}{2}}}lim_{r_1\rightarrow\infty}lim_{r_2\rightarrow\infty}lim_{r_3\rightarrow\infty}\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\int_{-r_3}^{r_3}f(\bar{x})e^{-i\bar{k}.\bar{x}}dx_1dx_2dx_3$$

$$=ik_1\mathcal{F}(f)(\bar{k})\ (TT)$$

the limit interchange being justified by the calculation in [2]. It follows that, for $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$, we have that;

$$k_1\mathcal{F}(f)(\bar{k})=-i\mathcal{F}(\frac{\partial f}{\partial x})$$

and similarly;

$k_i\mathcal{F}(f)(\overline{k}) = -i\mathcal{F}(\frac{\partial f}{\partial x_i})$ (A), for $1 \leq i \leq 3$ and $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$.

It follows that, using the fact that;

$F(x_1, k_2, k_3) = lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} \frac{\partial f}{\partial x}(x_1, x_2, x_3) e^{-ik_2x_2} e^{-ik_3x_3} dx_2 dx_3$

is of moderate decrease, the DCT and the FTC, and the fact that $f_{y,z}$ is of very moderate decrease;

$lim_{\overline{k} \rightarrow 0, \overline{k} \notin (k_1=0 \cup k_2=0 \cup k_3=0)} k_1 \mathcal{F}(f)(\overline{k})$

$-ilim_{\overline{k} \rightarrow 0, \overline{k} \notin (k_1=0 \cup k_2=0 \cup k_3=0)} \mathcal{F}(f)(\frac{\partial f}{\partial x})(\overline{k})$

$= \frac{-i}{(2\pi)^{\frac{3}{2}}} lim_{\overline{k} \rightarrow 0, \overline{k} \notin (k_1=0 \cup k_2=0 \cup k_3=0)} lim_{r_1 \rightarrow \infty} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_1}^{r_1} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} \frac{\partial f}{\partial x}(\overline{x}) e^{-i\overline{k}.\overline{x}} dx_1 dx_2 dx_3$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{k_2 \rightarrow 0, k_3 \rightarrow 0, k_2 \neq 0, k_3 \neq 0} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} (lim_{k_1 \rightarrow 0} \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(\overline{x}) e^{-ik_1x_1} dx_1)$

$e^{-i(k_2x_2+k_3x_3)} dx_2 dx_3$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{k_2 \rightarrow 0, k_3 \rightarrow 0, k_2 \neq 0, k_3 \neq 0} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} (\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(\overline{x}) dx_1) e^{-i(k_2x_2+k_3x_3)} dx_2 dx_3$

$= \frac{1}{(2\pi)^{\frac{3}{2}}} lim_{k_2 \rightarrow 0, k_3 \rightarrow 0, k_2 \neq 0, k_3 \neq 0} lim_{r_2 \rightarrow \infty} lim_{r_3 \rightarrow \infty} \int_{-r_2}^{r_2} \int_{-r_3}^{r_3} ([f]_{-\infty}^{\infty}) e^{-i(k_2x_2+k_3x_3)} dx_2 dx_3$

$= 0$ (E)

Similarly;

$lim_{\overline{k} \rightarrow 0, \overline{k} \notin (k_1=0 \cup k_2=0 \cup k_3=0)} k_i \mathcal{F}(f)(\overline{k}) = 0$, $1 \leq i \leq 3$

As $f \in C^{41}(\mathcal{R}^3)$, we have, by the product rule, that $x_i \frac{\partial f}{\partial x_j} \in C^{40}(\mathcal{R}^3)$, $1 \leq i \leq j \leq 3$. As f is of very moderate decrease and;

$\{\frac{\partial f^{l+m+n}}{\partial x_1^l \partial x_2^m \partial x_3^m} : 1 \leq l+m+n \leq 40\}$

are of very moderate decrease, we have, by repeated application of the product rule again, that;

$\{\frac{\partial^{l+m+n} x_i \frac{\partial f}{\partial x_j}}{\partial x_1^l \partial x_2^m \partial x_3^n} : 0 \leq l+m+n \leq 40\}$, $1 \leq i \leq j \leq 3$

are bounded. By Lemma 0.4, there exists an inflexionary approximation sequence g_m for $x\frac{\partial f}{\partial x}$ with the properties that;

(i) $g_m \in C^{13,13,14}(\mathcal{R}^3)$

(ii). $g_m|_{[-m,m]^3} = x\frac{\partial f}{\partial x}|_{[-m,m]^3}$

(iii). $\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3} |g_m(\overline{x})d\overline{x} \leq \frac{E}{m}$

(iv). $g_m|_{\mathcal{R}^3\setminus[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3} = 0$

By the construction of g_m, we have that $f_m = \frac{g_m}{x}$ is an approximation sequence for $\frac{\partial f}{\partial x}$, with the property that;

$(i)'$ $f_m \in C^{13,13,14}(\mathcal{R}^3)$

$(ii)'$. $f_m|_{[-m,m]^3} = \frac{\partial f}{\partial x}|_{[-m,m]^3}$

$(iii)'$. $\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3} |f_m(\overline{x})d\overline{x} \leq \frac{E'}{m}$

$(iv)'$. $f_m|_{\mathcal{R}^3\setminus[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3} = 0$

Following through the proof of Lemma 0.13, as $\frac{\partial f}{\partial x}$ is quasi split normal of moderate decrease and, therefore, of very moderate decrease, we have that $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(\frac{\partial f}{\partial x})$ on compact subsets of $\mathcal{R}^3 \setminus (k_1 = 0 \cup k_2 = 0 \cup k_3 = 0)$, so that $\mathcal{F}(\frac{\partial f}{\partial x}) \in C(\mathcal{R}^3 \setminus (k_1 = 0 \cup k_2 = 0 \cup k_3 = 0))$, As $x_i x_j f_m \in L^1(\mathcal{R}^3)$, for $1 \leq i \leq j \leq 3$, we have that $\mathcal{F}(f_m)$ is twice differentiable, in particularly, $\mathcal{F}(f_m) \in C^1(\mathcal{R}^3)$. As f is quasi split normal, so is $\frac{\partial f}{\partial x}$ and $x\frac{\partial f}{\partial x}$. It follows that for $\{m,n\} \subset \mathcal{N}$, with $m \geq n$, differentiating under the integral sign, using the DCT, property (iii) of an inflexionary approximating sequence, and the fact that $x\frac{\partial f}{\partial x}$ is of very moderate decrease and quasi split normal, for $|k_1| \geq \epsilon_1 > 0$, $|k_2| \geq \epsilon_2 > 0$, $|k_3| \geq \epsilon_3 > 0$, we have that;

$$|\frac{\partial \mathcal{F}(f_m)}{\partial k_1} - \frac{\partial \mathcal{F}(f_n)}{\partial k_1}|$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}|\frac{\partial}{\partial k_1}(\int_{\mathcal{R}^3} f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x} - \frac{\partial}{\partial k_1}\int_{\mathcal{R}^3} f_n(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{\mathcal{R}^3} -ix_1 f_m(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x} - \int_{\mathcal{R}^3} -ix_1 f_n(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$$

$= \frac{1}{(2\pi)^{\frac{3}{2}}}|\int_{\mathcal{R}^3}(g_m-g_n)(\overline{x})e^{-i\overline{k}.\overline{x}}d\overline{x}|$

$\leq \frac{1}{(2\pi)^{\frac{3}{2}}}(\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_m(\overline{x})|d\overline{x}+\int_{[-m-\frac{1}{m^3},m+\frac{1}{m^3}]^3\setminus[-m,m]^3}|g_n(\overline{x})|d\overline{x}$

$+|\int_{[-m,m]^3\setminus[-n,n]^3}x_1\frac{\partial f}{\partial x_1}e^{-i\overline{k}.\overline{x}}d\overline{x}|)$

$\leq \frac{E}{m}+\frac{E}{n}+\frac{C(\overline{k})}{n}$ $(*)$

where $C(\overline{k})$ is uniformly bounded on the region $|k_1|\geq\epsilon_1>0$, $|k_2|\geq\epsilon_2>0$, $|k_3|\geq\epsilon_3>0$. It follows that the sequence $\{\frac{\partial\mathcal{F}(f_m)}{\partial k_1}:m\in\mathcal{N}\}$ is uniformly Cauchy on the region $|k_1|\geq\epsilon_1>0$, $|k_2|\geq\epsilon_2>0$, $|k_3|\geq\epsilon_3>0$, and converges uniformly. By considering inflexionary sequences for $y\frac{\partial f}{\partial x}$ and $z\frac{\partial f}{\partial x}$, we can similarly show that the sequences $\{\frac{\partial\mathcal{F}(f_m)}{\partial k_2}:m\in\mathcal{N}\}$ and $\{\frac{\partial\mathcal{F}(f_m)}{\partial k_3}:m\in\mathcal{N}\}$ are uniformly Cauchy on the region $|k_1|\geq\epsilon_1>0$, $|k_2|\geq\epsilon_2>0$, $|k_3|\geq\epsilon_3>0$, and converge uniformly. As $\mathcal{F}(f_m)$ converges uniformly to $\mathcal{F}(\frac{\partial f}{\partial x})$ on the regions $|k_1|\geq\epsilon_1>0$, $|k_2|\geq\epsilon_2>0$, $|k_3|\geq\epsilon_3>0$, it follows that $\mathcal{F}(\frac{\partial f}{\partial x})\in C^1(\mathcal{R}^3\setminus(k_1=0\cup k_2=0\cup k_3=0))$. The same result folds for $\mathcal{F}(\frac{\partial f}{\partial y})$ and $\mathcal{F}(\frac{\partial f}{\partial z})$, so by (A);

$\{k_1\mathcal{F}(f)(\overline{k}),k_2\mathcal{F}(f)(\overline{k}),k_3\mathcal{F}(f)(\overline{k})\}\subset C^1(\mathcal{R}^3\setminus(k_1=0\cup k_2=0\cup k_3=0))$

(B)

It follows that, changing to polars;

$\frac{\partial r\mathcal{F}(f)(\overline{k})}{\partial r}=(\frac{\partial}{\partial k_1}\frac{k_1}{r}+\frac{\partial}{\partial k_2}\frac{k_2}{r}+\frac{\partial}{\partial k_3}\frac{k_3}{r})(r\mathcal{F}(f)(\overline{k}))$

$=\frac{\partial k_1\mathcal{F}(f)(\overline{k})}{\partial k_1}+\frac{\partial k_2\mathcal{F}(f)(\overline{k})}{\partial k_2}+\frac{\partial k_3\mathcal{F}(f)(\overline{k})}{\partial k_3}$ (WW)

so that, for generic $\{\theta,\phi\}$, $r\mathcal{F}(f)(r)_{\theta,\phi}\in C^1(\mathcal{R}_{>0})$, by (B). Moreover;

$lim_{r\rightarrow 0}r\mathcal{F}(f)(r)_{\theta,\phi}.$

$=lim_{\overline{k}(\theta,\phi)\rightarrow 0}\frac{r}{k_1}lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0}k_1\mathcal{F}(f)(\overline{k})$

$=lim_{\overline{k}(\theta,\phi)\rightarrow 0}\frac{r}{k_2}lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0}k_2\mathcal{F}(f)(\overline{k})$

$=lim_{\overline{k}(\theta,\phi)\rightarrow 0}\frac{r}{k_3}lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0}k_3\mathcal{F}(f)(\overline{k})$

$= lim_{\overline{k}(\theta,\phi)\rightarrow 0} sign(k_1)(1+\frac{k_2^2}{k_1^2}+\frac{k_3^2}{k_1^2}) lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_1\mathcal{F}(f)(\overline{k})$

$= lim_{\overline{k}(\theta,\phi)\rightarrow 0} sign(k_2)(1+\frac{k_1^2}{k_2^2}+\frac{k_3^2}{k_2^2}) lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_2\mathcal{F}(f)(\overline{k})$

$= lim_{\overline{k}(\theta,\phi)\rightarrow 0} sign(k_3)(1+\frac{k_1^2}{k_3^2}+\frac{k_2^2}{k_3^2}) lim_{\overline{k}(\theta,\phi)\rightarrow\overline{0},k_1\neq 0,k_2\neq 0,k_3\neq 0} k_3\mathcal{F}(f)(\overline{k})$

$= 0$

as the cases $max(|k_2|,|k_3|)\leq |k_1|$, $max(|k_1|,|k_3|)\leq |k_2|$ and $max(|k_1|,|k_2|)\leq |k_3|$ are exhaustive.

Clearly, we can repeat the above arguments for $\frac{\partial f}{\partial x_i}$, $1\leq i\leq 3$, and $f\in C^{42}(\mathcal{R}^3)$, using the fact that $\frac{\partial f}{\partial x_i}$ is of moderate decrease, in particularly of very moderate decrease, with the higher derivatives $\frac{\partial^{l+m+n}\frac{\partial f}{\partial x_i}}{\partial x^l y^m z^n}$ of moderate decrease $l+m+n+2$, in particularly of moderate decrease $l+m+n+1$.

For the next claim, we have, $\mathcal{F}(f)\in L^1(\mathcal{R}^3)$, (R), by Lemma 0.1. A similar calculation shows that, as $\frac{\partial f}{\partial x}$ is of moderate decrease 2, that $f\in L^{\frac{3}{2}+\epsilon}(\mathcal{R}^3)$, for $\epsilon>0$. Applying the Haussdorff-Young inequality, $\mathcal{F}(\frac{\partial f}{\partial x})\in L^{3-\delta}(\mathcal{R}^3)$, for $\delta>0$. In particular, due to the decay again, $\mathcal{F}(\frac{\partial f}{\partial x})\in L^2(\mathcal{R}^3)$. Locally, on $B(\overline{0},1)$, for $\delta>0$;

$\int_{B(\overline{0},1)}\frac{1}{|\overline{k}|^{3-\delta}}d\overline{k}$

$= \int_{0\leq\theta\leq\pi,-\pi\leq\phi\leq\phi}\int_0^1\frac{r^2}{r^{3-\delta}}drd\theta d\phi$

$\leq 2\pi^2[r^\delta]_0^1$

$= 2\pi^2<\infty$

so that $\frac{1}{|\overline{k}|}\in L^{3-\delta}(B(\overline{0},1))$, in particularly $\frac{1}{|\overline{k}|}\in L^2(B(\overline{0},1))$. As $\mathcal{F}(\frac{\partial f}{\partial x})\in L^2(B(\overline{0},1))$, by the Cauchy Schwarz inequality, we obtain that $\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{|\overline{k}|}\in L^1(B(\overline{0},1))$, and by the decay, we have that $\frac{\mathcal{F}(\frac{\partial f}{\partial x})(\overline{k})}{|\overline{k}|}\in L^1(\mathcal{R}^3)$. Similar arguments show that $\frac{\mathcal{F}(\frac{\partial f}{\partial x_i})(\overline{k})}{|\overline{k}|}\in L^1(\mathcal{R}^3)$, for $1\leq i\leq 3$. We also have, using the fact that $\frac{\partial^2 f}{\partial x_i\partial x_j}$ is of moderate decrease and quasi split normal, $1\leq i\leq j\leq 3$, using the argument (TT) twice, that for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$;

$\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j}) = (ik_i)(ik_j)\mathcal{F}(f)(\overline{k})$

$= -k_i k_j \mathcal{F}(f)(\overline{k})$

so that;

$\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j})(\overline{k})}{|\overline{k}|^2} = \frac{-k_i k_j}{|\overline{k}|^2}\mathcal{F}(f)(\overline{k})$

with, for $k_i \neq 0$, $k_j \neq 0$;

$|\frac{|-k_i k_j}{|\overline{k}|^2}| = |sign(k_1)sign(k_2||\frac{1}{(1+\frac{k_2}{k_1}^2+\frac{k_3}{k_1}^2)^{\frac{1}{2}}}||\frac{1}{(1+\frac{k_1}{k_2}^2+\frac{k_3}{k_2}^2)^{\frac{1}{2}}}| \leq 1$

so that;

$|\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j})(\overline{k})}{|\overline{k}|^2}| \leq |\mathcal{F}(f)(\overline{k})|$

and, by (R), $\mathcal{F}(f)(\overline{k}) \in L^1(\mathcal{R}^3)$, so that $\frac{\mathcal{F}(\frac{\partial^2 f}{\partial x_i \partial x_j})(\overline{k})}{|\overline{k}|^2} \in L^1(\mathcal{R}^3)$.

The last claim follows from the fact that, for $\overline{l}$, with $l_1 \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$, the translation $\mathcal{F}_{\overline{l}}(\frac{\partial f}{\partial x_i})(\overline{k}) \in C^1(B(\overline{0}, \epsilon'))$, for some $\epsilon' > 0$. In particular, given $\epsilon > 0$, there exists $\delta > 0$, such that;

$max(|\int_0^\delta r\mathcal{F}_{\theta,\phi,\overline{l}}(\frac{\partial f}{\partial x_j})(r)dr|, |\int_0^\delta \frac{d}{dr}(r\mathcal{F}_{\theta,\phi,\overline{l}}(\frac{\partial f}{\partial x_j})(r))dr|) < \epsilon$

uniformly in $\{\theta, \phi\}$.

□

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